DOI 10.1007/s10469-019-09522-6
Algebra and Logic, Vol. 58, No. 1, March, 2019
(Russian Original Vol. 58, No. 1, January-February, 2019)

COMBINATORICS ON BINARY WORDS AND CODIMENSIONS OF IDENTITIES IN LEFT NILPOTENT ALGEBRAS

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UDC 512.57

Keywords: *left nilpotent algebra, polynomial identity, codimension, subexponential function of combinatorial complexity,* PI-*exponent.*

Numerical characteristics of polynomial identities of left nilpotent algebras are examined. Previously, we came up with a construction which, given an infinite binary word, allowed us to build a two-step left nilpotent algebra with specified properties of the codimension sequence. However, the class of the infinite words used was confined to periodic words and Sturm words. Here the previously proposed approach is generalized to a considerably more general case. It is proved that for any algebra constructed given a binary word with subexponential function of combinatorial complexity, there exists a PI-*exponent. And its precise value is computed.*

INTRODUCTION

In the paper we study numerical characteristics of polynomial identities of left nilpotent algebras. Previously, we came up with a construction which, given an infinite binary word, allowed us to build a two-step left nilpotent algebra with specified properties of the codimension sequence. However, the class of the infinite words used was confined to periodic words and Sturm words. Here the previously proposed approach is generalized to a considerably more general case. It is proved that for any algebra constructed given a binary word with subexponential function of combinatorial complexity, there exists a PI-exponent. And its precise value is computed.

*Supported by Russian Science Foundation, project 16-11-10013.
**Supported by Slovenian Research Agency, project Nos. BI-RU/16-18-002, P1-0292, J1-8131, and J1-7025.

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Let F be a field of characteristic zero and A an algebra over F. With A we associate an integer-valued sequence $\{c_n(A)\}$, $n = 1, 2, \dots$, defined by its multilinear identities. In the general case the sequence $\{c_n(A)\}$ may grow superexponentially. For instance, if A is an absolutely free algebra of countable rank, then $\{c_n(A)\} = p(n)n!$, where $p(n) = \frac{1}{n}\binom{2n-2}{n-1}$ is the Catalan number. If A is a free associative algebra or a free Lie algebra, then $\{c_n(A)\} = n!$ or $(n-1)!$, respectively.

Nevertheless, there is a broad class of algebras for which $\{c_n(A)\}$ is exponentially bounded, i.e., $\{c_n(A)\} \le a^n$ for all $n \ge 1$ with some constant a. Among these are all finite-dimensional algebras [1], all associative PI-algebras [2], infinite-dimensional simple Lie algebras of Cartan type [3], affine Kac–Moody algebras [4, 5], Lie algebras with nilpotent commutant [6], and others. If the sequence $\{c_n(A)\}$ is exponentially bounded, then the root sequence $\sqrt[n]{c_n(A)}$ is bounded, and it is natural to ask whether its limit exists.

In the late 1980s, Sh. Amitsur conjectured that for any associative PI-algebra A, the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)} \tag{1}$$

exists and is an integer. That conjecture was confirmed in the 1990s [7, 8]. Later, a similar conjecture was validated for finite-dimensional Lie algebras [9], finite-dimensional Jordan algebras [10], Lie superalgebras with nilpotent commutant [11], and a number of others. For some infinite-dimensional Lie algebras, the conjecture received partial confirmation: it was proved that limit (1) exists, but it turned out to be fractional [12, 13]. The existence of limit (1) was also proved for all finite-dimensional simple algebras [14]. Moreover, examples of simple finite-dimensional algebras with fractional limit (1) were constructed [15]. If limit (1) exists, then it is conventionally called a PI-*exponent* of A and is denoted by $\exp(A)$.

At present, we know of just one work where a series of algebras with the missing PI-exponent is constructed [16]. In this connection, it seems important to extend the class of algebras in which the PI-exponent exists. In [17], we can find a family of nonassociative algebras whose PI-exponents run over all real values in an infinite interval $(1; \infty)$. All algebras of the family are two-step left nilpotent, i.e., satisfy an identity of the form

$$x(yz) \equiv 0. \tag{2}$$

Examples in [17] were constructed based on combinatorial properties of infinite binary Sturm words and periodic words.

Recently, it has been proved that every finitely generated algebra A with identity (2) has exponentially bounded growth of the codimension sequence $\{c_n(A)\}$ [18]. The main objective of the present paper is to generalize one of the basic results in [17]—Theorem 5.1—to a much wider class of algebras associated with binary words. We introduce the concept of a *slope* for an arbitrary infinite binary word w and prove that in the case where the combinatorial complexity of w grows subexponentially, the PI-exponent of a corresponding algebra exists and is explicitly expressed via the slope of w.

All basic notions in the theory of algebras with identities are contained in [19-21].

1. BASIC NOTIONS AND CONSTRUCTIONS

Denote by $F\{X\}$ an absolutely free algebra over a field F with an infinite set X of free generators. Recall that a polynomial $f = f(x_1, \dots, x_n) \in F\{X\}$ is called an *identity* of an algebra A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. The set of all identities of A forms a two-sided ideal $\mathrm{Id}\,(A)$ in $F\{X\}$, which is stable under all endomorphisms of $F\{X\}$. Denote by P_n the subspace of all multilinear polynomials in variables $x_1, \dots, x_n$ in $F\{X\}$. Then the intersection $P_n \cap \mathrm{Id}\,(A)$ consists of all multilinear identities of A of degree n. Put

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}\,(A)}, \quad c_n(A) = \dim P_n(A).$$

The symmetric group S_n acts naturally on P_n,

$$\sigma \circ f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{3}$$

Since an ideal $\mathrm{Id}\,(A)$ is stable under an automorpism induced on $F\{X\}$ by formula (3), $P_n(A)$ is also an FS_n-module.

Representations of the symmetric group play a key role in the quantitative PI-theory. We therefore recall the basic concepts and constructions used in what follows. Let $\lambda \vdash n$ be a partition of a natural number n, i.e., $\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \dots \lambda_k > 0$ are integers, with $\lambda + \dots + \lambda_k = n$. A *Young diagram* D_λ is a tableau with n boxes arranged in k rows. The first row contains λ_1 boxes, the second row contains λ_2 boxes, and so on. A *Young tableau* T_λ is a diagram D_λ with boxes occupied by numbers from 1 to n. The *row stabilizer* R_{T_λ} of T_λ is a subgroup of S_n that is isomorpic to $S_{\lambda_1} \times \dots \times S_{\lambda_k}$ and consists of all permutations that permute numbers within rows. Similarly, the *column stabilizer* C_{T_λ} consists of permutations that permute symbols within columns.

Denote by $R(T_\lambda)$, $C(T_\lambda)$, and e_{T_λ} the following elements of the group ring FS_n:

$$R(T_\lambda) = \sum_{\sigma \in R_{T_\lambda}} \sigma, \quad C(T_\lambda) = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\ \tau)\, \tau, \quad e_{T_\lambda} = R(T_\lambda) C(T_\lambda).$$

The element e_{T_λ} is a quasi-idempotent, i.e., $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$, where γ is a nonzero scalar. In particular, $C(T_\lambda) e_{T_\lambda} \neq 0$. It is known that $FS_n e_{T_\lambda}$ is a minimal left ideal, and every irreducible representation of the group S_n is isomorphic to its representation on one of the ideals $FS_n e_{T_\lambda}$. To be familiar with the foundations of the representation theory of the symmetric group, we ask the reader to consult [22]; its applications in the PI-theory can be found in [19-21].

In studying numerical characteristics linked to identical relations, it is common practice to use the notation adopted in character theory. Let $\chi_\lambda = \chi(FS_n e_{T_\lambda})$ be a character of an irreducible module $FS_n e_{T_\lambda}$. Then the expression

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{4}$$

where M is some FS_n-module, means that in the decomposition of M into irreducible components: namely,

$$M = M_1 \oplus \ldots \oplus M_q, \tag{5}$$

the module isomorphic to $FS_n e_{T_\lambda}$ occurs m_λ times. A nonnegative integer m_λ is called the *multiplicity* of a character χ_λ in $\chi(M)$.

Now let M in (4) be a submodule of P_n, complementary to $P_n \cap \mathrm{Id}\,(A)$ and isomorphic to $P_n(A)$. We will identify M with $P_n(A)$. Consider one of the summands M_j in (5). It is generated by a multilinear polynomial of the form $e_{T_\lambda} g$, where $g = g(x_1, \ldots, x_n) \in P_n$. Since e_{T_λ} is a quasi-idempotent, $f = C(T_\lambda) e_{T_\lambda} g$ is a nonzero element of M_j. On the other hand, if $\lambda'_1, \ldots, \lambda'_t$ are heights of columns in D_λ (here $t = \lambda_1$), then the variable set $\{x_1, \ldots, x_n\}$ is decomposed into the union of disjoint subsets $X_1 \cup \ldots \cup X_t$ of orders $\lambda'_1, \ldots, \lambda'_t$, respectively, such that f is skew-symmetric with respect to each subset X_i, $i = 1, \ldots, t$. Thus the following lemma holds true.

LEMMA 1. Let M_j be one of the irreducible summands in (5) and let $\chi(M_j) = \chi_\lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Denote by $\lambda'_1, \ldots, \lambda'_t$ column heights in D_λ. Then M_j is generated as an FS_n-module by a multilinear polynomial $f = f(x_1, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\} = X_1 \cup \ldots \cup X_t$, $|X_j| = \lambda'_j$, $j = 1, \ldots, t$, and f is skew-symmetric in variables of each subset $X_1, \ldots, X_t$. Moreover, $m_\lambda \neq 0$ in (4) if and only if f is not an identity of A.

Consider several other quantitative characteristics associated with identities and representations of S_n. The quantity q in (5) is called the *length* of a module M, and if $M = P_n(A)$, then it is called the *colength* of an algebra A and is denoted by $l_n(A)$. Clearly,

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda, \tag{6}$$

where m_λ are multiplicities in (4). If we denote by $d_\lambda = \deg \chi_\lambda$ the dimension of an appropriate representation, then

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda. \tag{7}$$

From (6) and (7), we derive the upper estimate

$$c_n(A) \leq l_n(A) \max\{d_\lambda \mid m_\lambda \neq 0\}. \tag{8}$$

Instead of the dimension d_λ, it is more convenient to use a numerical characteristic close to it. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$. Put

$$\Phi(\lambda) = \frac{1}{\left(\frac{\lambda_1}{n}\right)^{\frac{\lambda_1}{n}} \cdots \left(\frac{\lambda_k}{n}\right)^{\frac{\lambda_k}{n}}}.$$

The quantities $\Phi(\lambda)^n$ and d_λ asymptotically coincide up to polynomial factor. We revise this statement for $k = 2$. In this case $\Phi(\lambda)$ is in fact a function of one variable:

$$\Phi(x) = \frac{1}{x^x (1-x)^{1-x}}, \quad 0 < x \leq \frac{1}{2}. \tag{9}$$

LEMMA 2 [17, Lemma 3.3]. Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of a number n. Then

$$\frac{1}{\sqrt{\pi n^3}}\Phi(\beta)^n < d_\lambda < \Phi(\beta)^n,$$

where $\beta = \frac{\lambda_2}{n}$, and $\Phi(x)$ is defined by (9).

We point out yet another property of a function Φ.

LEMMA 3. $\Phi(x)$ is continuous on an interval $(0; \frac{1}{2}]$ and $\Phi(a) < \Phi(b)$ for any $0 < a < b \leq \frac{1}{2}$.

Now we recall some notions from the combinatorial theory of infinite words. Let $w = w_1 w_2 \dots$ be an infinite word over a binary alphabet $\{0; 1\}$. The *combinatorial complexity* of a word w is a function $Comp_w(n)$ equal to the number of different subwords of length n in the word w (see [23]). It is known that if w is a periodic word, then $Comp_w(n) = \text{const}$, starting with some n. Otherwise $Comp_w(n) \geq n + 1$. A word w is called a *Sturm word* if $Comp_w(n) = n + 1$ for all n. Along with slow growth of the complexity function, periodic words and Sturm words possess yet another important statistical characteristic. In either case there exists a limit

$$\pi(w) = \lim_{n\to\infty} \frac{w_1 + \ldots + w_n}{n},$$

which is called the *slope* of w.

In order to generalize the result in [17] to a wider class of algebras, we extend the notion of a slope. First let $u = u_1 \dots u_n$ be a finite word over a binary alphabet. The *slope* of u is the parameter

$$\pi(u) = \frac{u_1 + \ldots + u_n}{n},$$

and the length of u is the number $|u| = n$. Now, for an infinite word w, we put

$$q_n = \min\{\pi(u) \mid u \text{ is a subword of } w \text{ of length } n\},$$

and define the slope $\pi(w)$ as

$$\pi(w) = \varliminf_{n\to\infty} q_n.$$

The parameter $\pi(w)$ will define PI-exponents of algebras corresponding to a word w under weaker conditions on the function $Comp_w$ than in the case of periodic words and Sturm words.

LEMMA 4. Let w be a word with slope $\pi(w) = \alpha < 1$. Then, for any natural T, w will contain a subword u of length T with slope $\pi(u) \leq \alpha$.

Proof. Fix an arbitrary sufficiently small $\varepsilon > 0$. By the definition of $\pi(w)$, w has finite subwords of unbounded length with a slope smaller than $\alpha + \varepsilon$. Let v be one of such words of sufficiently large length m. Divide m by T with a remainder, i.e., $m = NT + q$, and partition v into $N + 1$ subwords, i.e., $v = v_1 \dots v_{N+1}$, where the consecutive subwords $v_1, \dots, v_N$ have length T, and v_{N+1} is a subword of length q or an empty word (if $q = 0$). Obviously, for $q = 0$, among $v_1, \dots, v_N$ there is at least one word with a slope smaller than $\alpha + \varepsilon$; otherwise the slope of v would be no less than $\alpha + \varepsilon$.

Now we consider the general case $0 < q < T$ and suppose that $\pi(v_i) \geq \alpha+\varepsilon$ for all $i = 1, \ldots, N$. The slope of any word of length T may assume values only within the set

$$\left\{0, \frac{1}{T}, \frac{2}{T}, \ldots, \frac{T-1}{T}\right\}.$$

Consequently, there is an integer k such that

$$\frac{k}{T} < \alpha + \varepsilon, \ \frac{k+1}{T} \geq \alpha + \varepsilon.$$

By our assumption, each subword $v_1, \ldots, v_N$ has no less than $k+1$ ones, and there are no less than $N(k+1)$ ones in the word $v_1 \ldots v_N$. Even if $\pi(v_{N+1}) = 0$, i.e., v_{N+1} consists of zeros only, then v has no less than $N(k+1)$ ones. Therefore,

$$\pi(v) \geq \frac{(k+1)N}{NT+q} = \frac{k+1}{T+\frac{q}{N}}.$$

Since $\frac{k+1}{T} \geq \alpha + \varepsilon$, and $\frac{q}{N} \to 0$ as $N \to \infty$, the condition $\pi(v) < \alpha + \varepsilon$ is not satisfied with a sufficiently large length m of a word v. This means that for $\varepsilon > 0$ as small as is wished, there exists a subword u of length T with $\pi(u) < \alpha + \varepsilon$. For a word u of length T, $\pi(u)$ may assume a discrete set of values of the form $\frac{k}{T}$, $0 \leq k \leq T-1$; therefore, there also exists a subword u of length T with $\pi(u) \leq \alpha$. $\square$

Lemma 4 implies that the upper and lower limits of the sequence q_n coincide, i.e., it has an ordinary limit.

2. ALGEBRAS OF BINARY WORDS

Let $K = \{k_1, k_2, \ldots\}$ be an infinite integer-valued sequence in which $k_i \geq 2$ for all $i \geq 1$. An algebra $A(K)$ is defined as follows. Its basis is the infinite union

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \ldots,$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \leq j \leq k_i\}, \quad i = 1, 2, \ldots,$$

and nonzero products of basic elements are defined via relations

$$\begin{gathered} z_2^{(i)} a = z_3^{(i)}, \ldots, z_{k_i-1}^{(i)} a = z_{k_i}^{(i)}, z_{k_i}^{(i)} a = z_1^{(i)}, \ i = 1, 2, \ldots, \\ z_1^{(i)} b = z_2^{(i+1)}, \ i = 1, 2, \ldots. \end{gathered} \tag{10}$$

Obviously, the algebra $A(K)$ satisfies identity (2), and hence only products with left-normed arrangement of parentheses may be nonzero. Therefore, we omit parentheses in representations of monomials. Furthermore, the linear hull $\langle Z_1 \cup Z_2 \cup \ldots \rangle$ is an ideal of codimension two with zero multiplication. Consequently, every monomial f, which is skew-symmetric in four variables or in two sets of variables of cardinality three, is an identity of the algebra $A(K)$. From this, in view of Lemma 1, we conclude that the following lemma holds true.

LEMMA 5. Let

$$\chi_n(A(K)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.$$

Then nonzero multiplicities m_λ may occur only in partitions $\lambda = (n)$, $\lambda = (n-k, k)$, and $\lambda = (n-k-1, k, 1)$.

We specify the form of the sequence K. Let w be an infinite binary word and $m \geq 2$ be an integer. Put

$$k_i = \begin{cases} m & \text{if } w_i = 0, \\ m+1 & \text{if } w_i = 1. \end{cases} \tag{11}$$

The resulting algebra depending on m and w is denoted by $A(m, w)$.

LEMMA 6 [17, Lemma 4.2]. For an arbitrary $m \geq 2$ and for any word w, the colength of an algebra $A(m, w)$ with all n satisfies the inequality

$$l_n(A(m, w)) \leq 3(m+1)n^3 Comp_w(n).$$

As distinct from [17], we consider algebras, not with the condition $Comp_w(n) \leq n+1$ on w, but with any subexponential growth of the complexity function. A function of natural argument $\varphi(n)$ is said to be *subexponential* if

$$\lim_{n \to \infty} \frac{\varphi(n)}{a^n} = 0$$

for any real $a > 1$. Lemma 6 and formula (8) show that for a word w with a subexponential complexity function, the values

$$\sqrt[n]{c_n(A(m, w))} \text{ and } \sqrt[n]{\max\{d_\lambda \mid m_\lambda \neq 0\}}$$

are asymptotically close. Note that the complexity function of a word w is exactly a function of combinatorial complexity in a factor language consisting of all finite subwords of w, and the class of languages with subexponential combinatorial complexity is quite rich (see, e.g., [24, 25]).

We find an upper estimate for codimensions $c_n(A(m, w))$.

LEMMA 7. Let $A = A(m, w)$, where $m \geq 2$, and w be an infinite binary word with slope $\pi(w) = \alpha$; $\beta = \frac{1}{m+\alpha}$. Then for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geq N$, the multiplicity m_λ in the decomposition of a cocharacter $\chi_{n+1}(A)$ is equal to zero if $\frac{\lambda_2}{n} > \beta + \varepsilon$.

Proof. First we analyze all possible values for multilinear monomials in the algebra A under the substitution of basic elements for variables. Suppose that there exists an associative monomial $g = g(a, b)$ of degree n such that $z_j^{(i)} g(a, b) \neq 0$ (in fact, $g(a, b)$ is a monomial from right multiplications by a and b). Then g has the form

$$g = a^{i_0} b a^{i_1} b \ldots b a^{i_{r+1}},$$

where $0 \le i_0, i_{r+1} \le m$ and $i_0, \dots, i_r \in \{m-1, m\}$. Moreover,

$$i_1 + \ldots + i_r = (m-1)r + r\pi(w_{i+1} \ldots w_{i+r}), \tag{12}$$

where $\pi(w_{i+1} \ldots w_{i+r})$ is the slope of a subword $w_{i+1} \ldots w_{i+r}$ in w. Furthermore,

$$\begin{aligned} n = \deg g &= i_0 + i_{r+1} + i_1 + \ldots + i_r + r + 1 \\ &= i_0 + i_{r+1} + (m-1)r + r\pi(w_{i+1} \ldots w_{i+r}) + r + 1, \end{aligned} \tag{13}$$

as follows from (12). Since the slope of any word does not exceed 1, from (13) we derive

$$n \le 2m + (m-1)r + 2r + 1 \le (3m+2)r.$$

In particular,

$$r \ge \frac{n}{3m+2}, \tag{14}$$

i.e., r grows linearly with n. The definition of a slope $\pi(w) = \alpha$ implies that for any $\delta > 0$, the value $\pi(w_{i+1} \ldots w_{i+r})$ is strictly greater than $\alpha - \delta$ for all sufficiently large n and r. Therefore, (13) also yields

$$n \ge (m-1)r + r(\alpha - \delta) + r + 1 = r(m + \alpha - \delta) + 1,$$

whence

$$\frac{\deg_b g}{\deg g} = \frac{r+1}{n} \le \frac{1}{m+\alpha-\delta} - \frac{1}{n(m+1)} + \frac{1}{n}. \tag{15}$$

Now consider a partition $\lambda \vdash n+1$ with $\frac{\lambda_2}{n} \ge \beta + \varepsilon$ and suppose that $m_\lambda \ne 0$ in the decomposition of a cocharacter $\chi_{n+1}(A)$. Lemma 1 implies that there exists a multilinear polynomial $f = f(x_1, \dots, x_{n+1})$ depending on λ_2 skew-symmetric sets of order at least two, which is not an identity in A. This means that there is a substitution $\varphi : \{x_1, \dots, x_{n+1}\} \to \{z_j^{(i)}, a, b\}$ under which $\varphi(f) \ne 0$ and b occurs among $\varphi(x_1), \dots, \varphi(x_{n+1})$ as a minimum $\lambda_2 - 1$ times. Then $\varphi(f)$ is a linear combination of monomials like $z_j^{(i)} g(a,b)$, where g is an associative monomial of degree n, with $\deg_b g \ge \lambda_2 - 1$. Consequently,

$$\frac{\deg_b g}{\deg g} \ge \frac{1}{m+\alpha} + \varepsilon - \frac{1}{n}. \tag{16}$$

The limit of the right part in (15) equals $\frac{1}{m+\alpha-\delta}$, and the limit of the right part in (16) equals $\frac{1}{m+\alpha} + \varepsilon$; therefore, for ε fixed, we can choose δ such that (15) and (16) are simultaneously not satisfiable, a contradiction. $\Box$

LEMMA 8. Let w be an infinite binary word with slope $\pi(w) = \alpha$, $m \ge 2$, and $A = A(m, w)$. Then, for any $\nu > 0$ and for all sufficiently large n,

$$c_{n+1}(A) \le 3(m+1)(n+1)^4 Comp_w(n+1)(\Phi(\beta) + \nu)^{n+1},$$

where $\beta = \frac{1}{m+\alpha}$, and $\Phi(x)$ is defined by formula (9).

Proof. Fix $\nu > 0$. Since Φ is continuous (see Lemma 3), there exists ε such that $|\Phi(x)-\Phi(\beta)| < \nu$ if $|x-\beta| < \varepsilon$. Now let λ be a partition of $n+1$ with nonzero multiplicity m_λ in $\chi_{n+1}(A)$. By Lemma 7, we have $\frac{\lambda_2}{n} \le \beta+\varepsilon$. Define $\rho = \frac{\lambda_2}{n}$. In view of Lemma 5, the third component λ_3 in the partition λ equals 0 or 1.

If $\lambda = (\lambda_1, \lambda_2)$, i.e., $\lambda_3 = 0$, then

$$d_\lambda = \deg \chi_\lambda \le \Phi(\rho)^{n+1} \le (\Phi(\beta)+\nu)^{n+1}$$

by virtue of Lemma 2. For a single element partition $\lambda = (n+1)$, a similar inequality is straightforward.

Let $\lambda_3 = 1$. The hook length formula for dimensions of irreducible representations of the group S_n (see [22]) implies that

$$d_\lambda = d_\mu \frac{\lambda_2(\lambda_1+1)(n+1)}{(\lambda_2+1)(\lambda_1+2)} < (n+1)d_\mu,$$

where $\mu = (\lambda_1, \lambda_2) \vdash n$. Consequently, $d_\lambda < (n+1)\Phi(\rho)^n$ by Lemma 2. Thus the inequality

$$d_\lambda < (n+1)(\Phi(\beta)+\nu)^{n+1} \tag{17}$$

holds for all $\lambda \vdash (n+1)$ with $m_\lambda \ne 0$. From (17), (8) and Lemma 6, we derive

$$\begin{aligned} c_{n+1}(A) &\le l_{n+1}(A)(n+1)(\Phi(\beta)+\nu)^{n+1} \\ &\le 3(m+1)(n+1)^4 Comp_w(n+1)(\Phi(\beta)+\nu)^{n+1}. \ \square \end{aligned}$$

3. MAIN RESULT

We find an upper estimate for the codimension growth of the algebra $A(m,w)$.

LEMMA 9. Let w be a word with slope $\pi(w) = \alpha$, $m \ge 2$, and $A = A(m,w)$. Then for any $0 < \varepsilon < \frac{1}{m+\alpha}$ there exist r_0 and an increasing sequence $n_{r_0}, n_{r_0+1}, \ldots$ such that

$$c_{n_r+1}(A) \ge \frac{1}{\sqrt{\pi n_r^3}} \Phi\left(\frac{1}{m+\alpha} - \varepsilon\right)^{n_r}, \quad r \ge r_0. \tag{18}$$

In addition, $n_{r+1} - n_r \le 2m+1$ for all $r \ge r_0$.

Proof. We choose r_0 so that $\varepsilon r_0 > m$. Let $r \ge r_0$ be an arbitrary integer. By Lemma 4, w has a subword $v = w_{i+1} \ldots w_{i+r}$ with slope $\pi(v) \le \alpha$. As in Lemma 7, the algebra A contains a nonzero product like $z_j^{(i)} g(a,b)$, where $g = g(a,b)$ is an associative monomial from right multiplications by a and b,

$$g = a^{i_0} b a^{i_1} b \ldots b a^{i_r},$$

with $0 \le i_0 \le m$, and $i_1, \ldots, i_r \in \{m-1, m\}$. Furthermore,

$$i_1 + \ldots + i_r = (m-1+\pi(v))r,$$

$$n = \deg g = i_0 + i_1 + \ldots + i_r + r = i_0 + r(m + \pi(v)) \leq m + r(m + \alpha),$$

and the degree of g with respect to b equals r. From the previous inequality, by the choice of r, it follows that

$$\frac{r}{n} \geq \frac{1}{m+\alpha} - \varepsilon. \tag{19}$$

In the free algebra $F\{X\}$, consider a left-normed monomial

$$f = zx_1^0 \ldots x_{i_0}^0 y_1 x_1^1 \ldots x_{i_1}^1 y_2 \ldots y_r x_1^r \ldots x_{i_r}^r.$$

Under substitution $\varphi : X \to A$, with $\varphi(z) = z_j^{(i)}$, $\varphi(x_q^p) = a$, and $\varphi(y_s) = b$ for all admissible p, q, and s, the value $\varphi(f)$ equals $z_j^{(i)} g(a, b)$, i.e., $\varphi(f) \neq 0$. Now if we alternate f over pairs $\{x_1^k, y_k\}$, $1 \leq k \leq r$, we obtain a polynomial $\widetilde{f}$. The multiplication table of basic elements of the algebra A shows that $\varphi(\widetilde{f}) = \varphi(f) \neq 0$, i.e., $\widetilde{f}$ is not an identity in A.

We look at how the symmetric group S_n acts on variables $\{x_p^q, y_s\}$ and consider an FS_n-submodule of P_{n+1} generated by $\widetilde{f}$. The structure of quasi-idempotents of a ring FS_n shows that the decomposition of $FS_n\widetilde{f}$ into irreducible components may give rise only to submodules with character χ_λ, where $\lambda = (n - t, t)$ and $t \geq r$. For one of such partitions λ, we obtain

$$c_{n+1}(A) \geq d_\lambda \geq \frac{1}{\sqrt{\pi n^3}} \Phi \left(\frac{t}{n} \right)^n$$

by virtue of Lemma 2. Since $\frac{t}{n} \geq \frac{r}{n}$,

$$c_{n+1}(A) \geq \frac{1}{\sqrt{\pi n^3}} \Phi \left(\frac{1}{m+\alpha} - \varepsilon \right)^n$$

in view of (19) and Lemma 3.

It is precisely this value $n = i_0 + i_1 + \ldots + i_r + r = i_0 + r(m + \pi(w_{i+1} \ldots w_{i+r}))$ that we take as n_r for chosen $r \geq r_0$.

We estimate the difference $n_{r+1} - n_r$. Note first that $n_{r+1} > n_r$ by the choice of words $w_{t+1} \ldots w_{t+r+1}$ and $w_{i+1} \ldots w_{i+r}$ with a minimal slope. The quantity n_r is defined by the subword $w_{i+1} \ldots w_{i+r}$ of length r in w having a minimal slope, and the product $r\pi(w_{i+1} \ldots w_{i+r})$ is equal to the number of ones among $w_{i+1}, \ldots, w_{i+r}$. Similarly, for n_{r+1}, there is a subword $w_{t+1} \ldots w_{t+r+1}$ with least number of ones, i.e.,

$$n_{r+1} = i_0' + (r + 1)(m + \pi(w_{t+1} \ldots w_{t+r+1})),$$

and so

$$n_{r+1} - n_r \leq 2m + (r + 1)\pi(w_{t+1} \ldots w_{t+r+1}) - r\pi(w_{i+1} \ldots w_{i+r}).$$

By the choice of $w_{t+1} \ldots w_{t+r+1}$ and $w_{i+1} \ldots w_{i+r}$, the number of ones in these subwords either is the same or differs by 1. Hence $n_{r+1} - n_r \leq 2m + 1$. $\square$

We are in a position to prove our main result, generalizing [17, Thm. 5.1].

THEOREM. Let w be an infinite binary word with slope $\pi(w) = \alpha$ and with a complexity function $Comp_w(n)$ of subexponential growth; $m \geq 2$ is an integer. Then the algebra $A = A(m, w)$ defined by relations (10) and (11) has a PI-exponent, in which case

$$\exp(A) = \Phi\left(\frac{1}{m+\alpha}\right), \tag{20}$$

where

$$\Phi(x) = \frac{1}{x^x(1-x)^{1-x}}.$$

Proof. For convenience, we define $a_n = \sqrt[n]{c_n(A)}$ and show that the upper and lower limits of the sequence a_n coincide and are equal to $\Phi\left(\frac{1}{m+\alpha}\right)$.

First we estimate the lower limit. To do this, we prove that for an arbitrarily small $\varepsilon > 0$ with all sufficiently large n, the following inequality holds:

$$c_{n+1}(A) \geq \frac{1}{2^{2m+1}\sqrt{\pi n^3}}\Phi\left(\frac{1}{m+\alpha} - \varepsilon\right)^n. \tag{21}$$

By Lemma 9, for a given ε, there exists a sequence of indices n_r, $r = r_0, r_0+1, \ldots$, for which inequality (18) is satisfied. For any $n \geq n_{r_0}$, there is an r such that $n_r \leq n < n_{r+1}$ and $n - n_r < 2m+1$.

Note that the right annihilator of any element $x \neq 0$ in A is equal to zero. This implies that $c_{t+1}(A) \geq c_t(A)$ for any t. Therefore,

$$c_{n+1}(A) \geq c_{n_r+1}(A) \geq \frac{1}{\sqrt{\pi n_{n_r}^3}}\Phi\left(\frac{1}{m+\alpha} - \varepsilon\right)^{n_r}. \tag{22}$$

Since $n \geq n_r$, $\Phi(x) \leq 2$, and $n - n_r \leq 2m+1$, the right part of (22) is not smaller than

$$\frac{1}{2^{2m+1}\sqrt{\pi n^3}}\Phi\left(\frac{1}{m+\alpha} - \varepsilon\right)^n,$$

which proves relation (21).

We have

$$\left(2^{2m+1}\sqrt{\pi n^3}\right)^{-\frac{1}{n}} \to 1 \text{ as } n \to \infty;$$

therefore, relation (21) implies the inequality

$$\varliminf_{n\to\infty} a_n \geq \Phi\left(\frac{1}{m+\alpha} - \varepsilon\right) \text{ for any } \varepsilon > 0.$$

Consequently,

$$\varliminf_{n\to\infty} a_n \geq \Phi\left(\frac{1}{m+\alpha}\right). \tag{23}$$

To derive an estimate for the upper limit of $\{a_n\}$, we use Lemma 8. Since $Comp_w(n)$ is a subexponential growth function, the limit of the quantity

$$\left(3(m+1)(n+1)^4 Comp_w(n+1)\right)^{\frac{1}{n}}$$

as $n \to \infty$ is equal to one. By Lemma 8, therefore,

$$\overline{\lim_{n\to\infty}}\, a_n \leq \Phi\left(\frac{1}{m+\alpha}\right). \tag{24}$$

Inequalities (23) and (24) mean that there exists an ordinary limit of the sequence $\{a_n\}$, i.e., a PI-exponent of the algebra $A(m, w)$, and that equality (20) holds. □

In conclusion we dub the conjecture that any two-step left nilpotent algebra with finitely many generators has a PI-exponent. Important particular cases are finite-dimensional algebras and relatively free algebras of finite rank. For these, the question posed is also open. The condition of being finitely generated is essential, as shown by the counterexample in [18].

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