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MATHEMATICS

Identities on Algebras and Combinatorial Properties of Binary Words

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Abstract—Polynomial identities and codimension growth of nonassociative algebras over a field of characteristic zero are considered. A new approach is proposed for constructing nonassociative algebras starting from a given infinite binary word. The sequence of codimensions of such an algebra is closely connected with the combinatorial complexity of the defining word. These constructions give new examples of algebras with abnormal codimension growth. The first important achievement of the given approach is that the algebras under study are finitely generated. The second one is that the asymptotic behavior of codimension sequences is widely different from all previous examples.

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We study numerical characteristics of identities of nonassociative algebras over a field of characteristic zero. Let A be an algebra over a field Φ and $\Phi\{X\}$ be an absolutely free algebra over Φ with an infinite set of generators X. The set $\mathrm{Id}(A)$ of all identities of A is an ideal in $\Phi\{X\}$. If P_n is the subspace of all multilinear polynomials in $x_1,\dots,x_n$ in $\Phi\{X\}$, then the sequence of codimensions $c_n(A) = \mathrm{codim}(P_n : P_n \cap Id(A))$, $n = 1$, 2, ..., is an important numerical characteristic of the family of identities of A. For a wide class of algebras, the growth of the sequence $\{c_n(A)\}$, which is called the codimension sequence of A, is bounded by an exponential function. For example, if $\dim A = d < \infty$, then $c_n(A) \le d^{n+1}$ [1, 2]. A similar constraint holds for any associative PI algebra [3]. A. Regev conjectured that, in the associative case, $c_n(A) \sim Cn^t d^n$, where C is a constant, t is a half-integer, and d is a nonnegative integer. More precisely, Regev's conjecture means the existence of three limits,

$$d = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \qquad t = \lim_{n\to\infty} \log_n \frac{c_n(A)}{d^n}, \qquad C = \lim_{n\to\infty} \frac{c_n(A)}{n^t d^n}, \tag{1}$$

which can be called the first, second, and third approximations. The first approximation is also known as Amitsur's conjecture.

The existence and integrality of the first limit in (1) were proved in [4, 5]. In the second approximation, Regev's conjecture was also confirmed [6, 7], while the third approximation is still an open question. In the nonassociative case, the first limit in (1) may be fractional [8] or may not exist at all [9] even if the codimension growth is exponentially bounded. Intermediate growth of type n^{n^β}, $0 < \beta < 1$, is also possible, as was shown in [10].

The goal of this work is to expand the class of functions of intermediate growth represented by growth functions of codimensions of some algebras. Additionally, we construct an example of an algebra A for which the first and second limits in (1) exist, while the third limit does not. New examples are constructed using results of the combinatorial theory of formal languages.

The combinatorial properties of binary words have been repeatedly used to construct various examples of asymptotic behavior of codimensions (see, e.g., [8, 10]). An alternative approach to the construction of algebras from binary words was proposed in [11]. This approach is used below.

Let $w_1w_2\dots$ be an infinite binary word. The combinatorial complexity of w is the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of length n in w. A subword u of w is called proper if at least one of its occurrences in w starts at the kth position, where $k \ge 3$. The proper subwords in w

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are divided into two categories. A subword u is said to be of the first type if it occurs in w only after 0 or only after 1. If u occurs in w after both 0 and 1, then it is called a subword of the second type.

Consider algebra $A(w)$ with basis $\{a, b_0, b_1, \ldots\}$ in which multiplication is defined as follows. For any $i \geq 0$, let $b_{i+1} = ab_i$ if $w_{i+1} = 1$ and $b_{i+1} = b_i a$ if $w_{i+1} = 0$. The other products of basis elements are set to zero. Let w^* denote an infinite subword $w_3 w_4 \ldots$ in w. The combinatorial complexity of w and the codimensions of $A(w)$ are connected by the following relation.

Theorem 1. *For the algebra $A(w)$, the nth codimension for $n \geq 3$ is*

$$c_n(A(w)) = k_{n-2}^{(1)} n + k_{n-2}^{(2)}(2n-1),$$

where $k_m^{(1)}$, $k_m^{(2)}$ are the numbers of type one and two subwords of length m in w. Specifically, $\mathrm{Comp}_{w^*}(n-2) \leq c_n(A(w)) \leq 2\mathrm{Comp}_{w^*}(n-2)$.

Relying on the results of [12] and Theorem 1, we can obtain a new wide class of algebras with codimensions of intermediate growth.

Theorem 2. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable function on $(0;\infty)$ such that*

(i) $\varphi(t) \gg \log_2 t$;

(ii) $\varphi'(t) \ll t^{-\beta}$ *for some constant* $\beta > 0$;

(iii) φ' *is a decreasing function.*

Then there exists a binary word u for which $\log_2 \mathrm{Comp}_u(n) \sim \varphi(n)$. *In particular, there exists an algebra A such that $c_n(A) \sim 2^{\varphi(n)}$.*

By using the results of [13], examples of algebras with a codimension function exhibiting sharper oscillations can be obtained.

Theorem 3. *There exists an algebra A for which an increasing sequence n_k, $k = 1, 2, \ldots$, can be chosen such that*

(a) $c_{n_k}(A) < n_k + \ln\ln n_k$ *if k is odd,*

(b) $c_{n_k} > 2^{\frac{n_k}{\ln\ln n_k}}$ *if k is even.*

Theorem 1 makes it possible to construct an example of an algebra for which the first and second limits in (1) exist, while the third limit does not. In the formal language theory, a language E_0 is well known consisting of all words over a two-letter alphabet $\{a, b\}$ that do not contain the subwords a^2, b^4 and ab^2a. It is easy to construct an infinite binary word w_0 for which the language of all finite subwords of both w_0 and w_0^* coincides with E_0. Then, for the algebra $A(w_0)$, the first and second limits in (1) are

$$\lim_{n\to\infty} \sqrt[n]{c_n(A(w))} = \sqrt{\varphi}, \quad \lim_{n\to\infty} \log_n \frac{c_n(A(w))}{\sqrt{\varphi}^n} = 1,$$

respectively, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. At the same time,

$$\lim_{n=2t\to\infty} \frac{c_n(A(w))}{n\sqrt{\varphi}^n} = \frac{\varphi^2 + \varphi + 2}{\varphi(2\varphi - 1)},$$

$$\lim_{n=2t+1\to\infty} \frac{c_n(A(w))}{n\sqrt{\varphi}^n} = \frac{\varphi^3 + \varphi + 2}{\varphi(2\varphi - 1)}.$$

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