ISSN 0037–4466, Siberian Mathematical Journal, 2020, Vol. 61, No. 2, pp. 255–260.

THE THEORY OF FORMAL LANGUAGES AND IDENTITIES OF NONASSOCIATIVE ALGEBRAS

M. V. Zaicev and D. D. Repovš

UDC 512.572

Abstract: We consider the numerical characteristics of identities of nonassociative algebras and propose a method for constructing some algebra $A(w)$ with prescribed properties of the codimension growth function. The growth of codimensions of $A(w)$ is completely determined by the combinatorial complexity of the language of subwords of w.

DOI: 10.1134/S003744662002007X

Keywords: identity, codimension, binary word, combinatorial complexity

In the present paper we use some results of the theory of formal languages to solve a whole series of problems of PI-theory, the theory of polynomial identities of algebras. The combinatorial properties of the languages consisting of the subwords of an infinite word have been used several times to construct various examples of the asymptotic behavior of numerical characteristics related to identities; see [1, 2] for instance. Some alternative construction of the algebras arising from an infinite binary word was proposed in [3]. We modernize the approach of [3], which enables us to relate the numerical invariants of identities of the constructed algebra to the combinatorial complexity of the language determined by the original word.

To start with, we recall the necessary concepts of PI-theory. Take a field Φ of characteristic zero and some (not necessarily associative) algebra A over Φ. Denote the absolutely free algebra over Φ with an infinite set of generators X by $\Phi\{X\}$. Then the collection of all identities of A constitutes the two-sided ideal $\mathrm{Id}(A)$ of $\Phi\{X\}$.

It is known that the ideal $\mathrm{Id}(A)$ is completely determined by its multilinear components; i.e., the collection of subspaces $\mathrm{Id}(A) \cap P_n$ for $n = 1, 2, \dots$, where P_n is the subspace of multilinear polynomials on $x_1, \dots, x_n$ in $\Phi\{X\}$.

All necessary definitions and concepts of PI-theory can be found in [4]. Put

$$P_n(A) = \frac{P_n}{\mathrm{Id}(A) \cap P_n}, \quad c_n(A) = \dim P_n(A).$$

The value $c_n(A)$ is called the *nth codimension* of A. It is established in [5] that the sequence $\{c_n(A)\}$ is exponentially bounded for every associative PI-algebra. At the end of the 1980s Amitsur conjectured that the sequence $\sqrt[n]{c_n(A)}$ has some limit that is a nonnegative integer. Later Regev proposed the stronger property $c_n(A) \sim Cn^t d^n$, where C is a constant, while the relation $f(n) \sim g(n)$ means that $\lim_{n\to\infty}(f(n)/g(n)) = 1$.

Moreover, d must be an integer, whereas t must be a half-integer. Regev's conjecture means that the three limits

$$d = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \quad t = \lim_{n\to\infty} \log_n \frac{c_n(A)}{d^n}, \quad C = \lim_{n\to\infty} \frac{c_n(A)}{n^t d^n} \tag{1}$$

exist. We will call them the first, second, and third approximations. Furthermore, the number $\exp(A) = d$ is called the *PI-exponent* of A. Note that for associative algebras Regev's conjecture is confirmed in the first and second approximations [6–9], while the question of the third approximation is still open.

M. V. Zaicev is partially supported by the Russian Science Foundation (Grant 16–11–10013), and D. D. Repovš is partially supported by the Slovenia Research Agency (Grant P1–0292).

Original article submitted June 28, 2019; revised June 28, 2019; accepted October 18, 2019.

Moreover, it turned out that the sequence $\{c_n(A)\}$ is polynomially bounded in the case $\exp(A) = 1$; i.e., $\{c_n(A)\}$ cannot be of intermediate growth.

In the general nonassociative case, there are examples of algebras with $c_n(A)$ growing asymptotically as α^n for every real $\alpha > 1$ (see [2]). Moreover, there are examples with $\{c_n(A)\}$ exponentially bounded but the first limit in (1) nonexistent [10]. In the second approximation, Regev's conjecture without associativity is also refuted [11]. There are also examples of algebras with intermediate growth of the sequence of codimensions [1]. However, in all these examples $c_n(A)$ grows asymptotically as n^{n^β} with $0 < \beta < 1$.

The main goal of this article is to show that the class of functions of intermediate growth realizable as sequences $\{c_n(A)\}$ is much wider, and that Regev's conjecture is false either in the third approximation on assuming the existence of the first and second limits in (1).

Let us proceed to the main construction, namely, to constructing an algebra from a prescribed infinite binary word. Recall that the *formal language* defined by the word w is the collection of all finite subwords of w, while the *combinatorial complexity* of this language and the word w itself is the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$ yielding the number $\mathrm{Comp}_w(n)$ of distinct subwords of length n in w.

Take an infinite word $w = w_1 w_2 \dots$ in the alphabet $\{0, 1\}$. Denote by $A(w)$ the algebra generated by two elements a and b_0 with basis $\{a, b_0, b_1, \dots\}$ and multiplication defined as follows: If $w_1 = 1$ then $b_1 = ab_0$ and if $w_1 = 0$ then $b_0 a = b_1$. Suppose that $b_1, \dots, b_{k-1}$ are already defined. Then

$$b_k = \begin{cases} ab_{k-1} & \text{if } w_k = 1, \\ b_{k-1}a & \text{if } w_k = 0. \end{cases} \tag{2}$$

All remaining products of the basis elements vanish.

The algebra $A(w)$ satisfies the identity

$$(x_1x_2)(x_3x_4) \equiv 0 \tag{3}$$

for every w. Thus, we may count codimensions not only in $\Phi\{X\}$, but also in the free metabelian algebra $M\{X\}$; i.e., in the relatively free algebra of the variety defined by (3).

Given an element x of the metabelian algebra, denote by R_x and L_x the operators of right and left multiplication by x, respectively. We will write both operators on the right; i.e., $yR_x = yx$ and $yL_x = xy$. Given a binary word $u = u_1 \dots u_m$ and $y, x_1, \dots, x_m \in X \subset M\{X\}$, denote by $yu(x_1, \dots, x_m)$ the monomial $yT_1 \dots T_m$, where $T_i = R_{x_i}$ if $u_i = 0$ and $T_i = L_i$ if $u_i = 1$. We can uniquely express each multilinear monomial in $x_1, \dots, x_n$ in $M\{X\}$ as

$$(x_ix_j)u(x_{i_1}, \dots, x_{i_{n-2}}), \tag{4}$$

where u is a binary word of length $n-2$, while $\{i_1, \dots, i_{n-2}\} = \{1, \dots, n\} \setminus \{i, j\}$. The following lemma ensures uniqueness.

Lemma 1. *The elements of the form* (4) *are linearly independent.*

Proof. Given an infinite binary word w, alongside $A(w)$ consider the algebra $C(w)$ with basis $\{c_0, c_1, \dots, c_m, \dots\}$ and the multiplication table

$$c_{k+1} = \begin{cases} c_kc_{k-1} & \text{if } w_k = 1, \\ c_{k-1}c_k & \text{if } w_k = 0 \end{cases}$$

for all $k \geq 1$. Assume that all other products of basis elements vanish. Fix an element z of the form (4) and consider a word w whose initial length $n-1$ subword equals $1u$. Then the substitution φ such that $\varphi(x_i) = c_1, \varphi(x_j) = c_0, \varphi(x_{i_1}) = c_2, \dots, \varphi(x_{i_{n-2}}) = c_{n-1}$ yields a nonzero value of z:

$$\varphi(z) = \varphi((x_ix_j)u(x_{i_1}, \dots, x_{i_{n-2}})) = c_{n-1},$$

while $\varphi((x_kx_l)u'(x_{t_1}, \dots, x_{t_{n-2}})) = 0$ whenever $(k, l, t_1, \dots, t_{n-2}) \neq (i, j, i_1, \dots, i_{n-2})$ or $u \neq u'$. $\square$

Call a subword u of an infinite word w *proper* whenever at least one of the occurrences of u in w starts in position $k \geq 3$.

Lemma 2. *The multilinear monomial $(y_1y_2)u(x_1,\dots,x_m)$ is not an identity of $A(w)$ if and only if u is a proper subword of w.*

Proof. If $u = w_i\dots w_{i+m-1}$ and $i \geq 3$ then $b_{i-2}T_a u(a,\dots,a) = b_{i+m-1} \neq 0$, where $T_a = R_a$ for $w_{i-1} = 0$ or $T_a = L_a$ for $w_{i-1} = 1$. If u is not a proper subword of w then every substitution of the basis elements of $A(w)$ into y_1, y_2, x_1, …, x_m yields the zero value. □

Lemma 3. *For every binary word u and every substitution $\sigma \in S_m$ in $A(w)$, we have*

$$(y_1y_2)u(x_{\sigma(1)},\dots,x_{\sigma(m)}) - (y_1y_2)u(x_1,\dots,x_m) \equiv 0. \tag{5}$$

Proof. If u is not a proper subword of w then both monomials in (5) vanish identically in $A(w)$ by Lemma 2. If u is a proper subword of w then each monomial in (5) can take a nonzero value only for $x_1 = \dots = x_m = a$. □

Subdivide the proper subwords of w into the two categories: a subword u is called a *subword of type* 1 if u appears in w only after 0 or 1. If u occurs in w both after 0 and 1, then w is a *subword of type* 2.

Lemma 4. *Suppose that u is a subword of type 1 in w. If u is always after zero, then the identities*

$$(x_ix_j)u(x_{i_1},\dots,x_{i_{n-2}}) \equiv (x_ix_1)u(x_2,\dots,x_{i-1},x_{i+1},\dots,x_n) \tag{6}$$

hold in $A(w)$ for $i > 1$ and $\{i_1,\dots,i_{n-2}\} = \{1,\dots,n\} \setminus \{i,j\}$; moreover,

$$(x_1x_i)u(x_{i_1},\dots,x_{i_{n-2}}) \equiv (x_1x_2)u(x_3,\dots,x_n), \tag{7}$$

where $\{i_1,\dots,i_{n-2}\} = \{2,\dots,n\} \setminus \{i\}$.

If u is always after 1 then the identities

$$(x_jx_i)u(x_{i_1},\dots,x_{i_{n-2}}) \equiv (x_1x_i)u(x_2,\dots,x_{i-1},x_{i+1},\dots,x_n) \tag{8}$$

hold in $A(w)$ for $i > 1$ and $\{i_1,\dots,i_{n-2}\} = \{1,\dots,n\} \setminus \{i,j\}$. Moreover,

$$(x_ix_1)u(x_{i_1},\dots,x_{i_{n-2}}) \equiv (x_2x_1)u(x_3,\dots,x_n), \tag{9}$$

where $\{i_1,\dots,i_{n-2}\} = \{2,\dots,n\} \setminus \{i\}$. Furthermore, the elements on the right-hand side of (6) with $2 \leq i \leq n$ and the elements on the right-hand side of (7) are linearly independent. The same independence holds for the right-hand sides of (8) and (9).

Proof. Let us verify (6). Take the substitution $\varphi : X \to A(w)$ of the basis elements of $A(w)$ for the generators. By (3), the left- and right-hand sides of (6) vanish if at least two generators come from $\{b_0, b_1, \dots\}$.

If $\varphi(x_i) = a$ then $\varphi(x_ix_j) = 0$ for all $j \neq i$ since u is always before zero in w. The value φ at the right- and left-hand sides of (6) can be nonzero only if $\varphi(x_i) = b_k$ and $\varphi(x_s) = a$ for all remaining s. But then both sides of (6) become b_{k+n-1}, which proves (6). Relations (7)–(9) are proved similarly.

Verify that the monomials $f_1 = (x_1x_2)u(x_3,\dots,x_n)$ and $f_i = (x_ix_1)u(x_2,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $2 \leq i \leq n$, are linearly independent modulo the ideal of identities of $A(w)$, provided that u is always after some zero in w. Assume that $f = \lambda_1 f_1 + \dots + \lambda_n f_n \equiv 0$ is an identity. Suppose for instance that $\lambda_1 \neq 0$. Then for every occurrence of u in w there exists $k \geq 0$ such that the substitution $\varphi(x_1) = b_k$, $\varphi(x_2) = \dots = \varphi(x_n) = a$ yields the values $\varphi(f_1) = b_{k+n-1}$ and $\varphi(f_2) = \dots = \varphi(f_n) = 0$, which contradicts the assumption that f is the identity of our algebra. By analogy, for every $i > 1$ we can find a substitution φ with $\varphi(f_i) \neq 0$ and $\varphi(f_j) = 0$ for all $j \neq i$. The independence of the right-hand sides of (8) and (9) is established similarly. □

Lemma 5. *Suppose that u is a subword of type 2 in w. Then the linear span of the monomials*

$$\begin{gathered} f_{ij} = (x_ix_j)u(x_{l_1},\dots,x_{l_{n-2}}), \quad i \neq j, l_1 < \dots < l_{n-2}, \\ \{l_1,\dots,l_{n-2}\} = \{1,\dots,n\} \setminus \{i,j\}, \end{gathered} \tag{10}$$

in $M\{X\}$ modulo the identities of $A(w)$ is of dimension r_n, where r_n is the rank of the system

$$\sum_i z_{ij} = 0,\ 1 \leq j \leq n, \quad \sum_j z_{ij} = 0,\ 1 \leq i \leq n, \tag{11}$$

of $2n$ equations in $n^2 - n$ unknowns z_{ij} for $1 \leq i \neq j \leq n$.

PROOF. Verify that the linear combination $f = \sum_{i,j} \lambda_{ij} f_{ij}$ is an identity in $A(w)$ if and only if the tuple of coefficients $\{\lambda_{ij}\}$ satisfies (11). Suppose firstly that one equality is violated: for instance, $\lambda = \lambda_{i1} + \cdots + \lambda_{in} \neq 0$. The definition of u shows that w has an initial subword of the form $w_1 \dots w_{k+1}u$ with $w_{k+1} = 0$ and $w_{k+2} \dots w_{k+n-1} = u$.

Denote $w_1 \dots w_k$ by v. Then the definition of multiplication in $A(w)$ yields $b_0 v(a, \dots, a)au(a, \dots, a) = b_{k+n-1}$. Consider the substitution $\varphi : \varphi(x_i) = b_k, \varphi(x_t) = a$ for all $t \neq i$. Then $\varphi((x_i x_j)u(x_{l_1}, \dots, x_{l_{n-2}}))$ $= b_{k+n-1}$ for every $j \neq i$, while $\varphi(f) = \lambda b_{k+n-1} \neq 0$. Similarly, $\lambda_{1j} f_{1j} + \cdots + \lambda_{nj} f_{nj}$ is not an identity if $\lambda_{1j} + \cdots + \lambda_{nj} \neq 0$.

Verify now that $f \in \mathrm{Id}(A(w))$ whenever $\{\lambda_{ij}\}$ is a solution to (11). The values of all monomials f_{ij} vanish for arbitrary substitutions φ such that among $\varphi(x_1), \dots, \varphi(x_n)$ either there is no basis vector b_k or there are at least two vectors b_r and b_m. Suppose that $\varphi(x_i) = b_k$ and $\varphi(x_j) = a$ for all $i \neq j$. Suppose again that $u = w_{k+2} \dots w_{k+n-1}$. If $w_{k+1} = 0$ in w then $b_k a = b_{k+1}$ and $ab_k = 0$. Thus, $\varphi(x_i x_1) = \cdots = \varphi(x_i x_n) = b_k a = b_{k+1}$ and $\varphi(f_{ij}) = b_{k+n-1}$ for all $j \neq i$. At the same time, $\varphi(f_{rt}) = 0$ for $r \neq i$.

Consequently, $\varphi(f) = \left(\sum_j \lambda_{ij}\right) b_{k+n-1} = 0$. By analogy, if $w_{k+1} = 1$ then $\varphi(x_1 x_i) = \cdots = \varphi(x_n x_i) = ab_k = b_{k+1}$ and $\varphi(f) = \left(\sum_j \lambda_{ji}\right) b_{k+n-1} = 0$.

The space generated by all f_{ij} of (10) in $M\{X\}$ is of dimension $n^2 - n$. Their linear combinations, which are identities of $A(w)$ constitute a space of dimension $n^2 - n - r_n$. Consequently, the codimension of the intersection of the latter space with $\mathrm{Id}(A(w))$ is r_n. $\square$

REMARK 1. To show that $r_n = 2n - 1$ is an easy linear algebra exercise.

Theorem 1. *For the algebra $A(w)$, the nth codimension for $n \geq 3$ equals*

$$c_n(A(w)) = k^{(1)}_{n-2} n + k^{(2)}_{n-2}(2n - 1), \tag{12}$$

where $k^{(1)}_m$ and $k^{(2)}_m$ are the numbers of length m subwords of type 1 or 2 in w respectively. In particular,

$$\mathrm{Comp}_{w*}(n-2) \leq c_n(A(w)) \leq 2\,\mathrm{Comp}_{w*}(n-2), \tag{13}$$

where $w* = w_3 w_4 \dots$.

PROOF. Denote by $P_{n,u}$ the linear span in $M\{X\}$ of all monomials (4) for every binary word u of length $n - 2$. Then

$$P_n(M\{X\}) = \bigoplus_{|u|=n-2} P_{n,u}. \tag{14}$$

Lemma 2 implies that, modulo the ideal $\mathrm{Id}(A(w))$, the space $P_n(M\{X\})$ contains only those terms of (14) for which u is a proper subword of w. Suppose that $u_1, \dots, u_t$ are all length $n - 2$ proper subwords and $f_1 \in P_{n,u_1}, \dots, f_t \in P_{n,u_t}$. It is not difficult to observe that if $f_1 + \cdots + f_t$ is an identity in $A(w)$ then so are all $f_1, \dots, f_t$. Now (14) follows from Lemmas 3–5, while (13) from the obvious relation $k^{(1)}_{n-2} + k^{(2)}_{n-2} = \mathrm{Comp}_{w*}(n-2)$. $\square$

Theorem 1 enables us to substantially enlarge the class of algebras with intermediate growth of codimensions. For instance, in [12] for every function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

(i) $\varphi(t) \gg \log t$,

(ii) $\varphi(t)$ is differentiable on $(0, \infty)$,

(iii) $\varphi'(t) \ll t^{-\beta}$ for some constant $\beta > 0$,

(iv) φ' is a decreasing function,

there exists a binary word u with $\log \mathrm{Comp}_u(n) \sim \varphi(n)$. Here the relation $f(t) \ll g(t)$ means that $\lim_{n\to\infty}(f(t)/g(t)) = 0$, while log stands for the base 2 logarithm. Theorem 1 yields the following series of propositions:

Theorem 2. *For every function $\varphi(t)$ satisfying* (i)–(iv) *there is an algebra A with $c_n(A) \sim 2^{\varphi(n)}$.* $\square$

The new class of functions of intermediate growth realized as the growth of codimensions includes, for instance, all functions $a^{\sqrt{n}}$ for $a > 1$.

Some more exotic example, presented in [12], corresponds to $\varphi(t) = (t+10)^{\frac{1}{2}+\frac{1}{4}\cos(\log\log(t+1))}$ which oscillates very slowly between $n^{1/4}$ and $n^{3/4}$.

Resting on other results of the theory of formal languages, we can construct examples of algebras with even sharper oscillations of the function of codimensions; see [13, Theorem 9].

Theorem 3. *There exists an algebra A for which we can choose an increasing sequence n_k, for $k = 1, 2, \dots$, such that*

(a) $c_{n_k}(A) < n_k + \log\log n_k$ *for odd k,*

(b) $c_{n_k} > 2^{\frac{n_k}{\log\log n_k}}$ *for even k.* $\square$

Theorem 1 enables us to construct some example of an algebra for which the first and second limits in (1) exist, but the third limit does not. It therefore refutes Regev's conjecture in the third approximation. To this end, consider the language E_0 consisting of all words in the two-letter alphabet $\{a, b\}$ which avoid the subwords a^2, b^4, and ab^2a. The combinatorial complexity of the language E_0 is calculated in [14]:

$$\mathrm{Comp}_{E_0} = \begin{cases} 2F_{k+2} & \text{if } n = 2k, \\ F_{k+4} & \text{if } n = 2k+1, \end{cases} \tag{15}$$

where F_m is the mth Fibonacci number. Construct a word w whose language of all subwords coincides with E_0. To this end, write out successively all words in E_0 of length 1, then of length 2, and so on. Moreover, list the words starting with ba twice. Then, in order to avoid forbidden subwords, we insert, if need be, an intermediate word of length at most 3 between adjacent words. For instance, between a and b^2a we can insert b, while between ab and ba, either a or ab^2. Moreover, since the words with the prefix ba occur twice, we can make additional insertions so that in one case this word v is preceded by the letter a, while in the other by b. This is possible since between a and ba we can insert both ba and bab^2; between ab and ba, both a and ab^2; between ab^2 and ba, both the empty word and ba; between ab^3 and ba, either a or ab^2. This enables us to construct a word w whose language coincides with E_0 and all words beginning with ba are subwords of type 2. The words beginning with a, b^2a, and b^3a cannot be subwords of type 2; i.e., they are of type 1.

Observe that $w* = w_3w_4\dots$ includes all words of E_0 by construction; thus, in order to apply Theorem 1, it suffices to count the number of subwords of type 1 or 2 in the word w itself.

Denote by α_k the number of length k subwords of w beginning with a. Then every subword like of the form either $abav$ or ab^3au. Thus, $\alpha_k = \alpha_{k-2} + \alpha_{k-4}$ for $k \geq 5$. Appreciating that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 1, 1, 2, 2$, we see that $\alpha_{2k} = \alpha_{2k-1} = F_{k+1}$ for all $k \geq 1$.

Denote the numbers of length m subwords of w beginning with ba, b^2a, and b^3a by β_m, γ_m, and δ_m. Then

$$\beta_{2k} = \alpha_{2k-1} = F_{k+1}, \quad \gamma_{2k} = \alpha_{2k-2} = F_k, \quad \delta_{2k} = \alpha_{2k-3} = F_k$$

for even indices, while

$$\beta_{2k+1} = \alpha_{2k} = F_{k+1}, \quad \gamma_{2k+1} = \alpha_{2k-1} = F_{k+1}, \quad \delta_{2k+1} = \alpha_{2k-3} = F_k$$

for odd indices. Hence,

$$k^{(1)}_{n-2} = \begin{cases} F_{k-1} + F_{k+1} & \text{for } n = 2k, \\ F_{k-1} + F_{k+2} & \text{for } n = 2k+1, \end{cases}$$

while $k^{(2)}_{n-2} = \beta_{n-2} = F_k$ for $n = 2k$ and $n = 2k+1$. Combining (16) and (17) with Theorem 1, we infer that for the word w and the algebra $A(w)$ constructed from it we have

$$c_n(A(w)) = n(F_{t-1} + F_{t+1}) + (2n-1)F_t$$

for $n = 2t$ and

$$c_n(A(w)) = n(F_{t-1} + F_{t+2}) + (2n-1)F_t$$

for $n = 2t + 1$. Since

$$F_t = \frac{\varphi^t + (-\varphi)^{-t}}{2\varphi - 1} \sim \frac{\varphi^t}{1+\sqrt{5}}, \quad \text{where } \varphi = \frac{1+\sqrt{5}}{2},$$

while the coefficient of n in $c_n(A(w))$ equals $F_{t-1}+2F_t+F_{t+1}$ for $n = 2t$ and $F_{t-1}+2F_t+F_{t+2}$ for $n = 2t+1$, it follows that

$$\lim_{n\to\infty} \sqrt[n]{c_n(A(w))} = \sqrt{\varphi}, \quad \lim_{n\to\infty} \log_n \frac{c_n(A(w))}{\sqrt{\varphi}^n} = 1.$$

At the same time, the third limit in (1) fails to exist because

$$\lim_{n=2t\to\infty} \frac{c_n(A(w))}{n\sqrt{\varphi}^n} = \frac{\varphi^2+\varphi+2}{\varphi(2\varphi-1)}, \quad \lim_{n=2t+1\to\infty} \frac{c_n(A(w))}{n\sqrt{\varphi}^n} = \frac{\varphi^3+\varphi+2}{\varphi(2\varphi-1)}.$$

The main results of this article were announced in [15].

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M. V. Zaicev
Moscow State University, Moscow, Russia
E-mail address: zaitsev@mail.ru

D. D. Repovš
University of Ljubljana, Ljubljana, Slovenia
E-mail address: dusan.repovs@guest.arnes.si