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Colength Growth Functions of Nonassociative Algebras

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Abstract—Numerical characteristics of identities of nonassociative algebras are considered. A series of algebras with subexponential colength growth is constructed.

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1. INTRODUCTION

The application of asymptotic methods in various domains of algebra and discrete mathematics is common practice in modern studies. Examples are the study of diverse growth functions in the theory of formal languages [1], in group theory [2], and in the theory of polynomial identities [3]. In the theory of identity relations of linear algebras, the role of the study of quantitative characteristics has increased in recent years (see, e.g., [4] and references therein). Most important among such characteristics are the sequences $\{c_nA\}$ of codimensions and $\{l_nA\}$ of colengths of a given algebra A (we recall the basic definitions and notions in the next section).

The former characteristic has been studied in much more detail. Nevertheless, there have presently appeared many papers analyzing the behavior of the sequence $\{l_n(A)\}$ for various algebras A. One of the first papers in this direction was [5], in which it was proved that the growth of the sequence $\{l_n(A)\}$ is polynomial for any associative algebra with a nontrivial identity, i.e., a PI-algebra. This result is important, because many other characteristics grow exponentially, so that the influence of the growth of colengths can be ignored. Note also that if A is a free associative algebra, then the growth of $\{l_n(A)\}$ is overexponential.

In the case of Lie algebras, the behavior of the sequence $\{l_n(A)\}$ is more complicated. On the one hand, the class L of Lie algebras with polynomial growth of $\{l_n(L)\}$ is fairly large. It includes all finite-dimensional algebras, Lie algebras with nilpotent commutator subgroup, affine Kac–Moody algebras, and a number of other algebras. On the other hand, there exist examples (see [6]) of Lie algebras L with $l_n(L) \sim (\sqrt{b}\,)^n$ for any integer $b \geq 2$ and even of algebras with overexponential growth of colength. For example, if L is the free class-2 solvable Lie algebra of countable rank, then

$$l_n(L) \sim \frac{n!}{(\ln n)^n}$$

(see [6]). There also exist examples of intermediate growth, but they are few. One of them is as follows: if L generates the variety $\mathbf{A}N_2$, then, according to [7],

$$l_n(L) \sim \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

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In the general nonassociative case, there are only some scattered results. Thus, if A is a finite-dimensional algebra of any signature with $\dim A = d$, then, as proved in [8],

$$l_n(A) \le d(n+1)^{d^2+d}.$$

In [8] and [9], a family of examples of infinite-dimensional class-2 left nilpotent algebras with polynomially growing sequences $\{l_n\}$ was constructed. Yet another curious example is as follows. In [10], it was shown that there exist precisely three varieties $\mathbf{V} = \operatorname{var} A$ with $l_n(\mathbf{V}) = 1$ for all $n = 1, 2, \dots$. One of them is generated by the commutative associative polynomial algebra $F[t]$, another one, by a two-dimensional meta-Abelian Lie algebra, and the third one, by the infinite-dimensional Jordan algebra J constructed by Shestakov [11, p. 104, Example 2].

The main objective of this paper is to construct a family of examples with subexponentially growing sequence $\{l_n(A)\}$ (see Theorem 1 and its corollaries). The character of the asymptotic behavior of $\{l_n(A)\}$ may be both monotone and strongly oscillating. Examples are based on a new approach to constructing nonassociative algebras by using infinite binary words, which was proposed in [12], [13] and developed in [14], [15]. The new construction makes it possible to connect numerical invariants of algebras with combinatorial characteristics of infinite words and use results of the well developed theory of formal languages.

2. BASIC NOTIONS AND DEFINITIONS

Let F be a field of characteristic zero. We denote by $F\{X\}$ the absolutely free algebra over F with an infinite set X of generators. Given an F-algebra A, a polynomial

$$f = f(x_1, \dots, x_n) \in F\{X\}$$

is called an *identity* of A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. The necessary information from the theory of identity relations can be found in [3] or in [16]. The set of all identities of an algebra A forms the ideal $\operatorname{Id}(A)$ in $F\{X\}$, which is stable with respect to all endomorphisms of $F\{X\}$, i.e., is a T-ideal. Let P_n denote the subspace of all multilinear polynomials in $x_1, \dots, x_n$ in $F\{X\}$. Then $P_n \cap \operatorname{Id}(A)$ is the set of all nth-degree multilinear identities of the algebra A. It is well known that, in the case of a field of characteristic zero, any T-ideal is uniquely determined by its multilinear components. Therefore, studying the identities of A largely reduces to studying the family of subspaces $P_n \cap \operatorname{Id}(A)$, $n = 1, 2, \dots$. As a rule, it is more convenient to consider the family of quotient spaces

$$P_n(A) = \frac{P_n}{P_n \cap \operatorname{Id}(A)}\,.$$

In the study of multilinear identities, an important role is played by the representation theory of the symmetric group S_n. The action of S_n on the multilinear monomials is defined by

$$\sigma \circ f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

and turns P_n into an FS_n-module. The space $P_n \cap \operatorname{Id}(A)$ is invariant with respect to the action of S_n; therefore, the space $P_n(A)$ is endowed with the structure of an FS_n-module as well. Its character $\chi(P_n(A))$ is called the *nth cocharacter of A* and denoted by $\chi_n(A)$. The necessary information from the representation theory of symmetric groups can be found in the monograph [17]. It is convenient to write the decomposition of $P_n(A)$ into a sum of irreducible summands in terms of characters as

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{2.1}$$

where χ_λ is the irreducible character corresponding to the partition λ of the number n and the nonnegative integer m_λ is the number of occurrences of χ_λ in $\chi_n(A)$. We must recall that a *partition* λ of a number n is a set $\lambda = (\lambda_1, \dots, \lambda_k)$ of integers satisfying the conditions

$$\lambda_1 \ge \dots \ge \lambda_k > 0, \qquad \lambda_1 + \dots + \lambda_k = n.$$

The dimension of the corresponding irreducible representation (or the degree of the character) is denoted by d_λ or $\deg \chi_\lambda$. The number

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

is called the *nth colength* of A. In other words, $l_n(A)$ is the number of terms in the decomposition of the FS_n-module $P_n(A)$ into a sum of irreducible components.

We need yet another quantitative characteristic related to the identities of the algebra A. Recall that the *nth codimension* of the identities of an algebra A equals

$$c_n(A) = \dim P_n(A).$$

Obviously,

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda \deg \chi_\lambda, \tag{2.2}$$

where m_λ is the same as in (2.1).

Since we consider nonassociative algebras, an important role is played by the parenthesizations of monomials in different algebras. By T we denote a parenthesization of a word of length n and by $[a_1 \cdots a_n]_T$, the product of n elements of a nonassociative algebra with this parenthesization. For example, if $n = 4$ and $T = (\,\cdot\,)(\,\cdot\,)$, then $[x_1, \dots, x_4]_T = (x_1x_2)(x_3x_4)$. An algebra A with the identity

$$(x_1x_2)(x_3x_4) \equiv 0 \tag{2.3}$$

is said to be *meta-Abelian*.

Given a parenthesization T, we can consider the subspace P_n^T of P_n generated by all monomials $[x_{\sigma(1)} \cdots x_{\sigma(n)}]_T$, $\sigma \in S_n$. Clearly,

$$P_n = \bigoplus_T P_n^T, \tag{2.4}$$

where the summation is over all possible parenthesizations, i.e., contains

$$\frac{1}{n}\binom{2n-2}{n-1}$$

summands. Each of the subspaces P_n^T, as well as $P_n \cap \operatorname{Id}(A)$, is an FS_n-submodule in P_n. Therefore, the quotient module

$$P_n^T(A) = \frac{P_n^T}{P_n^T \cap \operatorname{Id}(A)} \tag{2.5}$$

has the structure of an FS_n-module as well. We denote its character by $\chi_n^T(A)$.

We need the following result of [12]. Let M_1 denote the free meta-Abelian algebra with one generator z. Then

$$\chi_n^T(M_1) = \chi_n + 2\chi_{(n-1,1)} \tag{2.6}$$

for any parenthesization T with the property $[z_1 \cdots z_n]_T \neq 0$ in M_1.

3. MAIN RESULTS

Recall the construction of the algebra associated with an infinite binary word. Let $w = w_1w_2\dots$, where all w_i equal 0 or 1. The *combinatorial complexity* of the word w is the function $\operatorname{Comp}_w\colon \mathbb{N} \to \mathbb{N}$, where $\operatorname{Comp}_w(n)$ is the number of different subwords of length n in w.

Let w be an infinite word in the alphabet $\{0; 1\}$. By $A(w)$ we denote the nonassociative algebra with basis $\{a, b_0, b_1, \dots\}$ in which multiplication is defined as

$$b_k = \begin{cases} ab_{k-1} & \text{if } w_k = 1, \\ b_{k-1}a & \text{if } w_k = 0 \end{cases}$$

for all $k \geq 1$, and all the remaining products are zero. For any word w, the algebra $A(w)$ satisfies identity (2.3); therefore, all FS_n-decompositions for $P_n(A)$ can be considered modulo relation (2.3), i.e., in the free meta-Abelian algebra $M\{X\}$ rather than in the algebra $F\{X\}$.

Given an element x of any meta-Abelian algebra, by R_x and L_x we denote, respectively, the operators of right and left multiplication by x. It is convenient to write both operators on the right, as $yR_x = yx$ and $yL_x = xy$. For any binary word $u = u_1 \dots u_m$ and any $y, x_1, \dots, x_m \in X \subset M\{X\}$, by $yu(x_1, \dots, x_m)$ we denote the monomial $yT_1 \cdots T_m$, where

$$T_i = \begin{cases} R_{x_i} & \text{if } u_i = 0, \\ L_{x_i} & \text{if } u_i = 1, \end{cases} \qquad i = 1, \dots, m.$$

It is easy to see that any monomial of degree n in $M\{X\}$ can be written in the form

$$(x_i x_j)u(x_{i_1}, \dots, x_{i_{n-2}}), \tag{3.1}$$

where u is a binary word of length $n - 2$.

Let $w = w_1 w_2 \dots$ be an infinite binary word. We say that a finite word u is a *proper subword* of w if u is a subword of the word $w_2 w_3 \cdots$.

Lemma 1. *A multilinear monomial $(y_1y_2)u(x_1, \dots, x_m)$ is not an identity of the algebra $A(w)$ if and only if u is a proper subword of w.*

Proof. Let $u = w_i \dots w_{i+m-1}$, where $i \geq 2$. Then

$$b_{i-2}T_a u(a, \dots, a) = b_{i+m-1} \neq 0, \qquad T_a = \begin{cases} R_a & \text{if } w_{i-1} = 0, \\ L_a & \text{if } w_{i-1} = 1. \end{cases}$$

Therefore, $(y_1y_2)u(x_1, \dots, x_m) \notin \mathrm{Id}(A(w)$. On the other hand, at the basis elements of $A(w)$, y_1y_2 takes only the values $b_1, b_2, \dots$. Thus, for $k \geq 1$, the product $b_k u(a, \dots, a)$ is nonzero only for $u = w_{k+1} \dots w_{k+m}$. $\square$

Since $A(w)$ is not one-generated, we cannot constrain the colength of this algebra by directly applying relation (2.6). Let us denote the subalgebra of $A(w)$ generated by the element $a + b_0$ as $\widetilde{A}(w)$.

Lemma 2. *The algebras $A(w)$ and $\widetilde{A}(w)$ satisfy the same identities, i.e.,* $\mathrm{Id}(A(w)) = \mathrm{Id}(\widetilde{A}(w))$.

Proof. First, note that $\widetilde{A}(w)$ is the linear span of the elements $a + b_0, b_1, b_2, \dots$. Since the characteristic of the field F is zero, it suffices to compare the multilinear identities of these two algebras. The inclusion $\mathrm{Id}(A(w)) \subseteq \mathrm{Id}(\widetilde{A}(w))$ is obvious, because $\widetilde{A}(w)$ is a subalgebra of $A(w)$.

Let us show that any multilinear polynomial $f = f(x_1, \dots, x_n)$ which is not an identity of $A(w)$ does not identically vanish in $\widetilde{A}(w)$. Since both algebras are meta-Abelian, we can assume f to be a polynomial in the free meta-Abelian algebra $M\{X\}$. In this case, according to (3.1), f can be written as a linear combination

$$f = \sum_{i,k} \sum_{J} \sum_{u} \alpha_{i,k,J,u}(x_i x_k)u(x_{j_1}, \dots, x_{j_{n-2}}), \tag{3.2}$$

where $J = \{j_1, \dots, j_{n-2}\} = \{1, \dots, n\} \setminus \{1, 2\}$ and u is a binary word of length $n - 2$.

If f is not an identity in $A(w)$, then there exists a substitution

$$\varphi \colon X \to \{a, b_0, b_1, \dots\}$$

such that $\varphi(f) \neq 0$.

Clearly, we have $\varphi(x_{i_0}) = b_m$ for precisely one index i_0 and $\varphi(x_r) = a$ for all other r. Moreover, precisely one of the two products $x_{i_0}x_k$ and $x_k x_{i_0}$ takes the nonzero value b_{m+1}. Suppose that, say, $b_m a = b_{m+1}$ and $ab_m = 0$. Then, under the substitution φ, all monomials $(x_i x_k)u(x_{j_1}, \dots, x_{j_{n-2}})$ with $i \neq i_0$ vanish. Moreover, all $(x_{i_0}x_k)u(x_{j_1}, \dots, x_{j_{n-2}})$ with $u \neq u_0$, where $u_0 = w_{m+2} \cdots w_{n+m-1}$ is the subword of w beginning with the $m+2$th letter, vanish as well. Let us write f in the form $f = f_0 + f_1$, where

$$
\begin{aligned}
f_0 &= \sum_k \sum_J \alpha_{i_0,k,J,u_0}(x_{i_0}x_k)u_0(x_{j_1},\dots,x_{j_{n-2}}),\\
f_1 &= \sum_{i\neq i_0} \sum_k \sum_J \sum_u \alpha_{i,k,J,u}(x_i x_k)u(x_{j_1},\dots,x_{j_{n-2}})\\
&\quad + \sum_k \sum_J \sum_{u\neq u_0} \alpha_{i_0,k,J,u}(x_{i_0}x_k)u(x_{j_1},\dots,x_{j_{n-2}}).
\end{aligned}
$$

Then $\varphi(f_1) = 0$ and $\varphi(f_0) = \lambda(b_m a)u_0(a,\dots,a) = \lambda b_{m+n-1}$, where

$$\lambda = \sum_k \sum_J \alpha_{i_0,k,J,u_0} \neq 0.$$

Now we replace the substitution φ by $\widetilde{\varphi}$ such that

$$\widetilde{\varphi}(x_i) = a + b_0 \qquad \text{for all} \quad i = 1,\dots,n.$$

Then $\widetilde{\varphi}(x_{i_0}x_k) = b_m a = b_{m+1}$ and $\widetilde{\varphi}(x_i x_j) = 0$ for all $i \neq i_0$. In particular,

$$\widetilde{\varphi}(f_1) = 0 \qquad \text{and} \qquad \widetilde{\varphi}(f_0) = \varphi(f_0),$$

i.e., $\widetilde{\varphi}(f) = \varphi(f) \neq 0$. Since $\widetilde{\varphi}(x_i) \in \widetilde{A}(w)$ for all $i = 1,\dots,n$, it follows that f is not an identity of $\widetilde{A}(w)$, which proves the lemma. □

We proceed to the proof of the main result of this paper. Let us divide the proper subwords of w into two categories. A subword u is called a *subword of the first type* if it occurs in w only after 0 or only after 1. If u occurs in w both after 0 and after 1, then it is called a *subword of the second type*.

Theorem 1. *Let $w = w_1w_2\dots$ be an infinite binary word. Then*

$$l_n(A(w)) = 2k^{(1)}_{n-2} + 3k^{(2)}_{n-2}, \tag{3.3}$$

where $k^{(1)}_m$ and $k^{(2)}_m$ are, respectively, the numbers of proper subwords of length m of the first and second types in w. In particular,

$$2\,\mathrm{Comp}_{w^*}(n-2) \leq l_n(A(w)) \leq 3\,\mathrm{Comp}_{w^*}(n-2), \tag{3.4}$$

where $w^* = w_2w_3\cdots$.

Proof. First, we analyze the structure of the modules $P^T_n(A)$ of the form (2.5) and the decomposition of the space $P_n(A)$ into a sum of such $P^T_n(A)$ in the case $A = A(w)$. Note that all multilinear monomials of the form (3.1) with the same binary word u have the same parenthesization and do not vanish in the free meta-Abelian algebra $M\{X\}$. Moreover, the parenthesizations corresponding to different u are different. In particular, the space $P_{n,u}$, which is the linear span of multilinear monomials (3.1) with fixed u in $F\{X\}$, coincides with one of the subspaces P^T_n, in (2.4) and $P^T_n \not\subseteq \mathrm{Id}(M\{X\})$ for the corresponding parenthesizing T. And conversely, if $P^T_n \not\subseteq \mathrm{Id}(M\{X\})$, then, for the parenthesization T, there exists a word u for which $P^T_n = P_{n,u}$. Taking into account Lemma 1, we arrive at the conclusion

$$P_n \equiv \sum_u P_{n,u} \pmod{\mathrm{Id}(A(w))},$$

where the summation is over all proper subwords u of the word w.

Let $u_1,\dots,u_N$ be all proper subwords of length $n-2$ in w. We show that

$$P_n \cap \mathrm{Id}(A(w)) = P_{n,u_1} \cap \mathrm{Id}(A(w)) \oplus \cdots \oplus P_{n,u_N} \cap \mathrm{Id}(A(w)) + \sum_{T'} P^{T'}_n, \tag{3.5}$$

where T' ranges over all parenthesizations for which $P_n^{T'} \subset \mathrm{Id}(A(w))$. Obviously, the right-hand side of (3.5) is contained in the left-hand side. Therefore, it suffices to prove that if $f_1 + \cdots + f_N \equiv 0$ is an identity of the algebra $A(w)$, then all $f_1, \dots, f_N$ are also identities of $A(w)$.

Let, e.g.,

$$u_1 = w_{k+1} \cdots w_{k+n-2}, \qquad k \geq 1.$$

Then, under any substitution $\varphi\colon X \to \{a, b_0, b_1, \dots\}$ such that $\varphi(x_{i_0}) = b_{k-1}$ for some i_0 and $\varphi(x_t) = a$ for $t \neq i_0$, all elements (3.1) with $u \neq u_1$ vanish, because

$$b_k u_1(a, \dots, a) = b_{k+n-1}, \qquad b_k u(a, \dots, a) = 0.$$

In particular, $\varphi(f_2) = \cdots = \varphi(f_N) = 0$. Therefore, $\varphi(f_1) = 0$. If φ' is another substitution for which $\varphi'(x_0) = b_{m-1}$, $m \geq 1$, and $u_1 \neq u_{m+1} \dots u_{m+n-2}$, then $\varphi'(f_1) = 0$. Therefore, $f_1 \in \mathrm{Id}(A(w))$. Similarly, $f_2, \dots, f_N$ are identities of $A(w)$, which proves relation (3.5).

From (3.5), taking into account (2.5), we obtain the decomposition

$$P_n(A(w)) = \bigoplus_u P_{n,u}(A(w)), \tag{3.6}$$

in which the summation is over all proper subwords u of length $n-2$ in the word w and

$$P_{n,u}(A(w)) = \frac{P_{n,u}}{P_{n,u} \cap \mathrm{Id}(A(w))}.$$

Thus, to calculate the length of the FS_n-module $P_n(A(w))$, i.e., $l_n(A(w))$, it suffices to calculate and sum the values

$$l_{n,u}(A(w)) = \sum_{\lambda \vdash n} m_\lambda^{(u)}, \qquad \text{where} \quad \chi(P_{n,u}(A(w))) = \sum_{\lambda \vdash n} m_\lambda^{(u)} \chi_\lambda.$$

According to Lemma 1 and relation (2.6), we have

$$\chi(P_{n,u}(A(w))) = r\chi_{(n)} + s\chi_{n-1,1},$$

where $r = 0$ or 1 and $s = 0$, 1, or 2 for any proper subword u in w. Moreover,

$$\dim P_{n,u}(A(w)) = r + s(n-1), \tag{3.7}$$

because $\deg \chi_{(n)} = 1$ and $\deg \chi_{n-1,1} = n-1$.

In [15, Lemma 4], it was shown that $\dim P_{n,u}(A(w)) = n$ if u is a proper subword of the first type. It is easy to see that (3.7) can hold only for $r = s = 1$; hence $l_{n,u}(A(w)) = 2$. For a subword u of the second type, it was proved in the same paper (see Lemma 5 and Remark 1) that $\dim P_{n,u}(A(w)) = 2n-1$, whence $r = 1$, $s = 2$, and $l_{n,u}(A(w)) = 3$. This gives relation (3.3), which implies (3.4), because

$$k_{n-2}^{(1)} + k_{n-2}^{(2)} = \mathrm{Comp}_{w^*}(n-2). \quad \square$$

This result makes it possible to realize a large class of functions as colength growth functions. Below we give several examples of subexponential growth.

Corollary 1. *Let $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that*

(1) $\varphi(t) \gg \log_2(t)$;

(2) *φ is differentiable on $(0; \infty)$;*

(3) *$\varphi'(t) \ll t^{-\beta}$ for some constant $\beta > 0$;*

(4) *φ is a decreasing function.*

Then there exists an algebra A for which $l_n(A) \sim 2^{\varphi(n)}$.

Proof. In [18], it was proved that, for any $\varphi(t)$ satisfying conditions (1)–(4), there exists a binary word w for which $\mathrm{Comp}_w(n) \sim 2^{\varphi(n)}$. It remains to apply Theorem 1 to the algebra $A(w)$. $\square$

The relation $f(t) \ll g(t)$ in the statement of Corollary 1 means that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0.$$

Corollary 1 covers both monotone subexponential functions, such as 2^{n^α} or n^{n^α} with $\alpha \in (0;1)$, and weakly oscillating functions. The more exotic example given in [18] corresponds to the function

$$\varphi(t) = (t+10)^{1/2+(1/4)\cos(\ln\ln(t+1))},$$

which slowly oscillates between $n^{1/2}$ and $n^{3/4}$.

Using other results of the theory of formal languages, we can construct examples of algebras with much sharper oscillations of the sequence $\{l_n(A)\}$. Thus, in [19, Theorem 3], an example of a binary word w whose combinatorial complexity oscillates from "almost linear" to "almost exponential" was given. Using this example, we obtain the following result.

Corollary 2. *There exists an algebra A of the form $A(w)$ for which there is an increasing sequence n_k, $k = 1, 2, \dots$, such that*

(a) $l_{n_k} < n_k + \ln\ln n_k$ *if k is even*;

(b) $l_{n_k} > 2^{n_k/(\ln\ln n_k)}$ *if k is odd.*

In addition to the realization of functions with exotic asymptotics, Theorem 1 gives exact colength values in a number of cases. In the theory of factorial languages, the language E_0 consisting of all words in the two-letter alphabet $\{a, b\}$ that do not contain the subwords a^2, b^4, and $abba$ is well known (see, e.g., [1]). It is easy to construct a binary word $\overline{w}$ for which the set of proper subwords coincides with the language E_0. One of such examples was given in the paper [15]. In the same paper, it was shown that, for such a word $\overline{w}$,

$$k^{(1)}_{n-2} = \begin{cases} F_{k-1} + F_{k+1} & \text{if } n = 2k, \\ F_{k-1} + F_{k+2} & \text{if } n = 2k+1, \end{cases}$$

and

$$k^{(2)}_{n-2} = F_k$$

both for $n = 2k$ and for $n = 2k+1$, where

$$F_t = \frac{\varphi^t + (-\varphi)^{-t}}{2\varphi - 1} \quad \text{are the Fibonacci numbers,} \qquad \varphi = \frac{1+\sqrt{5}}{2}.$$

Applying Theorem 1, we obtain yet another corollary.

Corollary 3. *For the algebra $A(\overline{w})$,*

$$l_n(A(\overline{w})) = \begin{cases} 2F_{t+1} + F_{t+3} & \text{if } n = 2t, \\ F_{t+1} + 2F_{t+3} & \text{if } n = 2t+1. \end{cases}$$

In addition to the exact value, the asymptotics of the sequence $l_n(A(\overline{w}))$ can be estimated. Introducing the relation

$$f(x) \simeq g(x) \quad \Longleftrightarrow \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

between real functions, we obtain

$$l_n(A(\overline{w})) \simeq \begin{cases} C_0(\sqrt{\varphi})^n & \text{for even } n, \\ 2C_0(\sqrt{\varphi})^n & \text{for odd } n, \end{cases} \qquad \text{where} \quad C_0 = \frac{\varphi^2}{2} = \frac{3+\sqrt{5}}{4}.$$

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