Proceedings of the Steklov Institute of Mathematics, Vol. 221, 1998, pp. 60–70.
From Trudy Matematicheskogo Instituta imeni V.A. Steklova, Vol. 221, 1998, pp. 69–80.

A Generalization of the Sato–Levine Invariant

P.M. Akhmetiev and D. Repovš

Received October, 1997

1. INTRODUCTION

Let $f : L_1 \cup L_2 \subset \mathbb{R}^3$ be a two-component oriented link in the standard Euclidean space. We denote by $\mathrm{lk}(f)$ the linking number of the components $l_1 = f(L_1)$, $l_2 = f(L_2)$. We call the link f a semiboundary link if $\mathrm{lk}(f) = 0$. The Sato–Levine invariant, an integer invariant $\beta(f)$ of the isotopy class of a semiboundary link f, was defined in [1]. A higher-order generalization of the construction is found in [2]. We recall the definition.

Let S_i, $i = 1, 2$, be two Seifert surfaces bounded by the components l_i of the link. We take the surfaces to satisfy the condition $S_1 \cap l_2 = S_2 \cap l_1 = \varnothing$. Let $\Gamma = S_1 \cap S_2$ be the intersection curve (possibly disconnected) of the surfaces. The curve Γ is embedded in $\mathbb{R}^3$ and we have the canonical framing Ξ, i.e., a vector field $\vec{\xi}(x)$, $x \in \Gamma$, transversal to the curve at each point x. This frame Ξ is defined by the formula $\vec{\xi}(x) = \vec{\varkappa}_1(x) + \vec{\varkappa}_2(x)$, where $\vec{\varkappa}_i(x)$ is the normal vector to the surface S_i, $x \in S_1 \cap S_2$. Let Γ' be a curve obtained from Γ by a small shift $\Gamma \to \Gamma'$ along the framing Ξ. The selflinking number $\mathrm{lk}(\Gamma; \Xi) \in \mathbb{Z}$ is defined by $\mathrm{lk}(\Gamma; \Xi) = \mathrm{lk}(\Gamma; \Gamma')$. This number does not depend on S_i and is denoted by $\beta(f)$. Hence, the Sato–Levine invariant β is defined. A formula for computing $\beta(f)$ was obtained in [3] (see also [4]). This formula shows that the invariant β is defined as a Vassiliev invariant of order 3. This means that the invariant β should be defined by means of the cocycle on the cooriented stratum of codimension ≤ 3.

Let f_-, f_+ be a pair of links. We assume that the links coincide outside the ball D^3 of a small radius. In this ball we have two pairs of the branches of each link f_-, f_+. These branches are formed by segments with endpoints on ∂D^3. The segments are on the same component of the links. Let $f(t)$, $t \in [t_0 - \varepsilon; t_0 + \varepsilon]$, be a homotopy with support inside D^3 such that $f_- = f(t_0 - \varepsilon)$ and $f_+ = f(t_0 + \varepsilon)$. The homotopy has a single transversal selfintersection point. We use the notation "+", "−" as shown in Fig. 1.

Let x be a singular point of the homotopy $f(t)$. We define a sign $O(x) = \pm 1$ according to the following rule. Let $l_i(t_0)$ be the component of the curve $f(t_0)$ with a selfintersection point. We define an order of the branches λ_1 and λ_2 of the curve $l_i(t_0)$ in the neighborhood of the singular point. Let us consider the ordered bases $X(t_0 - \varepsilon)$ and $X(t_0 + \varepsilon)$ in $\mathbb{R}^2$ formed by the vectors $\vec{\chi}_1(t)$, $\vec{\chi}_2(t)$ tangent to the branches λ_i, and the vector $\vec{\chi}_3(t) = (x_1(t); x_2(t))$ with endpoints at $x_i(t) \in l_i(t)$, $i = 1, 2$. The origins of the vectors project to the intersection point x of the components. We define the value $O(x)$ as the sign of orientation of the basis $X(t_0 - \varepsilon)$ in the plane. We note that for the link f_- in Fig. 1 the basis $X(t_0 - \varepsilon)$ is right and we have $O(x) = +1$.

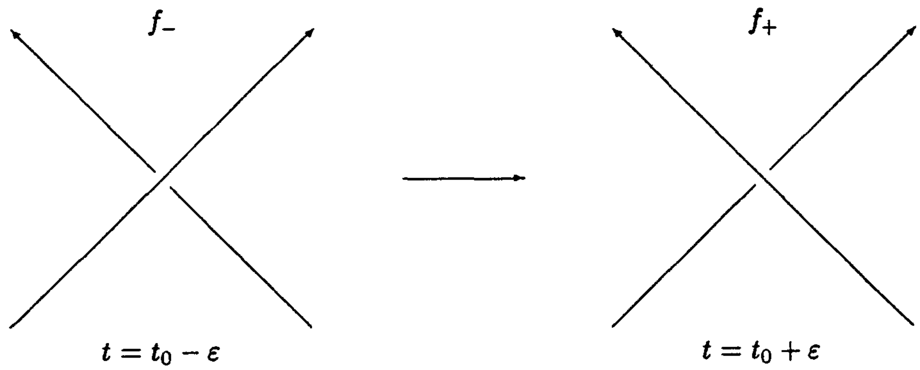

Fig. 1

For the semiboundary links f_- and f_+, the values of β are related by the formula

$$\beta(f_+) - \beta(f_-) = O(x)\,\mathrm{lk}^2(x), \tag{1}$$

where $\mathrm{lk}(x)$ is the linking number of a closed loop Λ with vertex at the point x, formed by one of the two branches of the curve l_i, with the last component l_{i+1}, $i(\mathrm{mod}2)$. For obvious reasons, the alternative choice of the loop Λ gives the same right-hand side in relation (1), because the other choice of Λ changes the sign of the integer $\mathrm{lk}(x)$.

We define the invariant β by relation (1) putting $\beta(f_e) = 0$, where f_e is a two-component link formed by the two standard circles embedded in two disjoint balls. Let $f(t)$ be a homotopy such that $f(0) = f_e$, $f(1) = g$, and components $f(t)(L_1)$ and $f(t)(L_2)$ are not intersected for an arbitrary t. We denote by $\Upsilon = \{t_1, \dots, t_i\}$ the set of critical values of the homotopy f, i.e., the values corresponding to the selfintersection of the component.

For every $t_j \in \Upsilon$ the links $f(t_j - \varepsilon)$ and $f(t_j + \varepsilon)$ are joined by the deformation depicted in Fig. 1. Therefore, the invariant $\beta(g)$ is well defined by relation (1) and the boundary condition $f(0) = f_e$, $\beta(f_e) = 0$. G.T. Jin has proved that $\beta(g)$ does not depend on the choice of the homotopy $f(t)$.

We generalize this approach. Let f be an arbitrary two-component link. We define the integer number $\beta(f)$. If f is a semiboundary link, the integer $\beta(f)$ is the Sato–Levine invariant. To define $\beta(f)$ for an arbitrary f, for every $k \in \mathbb{Z}$ we choose a link f_e such that $\mathrm{lk}(f_e) = k$ according to the following rule. Let f_e be the link formed by the two parallel loops on the standard torus of degree $(1, k)$ embedded in $\mathbb{R}^3$ (see Fig. 2).

We note that $f_e(0)$ is joined by an isotopy with the link f_e defined above. We generalize relation (1) for an arbitrary link and define the generalized Sato–Levine invariant such that $\beta(f_e) = 0$. The generalization is considered in [4] where the invariant $\beta(f)$ is defined only mod 4 under the assumption that $\mathrm{lk}(f) = 0 \bmod 2$. The question about the coincidence of the two invariants is open.

The application of the invariant β in Dynamo theory was constructed in [5]. We note that the notation β and W of the same Sato–Levine invariant in Dynamo theory and topology is distinguished.

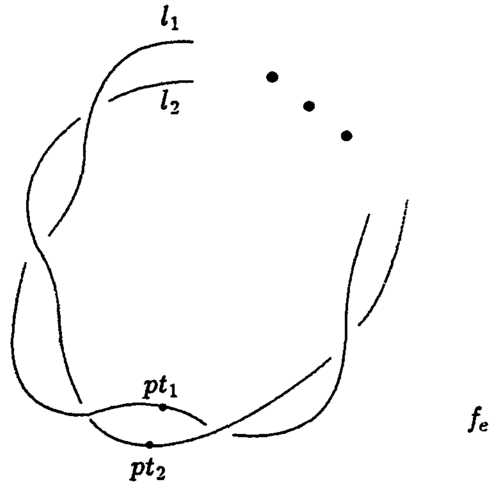

Fig. 2

2. MAIN RESULT

Let $R(k)$ be a space of mappings $f : L_1 \cup L_2 \to \mathbb{R}^3$, satisfying the following conditions:

(1) $l_1 \cap l_2 = \varnothing$;

(2) $\mathrm{lk}(f) = k$.

We denote $R = \cup_k R(k)$. Let us define $P(k) \subset R(k)$ as a subspace of links, generally speaking, with singularities. We denote $\cup_k R(k)$ by R. The considered spaces are equipped by C^∞-topology.

We generalize relation (1). Let two flat links $f_+, f_- \in P(k)$ be joined by the homotopy $f(t)$, $f(t_0 - \varepsilon) = f_-$, $f(t_0 + \varepsilon) = f_+$, with the support in the neighborhood D^3 of the point x (see Fig. 1). The link $f(t_0)$ is singular, $f(t_0) \in R(k) \setminus P(k)$. Let $\beta : P(k) \to \mathbb{Z}$ be the function under the following condition. The values $\beta(f_+)$ and $\beta(f_-)$ are related by the formulas

$$\tilde{\beta}(f_+) - \tilde{\beta}(f_-) = O(x)[(\mathrm{lk}(x) - k/2)^2 - k/4^2] = O(x)[\mathrm{lk}^2(x) - \mathrm{lk}(x)k] \tag{2}$$

and

$$\beta(f_e) = 0. \tag{3}$$

In relation (2) the integer $\mathrm{lk}(x)$ is defined correspondingly to relation (1). Evidently, the right-hand side of the relation is independent of the choice of Λ. The following theorem holds.

Theorem 1. *For an arbitrary k there exists one and only one function $\beta : P(k) \to \mathbb{Z}$ which satisfies conditions* (2) *and* (3).

The explicit construction of the generalized Sato–Levine invariant is the following. For an arbitrary two-component link $f \in P(k)$ we take the generic homotopy $\rho(t)$ in the space $R(k)$ satisfying the condition $\rho(0) = f_e(k)$, $\rho(1) = f$, $\rho(t)(L_1) \cap \rho(t)(L_2) = \varnothing$.

We denote by $\Upsilon(\rho) = \{t_1, \dots, t_i\}$ the set of the critical values of the homotopy ρ such that $\rho(t_j) \in R(k) \setminus P(k)$, $t_j \in \Upsilon(\rho)$. For every critical value t_j, we define

$$\Delta\beta(t_j) = \beta(\rho(t_j + \varepsilon)) - \beta(\rho(t_j - \varepsilon)).$$

Then we define

$$\Delta\beta(\rho) = \sum_j \Delta\beta(t_j), \qquad t_j \in \Upsilon(\rho).$$

For an arbitrary $f \in P(k)$ we calculate the invariant by the formula

$$\beta(f) = \Delta\beta(\rho).$$

3. PROOFS

Let us formulate Lemma 1 and Corollary 1 which are evident.

Lemma 1. *The space $R(k)$ is connected, i.e., $\pi_0(R(k)) = 0$. (Note that $R(k)$ is equipped with a marked point $f_e(k)$.)*

Corollary 1. *Two arbitrary invariants $\beta_1, \beta_2 : P(k) \to \mathbb{Z}$ which satisfy relations (2) and (3) coincide.*

Lemma 2. *For arbitrary two homotopies $g_t, h_t \in \pi_1(R(k))$, $[g_t] = [h_t]$, the following relation holds:*

$$\Delta\beta(g) = \Delta\beta(h).$$

Lemma 3. *For a certain (and therefore for any) system of generators $\{\rho_s\}$ of the group $\pi_1(R(k))$ the following relation holds:*

$$\Delta\beta(\rho_s) = 0.$$

We start by constructing the system of generators $\rho_s(t)$. We consider an auxiliary homotopy $g(t) : I \to R(k)$ which transfers the diagram in Fig. 3 into itself. The homotopy is shown in Fig. 4 by a sequence of projections of the links. We have $g(0) = g(1)$ and, generally speaking, $g(0) \neq f_e$. Let $\{s/5\} \in [0:1]$, $s = 0, 1, \ldots, 5$, be the values of the parameter t. The sequence of the corresponding diagrams is given. We shall first describe the diagram $g(0)$ given in Fig. 3.

Let l_1 be the curve with projection l_1', let m_1' and m_2' be two selfintersection points on the projection. (We denote below a corresponding projections by primes.) The point m_1' is of positive type (see Fig. 1) and the point m_2' is of negative type. Let n_1 and n_2 be a pair of points on l_1 in the small neighborhood of the upper inverse image m_1 of the point m_1', and the point n_1 is chosen to be the nearest to m_1. Let N be an arc with the ends $(n_1; n_2)$. We denote by n_3 the midpoint of the arc. Let $U \supset N$ be a small ε-neighborhood of this arc. The curve l_1 intersects ∂U at two pairs of points, $\partial U \cap l_1 = \{x_1; y_1\} \cup \{x_2; y_2\}$. The points in each pair $\{x_i; y_i\}$, $i = 1, 2$, are near the point n_i.

Let us describe the component l_2 of the link g_0 in the neighborhood U. Let $S \subset U$ be a circle of small radius close to the point n_3. We take this circle parallel to the plane of projection. We denote by $\{\gamma_i^1\}$, $i = 1, \ldots, s$, s copies of the "tendrils" with the initial points $c_i^1 \in S$. The "tendrils" are drawn out along N to the point n_1. We draw out the handles linked with the arc $(x_1; y_1)$ of the curve l_1 with the coefficient $+1$ or -1 along γ_i. Each handle is the boundary of a thin plate P_i with the central axis γ_i. Each plate P_i intersects the arc $(x_1; y_1)$ at an interior point. This means that the corresponding handle γ_i^1 links with the arc $(x_1; y_1)$. By analogy with the preceding construction, let $\{\gamma_j^2\}$ be q copies of handles corresponding to the "tendrils" along N toward the point n_2. The

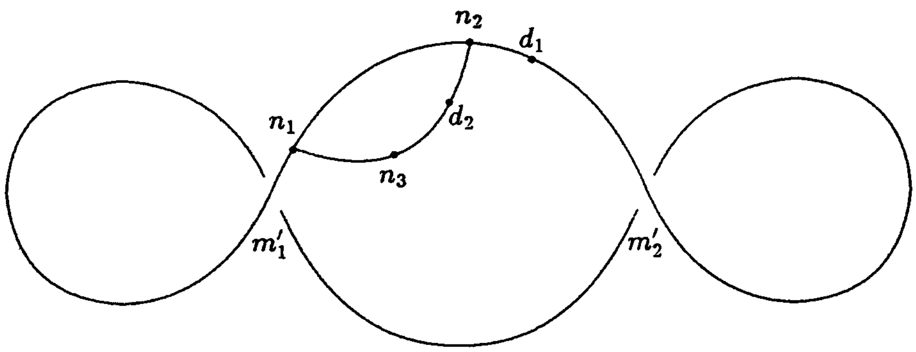

Fig. 3

initial points $\{c_i^1\}$ and $\{c_j^2\}$ of the two sets of "tendrils" can be mixed on the circle S. We define the component l_2 as the circle S with the set of handles described above. We take the handles under the condition $\gamma(l'_2) = +1$ and without selfintersection points on the projection. The link $g_0 \in P(k)$ is constructed. The linking number $k = \mathrm{lk}(l_1; l_2)$ is given by the sum of the linking numbers of the handles with the component l_1.

We construct a homotopy g, $g(t) \in R(k)$. We define $g(0) = g(1) = g_0$. The homotopy $g(t)$ is the identity in the neighborhood of two points d_1 and d_2 (see Fig. 3). For $0 < t < 3/5$ the homotopy of the curve l_2 is induced by the flat regular homotopy of the neighborhood U. Therefore it is sufficient to describe the deformation of the arc N. For any t the component l_1 is deformed along the axis of the projection in a small neighborhood of the point m'_1.

For $0 < t < 1/5$, the homotopy $g(t)$ transports the point $n_1(0)$ on the arc N to the point $n_1(1/5)$, and the trajectory of the projection of the point $n'_1(t)$ contains the point m'_1. For $1/5 < t < 2/5$ the order of the branches of the projection l'_1 is reversed in the neighborhood of m'_1 and the curve l_1 selfintersects. The curve l_2 is not deformed for these values of the parameter. Then for $2/5 < t < 3/5$ the point $n_1(t)$ is deformed along l_1 and the outer (upper) branch of its projection intersects with m'_1. Note that the projection $N'(3/5)$ of the arc has a selfintersection point. For $3/5 < t < 4/5$, the curve l_2 is deformed in line with the following rule. The point m'_1 divides the component $l'(3/5)$ into two closed curves. The left curve is embedded in the plane. We denote this curve by Λ'. Each "tendril" $\gamma_i^1(3/5)$ of the curve $l_2(3/5)$ is deformed to the "tendril" $\gamma_i^1(4/5)$, and the branch Λ passes through the "tendril" as being much smaller than the plate bounded by the handle. After the deformation, the curve $l_2(4/5)$ is, as for $t = 0$, in the small neighborhood of the arc N. But the projection of the central axe $N'(4/5)$ has the selfintersection point. At last, for $4/5 < t < 1$ the link is deformed to the original link g_0. The curve l_1 is deformed with the changing of the order of branches of the projection at the point m'_1. The description of the homotopy of the curve l_2 for $4/5 < t < 1$ is omitted. Note that the curves $l_2(4/5)$ and $l_2(1)$ are joined by the homotopy in the exterior $\mathbb{R}^3 \setminus l_1(0) \cup [m_1; \bar{m}_1]$ and $[m_1; \bar{m}_1]$ is the segment connecting the preimages of the point m_1. The homotopy $g(t)$ is constructed.

We define the homotopy $\rho(t)$ by means of $g(t)$. The homotopy $\rho(t)$ gives an element in $\pi_1(R(k))$. We consider an arbitrary homotopy $h(t)$. The homotopy joins g_0 with f_e in the space $R(k)$. We define $\rho(t) = h(t_3) \circ g(t_2) \circ h^{-1}(t_1)$ after a redenotation of the parameters. The homotopy $\rho(t)$

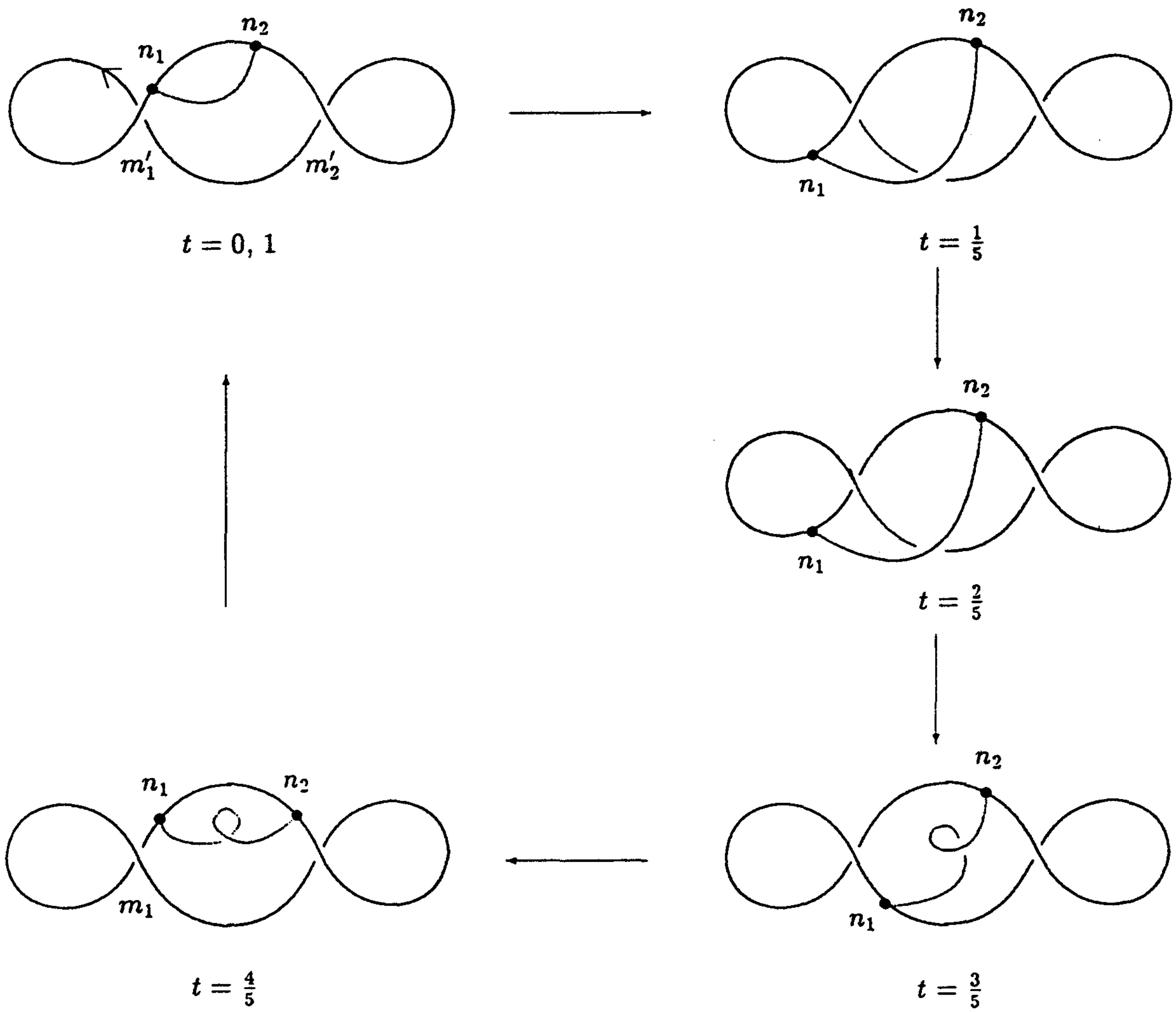

Fig. 4

is defined up to conjugation. Moreover, for simplicity we consider the homotopy ρ up to the composition with a flat selfisotopy of the link f_e. The homotopy ρ is constructed up to the choice of the handles $\{\gamma_i^1\}$, $\{\gamma_j^2\}$, $\gamma 1$, $\gamma 2$, up to the homotopy $h(t)$, and, at last, up to the isotopy combined with the homotopy $h(t_3) \circ g(t_2) \circ h^{-1}(t_1)$.

Lemma 4. *The set of homotopies $\{\rho_s\}$ determines a system of generators in the group $\pi_1(R(k))$.*

Proof. Let pt_1 and pt_2 be the marked points on the components of links (see Fig. 2). We consider the set of generic singularities of the flat homotopy $f(t) \in R(k)$, $t \in I = [0; 1]$. Let for $t = 1/2$ the link f has a generic singularity. The list of possible singularities, the singularity of the projection of a link being taken into account, is given below.

(1) The selfintersection point on a component of the link; $f(1/2) \in R(k) \setminus P(k)$.

(2) The generic degeneration of the projection diagram of the link $f(1/2)$. That is, the singular (self-tangent or triple) point on the link projection.

(3) The intersection point of the projection $f'(1/2)$ with projections pt'_1, pt'_2 of the marked points.

We begin with a sketch of the proof. For an arbitrary f we construct the homotopy $h(t_1) \in R(k)$, $t_1 \in [0;1]$, joining f and the link f_e; $h(0) = f$, $h(1) = f_e$. We call this homotopy a canonical homotopy. We assume that the links f and $\bar{f}$ are joined by the homotopy $g(t_2) \in R(k)$, $t_2 \in [1/3;2/3]$. For $t_2 = 1/2$ the homotopy has a singularity of type 1–3 and the homotopy is an isotopy for $t_2 \neq 1/2$. Let $h(t_1)$, $\bar{h}(t_3)$ be two canonical homotopies constructed from f and $\bar{f}$, respectively. For simplicity, we assume that $t_1 \in [0;1/3]$ and $t_3 \in [2/3;1]$. Let us consider the homotopy $\rho(t) = h(t_3) \circ g(t_2) \circ \bar{h}^{-1}(t_1)$, $t \in [0;1]$. Note that $\rho(0) = \rho(1) = f_e$. We prove that the homotopy ρ can be deformed (rel ∂) to a generator described above.

We start to define the canonical homotopy h. We define h as the composition $h(t) = h_3(t_3) \circ h_2(t_2) \circ h_1(t_1)$, $t_i \in [(i-1)/3; i/3]$, $i = 1,2,3$. Describe the homotopy $h_1(t_1)$. We deform the curve $l_1(t)$ so that the projection $l'_1(t)$ of the curve $l_1(t)$ is not deformed. We change the order of the branches of the curve in the neighborhood of the selfintersection point of the projection. Consider a motion along the curve l_1 with respect to the given orientation. We start from the point pt_1. Let m'_i be the selfintersection point of l'_1. Let $\lambda_1(i)$ and $\lambda_2(i)$ be the two branches of the curve l_1 projected in the small neighborhood of m'_i. We choose the order of branches with respect to the order of the motion. The homotopy h_1 deforms the branches of the curve parallel to the axis of the projection with respect to its order. More precisely, let us assume that the branch $\lambda_1(i)$ is upper and the branch $\lambda_2(i)$ is lower with respect to the projection. Then the homotopy h_1 is the identity in the neighborhood of the branches. Conversely, if the branch $\lambda_1(i)$ is lower and the branch $\lambda_2(i)$ is upper, then the homotopy h_1 changes the order of the branches. The homotopy h_1 is defined. Note that the first branch $\lambda_1(i)$ of the curve $l_1(1/3)$ is over the second branch $\lambda_2(i)$ with respect to the axis of the projection. Therefore the obtained curve l_1 is joined by an isotopy with the standard embedded circle.

We define the homotopy $h_2(t_2)$ as an isotopy of the curves $l_1(1/3)$ and the standard embedded circle. Then we continue the homotopy to the component l_2 by an arbitrary flat homotopy satisfying the condition $l_1 \cap l_2 = \varnothing$.

Define $h_3(t_3)$ as an arbitrary homotopy identical on l_1 and joining the curve $l_2(2/3)$ with the second component of the standard link f_e in the complement $\mathbb{R}^3 \setminus l_1(2/3)$. The homotopy h_3, as well as the homotopies $h(t)$, $\rho(t)$, are well defined.

We consider the homotopy $\rho(t)$ with respect to the type of the singular point of the homotopy g. For obvious reasons ρ can be deformed to the identical homotopy by a deformation except if $g(1/2)$ intersects pt'_1.

Let us consider the homotopy ρ with respect to the type of the singularity of g.

Note that the homotopy $\rho(t)$ is defined by

$$\rho(t) = e(t_5) \circ h(t_4) \circ g(t_3) \circ \bar{h}^{-1}(t_2) \circ \bar{e}^{-1}(t_1).$$

Here $g(t_3)$ is the homotopy with the singular point, h, $\bar{h}$ are the two canonical homotopies joining the corresponding link to the standard component of f_e. The homotopies e, $\bar{e}$ are identical on the component l_1 and deform l_2 to the standard component. We assume that the parameters of the

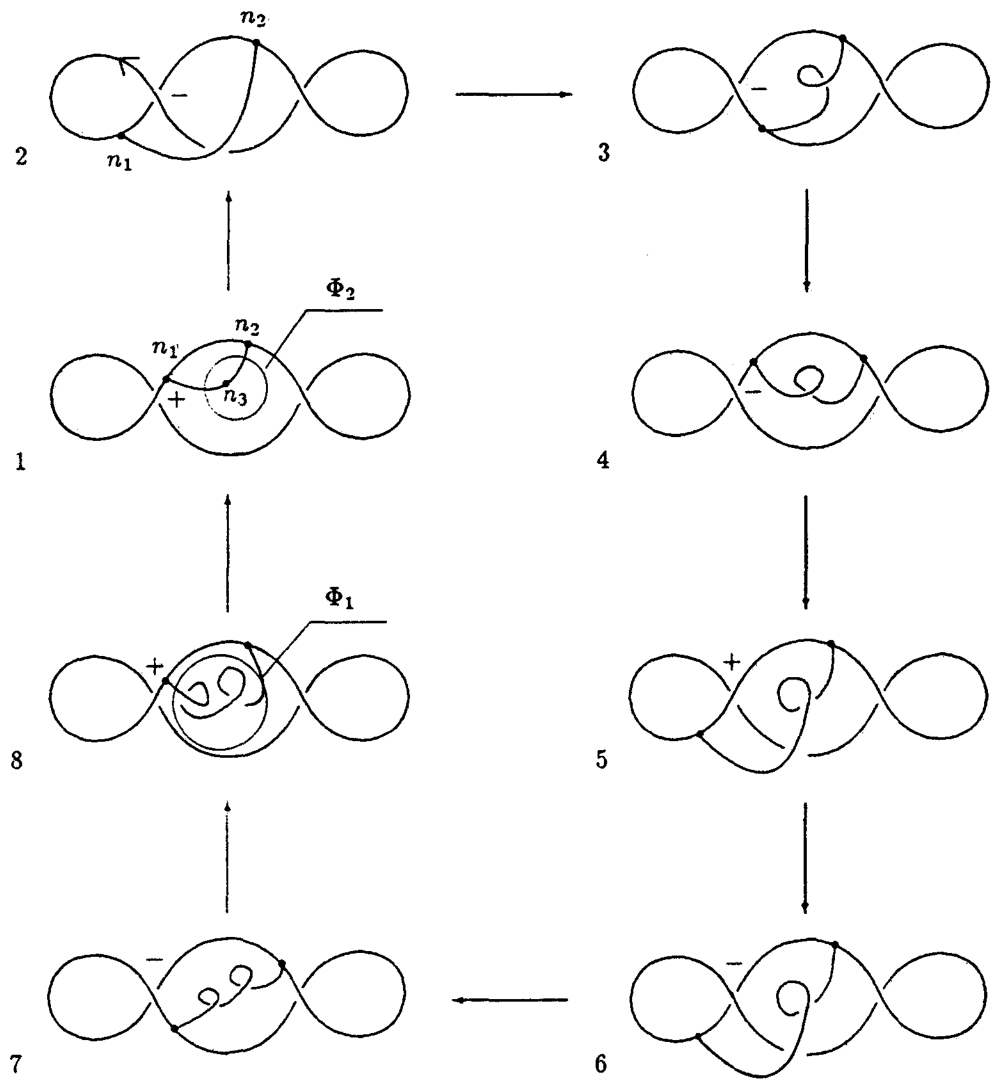

Fig. 5

homotopies belong to the corresponding segments, $t_i \in [i - 1/5; i/5]$. Moreover, we denote

$$\delta(t) = h(t_4) \circ g(t_3) \circ \bar{h}^{-1}(t_2), \quad t \in [2/5; 4/5].$$

Below, the homotopy of homotopies is called a deformation. Let us consider a deformation $\delta \to \delta'$ which is the identity on $\{2/5\}; \{3/5\}$. The deformation is also the identity on the component l_2. We start with the description of the homotopy δ'. Let us consider the homotopy $\delta(t)$ in the neighborhood of $t = 1/2$. By construction, this homotopy is identical on the component l_2. For $1/2 > t > 1/2 - 1/6$, $1/2 - 1/6 > t > 1/2$, the component l_1 is deformed by the vertical homotopy. The vertical homotopy orders the selfintersection points of the projection. Otherwise, the links $\delta(1/2 + 1/6)$, $\delta(1/2 - 1/6)$ are joined by the homotopy identical on l_2 and vertical on l_1. This homotopy has one selfintersection point with projection pt_1. We denote this homotopy by δ'. Obviously, δ and δ' are joined by the deformation $\delta \to \delta'$.

We construct a deformation $\delta' \to \delta''$ with support in the segment $[1/2-\varepsilon; 1/2+\varepsilon]$. Let us consider an isotopy of the singular component $l_1(1/2)$ to the component in Fig. 3. Such a homotopy identical in the neighborhood of the singular point m_1 exists, because the two loops Λ_1 and Λ_2 on the component $l_1(1/2)$ divided by the point m_1 are unlinked and unknotted. The restriction of the deformation is arbitrary on l_2. The homotopies δ' and δ'' have the same properties and $\delta''(l_1(1/2))$ is nice. The deformation $\delta' \to \delta''$ is induced by an isotopy of the component $l_1(1/2)$. The homotopy δ'' determines the homotopy ρ'' for $t \in [0;1]$.

Let us construct a deformation $\rho'' \to \rho'''$, identical on the homotopy of the component l_1. At the first step, we construct a homotopy satisfying the condition: for $t \in [1/2-\varepsilon; 1/2+\varepsilon]$ the component l_2 belongs to the small neighborhood of the point m_1. The component is depicted in Fig. 4 for $t = 1/5$. Moreover, for $t \approx 1/2$ the homotopy of l_2 is also depicted in Fig. 4. In particular, the links $\rho'''(1/2-\varepsilon)$ and $\rho'''(1/2+\varepsilon)$ coincide with the link in this figure for $t = 1/5$. For each $t \neq 1/2$, the complement to the small neighborhood of the unlinked component $l_1(t)$ is homotopy equivalent to a circle S^1. Therefore, we can deform the homotopy ρ''' on $l_2(t)$, $t \neq 1/2$, and for $t \in [2/5; 3/5]$ we obtain the homotopy ρ'''. This homotopy coincides with the generator in the lemma. For $t \in [0; 2/5] \cup [3/5; 1]$, the homotopy approximates the isotopy of f_e to itself. We note that the approximation can be taken flat. Lemma 4 is proved.

Proof of Lemma 2. Let $g(t), h(t) \in \pi_1(R(k))$ be the flat homotopies joined by the deformation $F(t,\tau)$, $t \times \tau \in K^2 = [0;1] \times [0;1]$, $F(t, \tau = 0) = g(t)$, $F(t, \tau = 1) = h(t)$. Let $\Delta_2(F)$ be a set of selfintersection points of the immersed 3-manifold $K^2 \times L_1 \cup K^2 \times L_2 \to \mathbb{R}^3 \times K^2$. For dimension reasons the set $\Delta_2(F)$ is an embedded curve in $\mathbb{R}^5$, and the boundary $\partial(\Delta_2(F))$ coincides with the sets $\Upsilon(g)$ and $\Upsilon(h)$ of the critical values of the homotopies g and h. Note that if $t_i \in \Upsilon(g)$, $t_j \in \Upsilon(h)$ are joined by a segment from $\Delta_2(F)$, then $\Delta\beta(t_i) = \Delta\beta(t_j)$. If t_i and t_j are joined by a segment and belong to $\Upsilon(g)$ or $\Upsilon(h)$ simultaneously, then $\Delta\beta(t_i) = -\Delta\beta(t_j)$. Note that the deformation $F(t \times \tau)$ may have critical points of Whitney umbrella type. These critical points belong to $\partial(\Delta_2(F))$. In the small neighborhood of this singularity the value of β does not change. Therefore $\Delta\beta(g) = \Delta\beta(h)$. Lemma 3 is proved.

Proof of Lemma 3. Evidently for an arbitrary isotopy g we have $\Delta\beta(g) = 0$. Let us prove that for a generator ρ the relation $\Delta\beta(\rho) = 0$ holds. Let us consider the double generator $\mu = 2\rho$. This homotopy is obtained by the composition of the two copies of ρ. We prove that

$$\Delta\beta(\mu) = 0. \tag{4}$$

We clarify the construction. Let us deform the homotopy $\mu \to \mu'$. The homotopy μ' is more simple and we prove that $\Delta\beta(\mu') = 0$. Then by Lemma 2 we have the obvious relation $\Delta\beta(\mu) = 2\Delta\beta(\rho)$.

We define the homotopy μ' by means of the set of the following diagrams 1–8 in Fig. 5.

The homotopy μ' coincides with μ everywhere exept the fragments 1–8 in Fig. 5. Note that the deformation μ' is identical on the component l_1 and is arbitrary on the component l_2. The homotopy inside the fragments is shown in Fig. 6.

The fragment Φ_1 is given in Fig. 6 by Diagram 1; the fragment Φ_2 is given by Diagram 6. On Diagram 1, the point n_3 marks the circle S. The circle is joined with the points n_2 and n_3 by handles $\{\gamma_i^1\}$ and $\{\gamma_j^2\}$. For the homotopy $1 \to 2$ each handle γ_i^1 intersects each handle γ_j^2. Then,

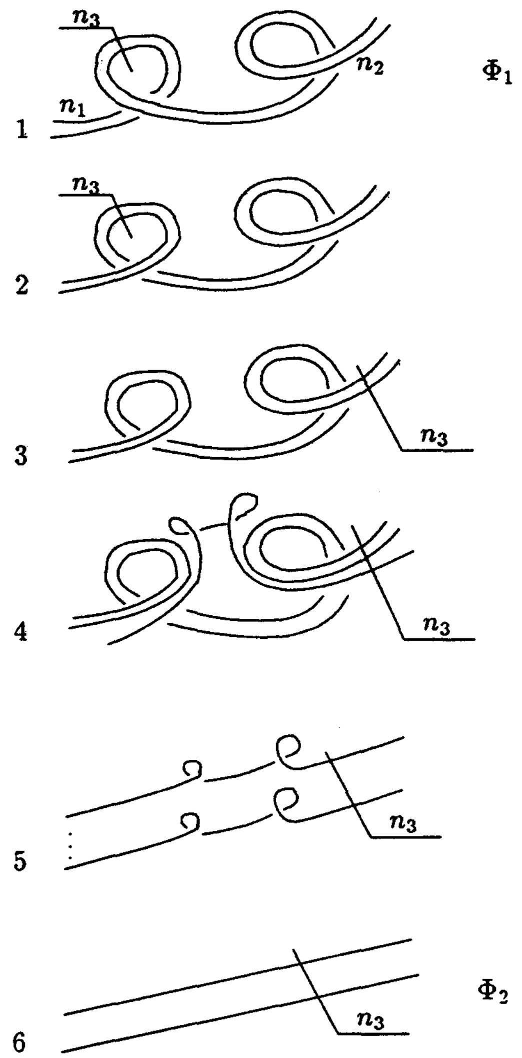

Fig. 6

for the homotopy $2 \to 3$ the circle S with the base points of the handles is deformed along the right loop outside the fragment by a flat isotopy. For the isotopy $3 \to 4$ the component l_2 is simplified and two small loops appear which generate the intersection points on the projection of the curve l_2. The isotopy $5 \to 6$ is outside of Fig. 6. For the isotopy each loop is deformed to a neighborhood of the point on the component l_2 and then is annihilated in corresponding pairs. To compute $\Delta\beta(\mu')$ we use the following lemma.

Lemma 5. *Let $g(t)$ be the homotopy depicted in Fig. 7. Then $\Delta\beta(g) = -2\varepsilon_1\varepsilon_2$, where ε_1 and ε_2 are the linking numbers of the handles on the component l_2 with the component l_1.*

Proof. The following relation is evident: $\Delta\beta(g) = -[\lambda + \varepsilon_1 + \varepsilon_2 - \mathrm{lk}/2]^2 - [\lambda - \mathrm{lk}/2]^2 + [\lambda + \varepsilon_1 - \mathrm{lk}/2]^2 + [\lambda + \varepsilon_2 - \mathrm{lk}/2]^2 = -2\varepsilon_1\varepsilon_2$. We denote by λ the linking number of a branch of the component l_2 (with a vertex a) with the last component. The lemma is proved.

We finish the proof of Theorem 1. Let us consider Fig. 5. We denote $\Delta\beta = \Delta\beta_1 + \Delta\beta_2$ with respect to the number of the selfintersection points of the component. For the homotopies

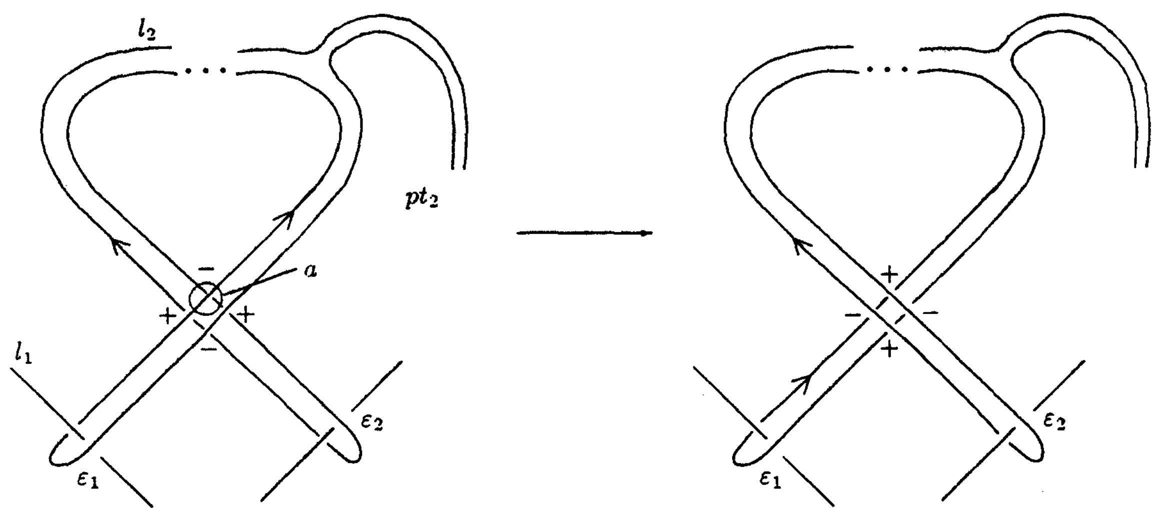

Fig. 7

$1 \to 2$, $5 \to 6$ we have $\Delta\beta_1(1) = 2[\nu_1 - k/2]^2 - k^2/2$, where ν_1 and ν_2 be two values of $\mathrm{lk}(x)$ for corresponding loops of Λ with the vertex x, $\nu_1 + \nu_2 = k$. For the homotopies $4 \to 5$, $7 \to 8$, we have $\Delta\beta_1(2) = 0$. Therefore $\Delta\beta_1 = 2\nu_1\nu_2$ with respect to relation $\nu_1 + \nu_2 = \mathrm{lk}$. By Lemma 4 we have $\Delta\beta_2 = -2\nu_1\nu_2$ because the product $\nu_1\nu_2$ coincides with the sum of the products $\varepsilon_1\varepsilon_2$ for every singularity of the intersection of the handles on the component l_2. Indeed, the homotopy $1 \to 2$ in Fig. 6 is composed of a number of the homotopies. Each composition is inverse to the homotopy from Fig. 7. Each handle γ_i^1 from n_3 to n_1 intersects ν_2 times with the handle γ_j^2 from n_3 to n_2. Note that the intersection points are counted with respect to the algebraic sign $\varepsilon_2(j)$. The algebraic number $\varepsilon_1(i)$ of the handles coincides with ν_1. The relation (4) and Theorem 1 are proved.

ACKNOWLEDGMENTS

The authors are grateful to A.A. Bolibrukh for the formulation of the problem (1993) and to Eiji Ogasa for the discussions.

This work was carried out under the financial support of the Russian Foundation for Basic Research (project no. 96-01-01166a) and under the partial support of the the Ministry of Science and Technology of the Republic of Slovenia (grant J1-7039-0101-95).

REFERENCES

1. Sato, N., Cobordism of Semi-Boundary Links, *Topol. and Appl.*, 1984, vol. 18, pp. 225–231.
2. Cochran, T.D., Geometric Invariants of Link Cobordism, *Comment. Math. Helv.*, 1985, vol. 60, pp. 291–311.
3. Jin, G.T., A Calculation of the Sato–Levine Invariant, *Preprint*, Brandeis, 1987.
4. Saito, M., On the Unoriented Sato–Levine Invariant, *J. Knot Theory and Ramif.*, 1993, vol. 2, no. 3, pp. 335–358.
5. Akhmetiev, P.M., A Higher-Order Analog of the Linking Number of the Closed Curves, in: *Topics in Finite-Type Invariant and Quantum Groups*, Providence (R.I.): Amer. Math. Soc., 1998 (in print). (*AMS Transl.*, Ser. 2).

The text is submitted by the authors in English