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A deleted product criterion for approximability of maps by embeddings

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Abstract

We prove the following theorem: *Suppose that $m \geqslant 3(n+1)/2$ and that $f: K \to \mathbb{R}^m$ is a PL map of an n-dimensional finite polyhedron K. Then f is approximable by embeddings if and only if there exists an equivariant homotopical extension $\Phi: \widetilde{K} \to S^{m-1}$ of the map $\tilde{f}: \widetilde{K}^f \to S^{m-1}$, defined by $\tilde{f}(x,y) = (f(x)-f(y))/(\|f(x)-f(y)\|)$, where $\widetilde{K}^f = \{(x,y) \in K \times K \mid f(x) \neq f(y)\}$.* Our result is a controlled version of the classical deleted product criterion of embeddability of n-dimensional polyhedra in $\mathbb{R}^m$. The proof requires additional (compared with the classical result) general position arguments, for which the restriction $m \geqslant 3(n+1)/2$ is again necessary. We also introduce the van Kampen obstruction for approximability by embeddings. © 1998 Published by Elsevier Science B.V.

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1. Introduction

The main goal of this paper is to prove a *controlled* version of the following classical result:

Theorem 1.1 [28,35,36]. *For every integer $m \geqslant 3(n+1)/2$, every n-dimensional finite polyhedron K is embeddable in $\mathbb{R}^m$ if and only if there exists a $\mathbb{Z}_2$-equivariant map*

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$\Phi:\widetilde{K}\to S^{m-1}$. Moreover, for each such Φ there exists an embedding $\varphi:K\to\mathbb{R}^m$ such that $\widetilde{\varphi}\underset{\text{eq}}{\simeq}\Phi$.

Here $\widetilde{K}=\{(x,y)\in K\times K\mid x\neq y\}$, and the involutions on $\widetilde{K}$ and $S^{m-1}\subset\mathbb{R}^m$ are given by $(x,y)\mapsto(y,x)$ and $x\mapsto -x$, respectively. For any embedding $\varphi:K\to\mathbb{R}^m$, the map $\widetilde{\varphi}:\widetilde{K}\to S^{m-1}$ is defined by

$$\widetilde{\varphi}(x,y)=\frac{\varphi(x)-\varphi(y)}{\|\varphi(x)-\varphi(y)\|}.$$

For any map $f:K\to\mathbb{R}^m$, let

$$\widetilde{K}^f=\{(x,y)\in K\times K\mid f(x)\neq f(y)\}$$

and define the map $\tilde{f}:\widetilde{K}^f\to S^{m-1}$ by $\tilde{f}(x,y)=(f(x)-f(y))/(\|f(x)-f(y)\|)$. By a polyhedron we always mean a *finite* polyhedron.

Theorem 1.2. *If a PL-map $f:K\to\mathbb{R}^m$ of an n-dimensional polyhedron K is approximable by (PL or TOP) embeddings then there exists an equivariant map $\Phi:\widetilde{K}\to S^{m-1}$ such that $\Phi|_{\widetilde{K}^f}\underset{\text{eq}}{\simeq}\tilde{f}$. For every $m\geqslant 3(n+1)/2$ this condition is also sufficient, whereas for $m<3(n+1)/2$ it need not be. Moreover, when $m\geqslant 3(n+1)/2$, f is approximable by embeddings φ such that $\widetilde{\varphi}\underset{\text{eq}}{\simeq}\Phi$, for each such Φ.*

Inverse limits criteria [25,29] reduce, roughly speaking, embeddability of *compacta* into $\mathbb{R}^m$ to embeddability of *PL-maps* between polyhedra in $\mathbb{R}^m$. A map $f:K\to M$ is said to be *embeddable*, or *realizable* in $\mathbb{R}^m$ via an embedding $\psi:M\to\mathbb{R}^m$, if $\psi\circ f$ is approximable by embeddings. Examples [29] show that this notion is rather geometric and is also interesting by itself (see also [1,2,16]). Suppose that $m\geqslant 3(n+1)/2$ and that $f:K\to M$ is a PL-map between polyhedra K and M of dimensions at most n. It follows by Theorem 1.2 that f is embeddable in $\mathbb{R}^m$ via an embedding $\psi:M\to\mathbb{R}^m$ if and only if there exists an equivariant map $\Phi:\widetilde{K}\to S^{m-1}$ such that $\Phi|_{\widetilde{K}^f}\underset{\text{eq}}{\simeq}\widetilde{\psi}\circ\tilde{f}$, where $\tilde{f}(x,y)=(f(x),f(y))$ (see diagram (1.1)).

$$\begin{array}{ccc}\widetilde{K} & \xrightarrow{\ \Phi\ } & S^{m-1}\\ \cup & & \uparrow{\scriptstyle\widetilde{\psi}}\\ \widetilde{K}^f & \xrightarrow{\ \tilde{f}\ } & \widetilde{M}\end{array}\qquad(1.1)$$

Perhaps this criterion can be used to study embeddability of compacta in $\mathbb{R}^m$, in particular to attack Borsuk's conjecture that every contractible locally contractible n-dimensional compactum (CAR) is embeddable into $\mathbb{R}^{2n}$. Approximability by embeddings of every map of a compactum K into $\mathbb{R}^m$ was studied in [6] as a general position property. Theorem 1.2 can be compared with [7,33] (for a short survey see [8; Introduction]): every map of an n-dimensional compactum K into $\mathbb{R}^m$ is approximable by embeddings if and only if every map $g:A\to S^{m-1}$ of a closed subset $A\subset K\times K$ is extendable over $K\times K$ (this is possible only for $m\geqslant 2n$).

Let us state an equivalent formulation of Theorem 1.2 which is convenient for applications. For a triangulation T of K, let

$$\widetilde{T} = \bigcup \{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \emptyset\}$$

and

$$\widetilde{T}^f = \bigcup \{\sigma \times \tau \in T \times T \mid f(\sigma) \cap f(\tau) = \emptyset\}.$$

If T is so small that f is linear on simplices of T, then the necessary (and for $m \geqslant 3(n+1)/2$ also sufficient) condition in Theorem 1.2 can be replaced by the requirement that there exists an equivariant extension of the map $\tilde{f} : \widetilde{T}^f \to S^{m-1}$ to $\widetilde{T}$. This is equivalent to Theorem 1.2, since $(\widetilde{T}, \widetilde{T}^f)$ is an equivariant retract of $(\widetilde{K}, \widetilde{K}^f)$ and because of the equivariant analogue of Borsuk's Extension Theorem.

The proof of necessity in Theorem 1.2 is easy. Take a triangulation T of K such that $f|_\sigma$ is linear for each $\sigma \in T$. Take

$$\varepsilon < \frac{1}{2} \min \{\operatorname{dist}(f(\sigma), f(\tau)) \mid f(\sigma) \cap f(\tau) = \emptyset\}$$

and any embedding $\varphi : K \to \mathbb{R}^m$, ε-close to f. Then for every pair $(x, y) \in \widetilde{T}^f$, $\widetilde{\varphi}(x, y)$ and $\tilde{f}(x, y)$ are not antipodal points of S^{m-1}. Hence $\widetilde{\varphi}|_{\widetilde{T}^f} \underset{\text{eq}}{\simeq} \tilde{f}$ and so $\widetilde{\varphi}$ is the required homotopical extension.

Example 1.3 [29]. The composition $f : S^1 \to S^1 \subset \mathbb{R}^2$ of the standard map of degree 2 and an arbitrary embedding is not approximable by embeddings.

Proof. We have $(\widetilde{S}^1, \widetilde{S}^{1f}) \underset{\text{eq}}{\simeq} (A, \partial A)$, where

$$A = \{(x, y) \in S^1 \times S^1 \mid \operatorname{dist}(x, -y) \leqslant \varepsilon\}$$

is an annulus and ∂A is its boundary. It is easy to see that both restrictions of $\tilde{f}$ to the two connected components of ∂A have degree 2. Hence $\tilde{f}$ is extendable over A. But the circle $A_0 = \{(x, -x) \mid x \in S^1\} \subset A$ is invariant under the involution on $\widetilde{S}^1$. Hence if $\tilde{f}$ extends to an equivariant map $\Phi : A \to S^1$, then $\Phi|_{A_0}$ has odd degree. Hence $\Phi|_{A_0}$ is homotopic to $\Phi|_*$ ($*$ is any connected component of ∂A). Contradiction. □

This proof shows that 'equivalent extension' in Theorem 1.2 cannot be replaced by just 'extension'. In Chapter 5 we present a generalization of this example—we prove that certain maps $S^n \to S^n$ are not embeddable into $\mathbb{R}^{n+k}$ via an embedding $S^n \subset \mathbb{R}^{n+k}$ [2, p. 4]. Here $k = 1, 3, 7$ and $n \geqslant 1$ (for $k = 1$) or $n \geqslant k + 1$ (for $k = 3, 7$). Hence any embedding $S^n \subset \mathbb{R}^{n+k}$ is TOP-standard.

Let us also state and discuss pre-limit formulation of Theorem 1.2. Denote

$$\widetilde{K}^f_\varepsilon = \{(x, y) \in K \times K \mid \operatorname{dist}(f(x), f(y)) \geqslant \varepsilon\}.$$

If a map $f : K \to \mathbb{R}^m$ of a *compactum* K is $\frac{1}{2}\varepsilon$-close to an embedding, then there exists an equivariant homotopical extension $\Phi_\varepsilon : \widetilde{K} \to S^{m-1}$ of the map $\tilde{f}|_{\widetilde{K}^f_\varepsilon}$. Thus if f is approximable by embeddings, then such Φ_ε exists, for each $\varepsilon > 0$. In general, there is no unique Φ for all ε, as an example from [21] shows: Set $m = 2$, $P =$ the pseudoarc, $i : P \hookrightarrow \mathbb{R}^2$ any embedding, $K = P \sqcup P$, and $f = i \sqcup i$. This necessary condition can be reformulated in spirit of [34]: $o \in \mathbb{R}^m$ must be an inessential point of the map $\overline{f} : \widetilde{K} \to \mathbb{R}^m$, defined by $\overline{f}(x, y) = f(x) - f(y)$. On the contrary, it follows from the proof of Theorem 1.2 that if $m \geqslant 3(n+1)/2$ and there exists an equivariant homotopical extension $\Phi_\varepsilon : \widetilde{K} \to S^{m-1}$ of the map $\overline{f}|_{\widetilde{K}^f_\varepsilon}$, then f is $C(n) \cdot \varepsilon$-close to an embedding.

Note that necessary conditions for embeddability of $X \times I$ in $\mathbb{R}^{m+1}$ (or even for the existence of an uncountable collection of disjoint copies of X in $\mathbb{R}^{m+1}$) [21,22] are partial cases of the pre-limit formulation of Theorem 1.2 for $K = X \sqcup X$ and $f = i \sqcup i$ (after [21] was published, the authors discovered that a stronger result than [21, Theorem 1.3] had been proved in [11], although that proof does not work under weaker assumptions than embeddability of $X \times I$ into $\mathbb{R}^{m+1}$). For another (simple) criterion for approximability of maps by embeddings see [30].

Let us construct a controlled analogue of the van Kampen obstruction $\vartheta(f) \in H^{2n}_S(\widetilde{T}, \widetilde{T}^f; \mathbb{Z})$ to approximability of an arbitrary PL-map $f : K \to \mathbb{R}^{2n}$ by embeddings (cf. [15,24]). Take a general position PL-map $g : K \to \mathbb{R}^{2n}$, sufficiently close to f. Fix orientations on every simplex of T. Fix an orientation of $\mathbb{R}^{2n}$. For any two disjoint oriented edges σ and τ of T, count every intersection where the orientation of $g(\sigma)$ followed by that of $g(\tau)$ agrees with that of $\mathbb{R}^{2n}$ as $+1$, and as -1 otherwise. Then $\vartheta(f)$ is the class of the cocycle $\vartheta_g(f)(\sigma, \tau)$ which counts algebraically the intersections of $g(\sigma)$ and $g(\tau)$ in this fashion. If f maps the entire K to a point, then $\vartheta(f)$ is the van Kampen obstruction to embeddability of K in $\mathbb{R}^{2n}$.

Theorem 1.4. *If a PL-map $f : K \to \mathbb{R}^{2n}$ of an n-dimensional polyhedron K is approximable by (PL or TOP) embeddings, then $\vartheta(f) = 0$. For $n \geqslant 3$, this condition is also sufficient, whereas for $n = 1, 2$ it need not be.*

For another cohomological reformulation of Theorem 1.2 see [4, Definition 4.2]. Besides the deleted product condition and van Kampen's obstruction for embeddability in $\mathbb{R}^m$, there are several other necessary conditions [20], e.g., the normal Whitney classes.

Problem 1.5. Find the controlled analogue of the normal Whitney classes.

Theorem 1.1 is a special case of Theorem 1.2 when f is the constant map. Therefore it follows from [9,14,18,26,27] that Theorem 1.2 is false for pairs (m, n) such that $3 < m < 3(n+1)/2$. Analogously, it follows from [9] that Theorem 1.4 is false for $n = 2$. Although both Theorem 1.1 and its cohomological version are true for $m = 2n = 2$, neither one of Theorems 1.2 and 1.4 is.

Example 1.6 (cf. [31, Example 1.5]). Let $K = S^1$ and $f : S^1 \to S^1 \subset \mathbb{R}^2$ be a composition of a degree 3 map and an embedding (on Fig. 1(a) the general position map $g : K \to \mathbb{R}^2$, close to f, is shown, so as to make $\vartheta(f) = 0$ evident). Then f is not

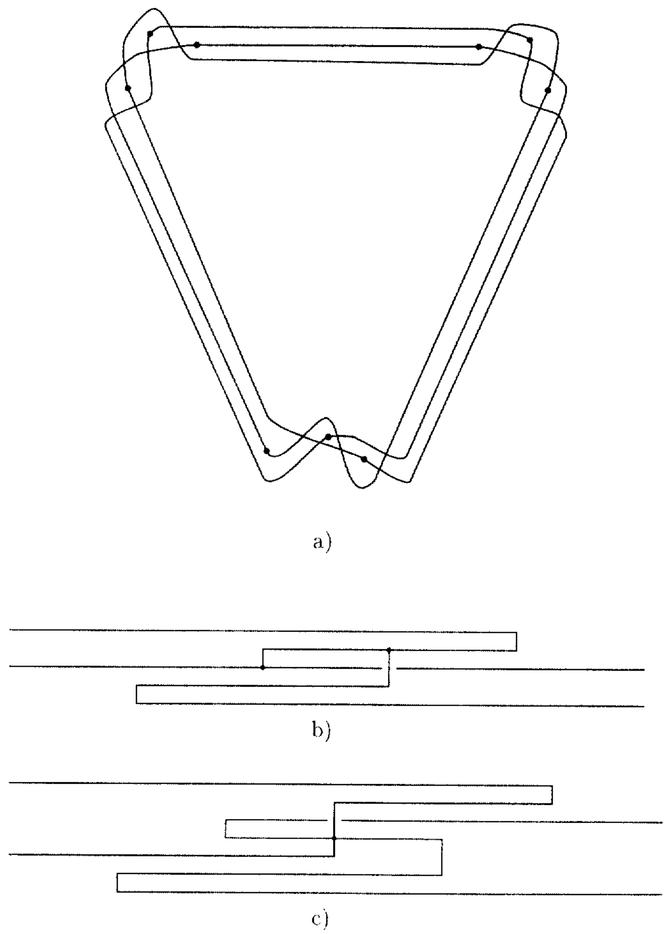

Fig. 1.

approximable by embeddings, even though $\vartheta(f) = 0$ and there exists an equivariant map $\Phi : \widetilde{K} \to S^1$ such that $\Phi|_{\widetilde{K}^f} \underset{\text{eq}}{\simeq} \tilde{f}$.

The reason for non-approximability of f by embeddings in Example 1.6 is that no such Φ is realizable by embeddings (i.e., there is no embedding $\varphi : K \to \mathbb{R}^2$ such that $\tilde{\varphi} \underset{\text{eq}}{\simeq} \Phi$). Note that for $m \geqslant 3(n+1)/2$, every equivariant map $\Phi : \widetilde{K} \to S^{m-1}$ is realizable by embeddings [35].

Example 1.7. Let K be either the 'letter H' or the 'letter X' and $f : K \to I \subset \mathbb{R}^2$ be either of the two maps, defined in [29] (in Figs. 1(b) and (c) general position maps $g : K \to \mathbb{R}^2$, close to f, are shown). Then $\vartheta(f) \neq 0$ even though there exists a map $\Phi : \widetilde{K} \to S^1$, realizable by embeddings and such that $\Phi|_{\widetilde{K}^f} \underset{\text{eq}}{\simeq} \tilde{f}$.

Conjecture 1.8. The sufficiency in Theorem 1.2 holds for $n = 1$ when K is a tree.

For further discussion see [4]. The inverse limits criterion from [2] and [25] also motivated the following:

Conjecture 1.9. Suppose that K is an n-dimensional polyhedron, $f : K \to \mathbb{R}^m$ is a PL-map, and $m > 3(n+1)/2$. Then:

(a) (cf. [17,13]) For each $\varepsilon > 0$, there exists $\delta > 0$ such that every two PL δ-close to f and δ-concordant embeddings $g_1, g_2 : K \to \mathbb{R}^m$ are ε-isotopic;

(b) (cf. [19, Theorem 11]; [35, Theorem 1′]) For each $\varepsilon > 0$, there exists $\delta > 0$ such that every two δ-close to f PL-embeddings $g_1, g_2 : K \to \mathbb{R}^m$ with $\tilde{g}_1 \underset{\mathrm{eq}}{\simeq} \tilde{g}_2$ rel $\widetilde{K}^f$ are ε-isotopic;

(c) (E.V. Ščepin) If $g : K \to \mathbb{R}^m$ is a PL-embedding and $\tilde{g}|_{\widetilde{K}^f} \underset{\mathrm{eq}}{\simeq} \tilde{f}$, then there is a pseudo-isotopy from g to f; and

(d) If f is approximable by embeddings, then there exists a pseudo-isotopy from an embedding to f.

Embeddings $f, g : K \to \mathbb{R}^m$ are said to be *δ-concordant* if there exists an embedding $F : K \times I \to \mathbb{R}^m \times I$ such that $F(x,0) = (f(x),0)$, $F(x,1) = (g(x),1)$ and $\operatorname{dist}(F(x,t), F(x,0)) < \delta$, for each $x \in K$, $t \in I$. Conjecture 1.9(b) follows from Conjecture 1.9(a) and the relative version of Theorem 1.2 (cf. [35, §7]).

A homotopy $F_t : K \to \mathbb{R}^m$ is said to be a *pseudo-isotopy* from an embedding, $F_0 : K \to \mathbb{R}^m$ to a map $F_1 : K \to \mathbb{R}^m$ if the map F_t is an embedding for each $t < 1$. Conjectures 1.9(c), (d) are corollaries of Conjecture 1.8(b) and the pre-limit version of Theorem 1.2.

In the controlled topology the situation when the distances are controlled not in the target space, but in the control space is often studied. In [25] a map $f : K \to M$ was said to be *embeddable* into $\mathbb{R}^m$ via an embedding $\psi : M \to \mathbb{R}^m$ and a cell-like map $p : \mathbb{R}^m \to \mathbb{R}^m$ if there exists an embedding $\varphi : K \to \mathbb{R}^m$ such that $\psi \circ f = p \circ \varphi$.

This motivated the following generalization of Theorem 1.2. Let $f : K \to \mathbb{R}^m$ be a PL-map of an n-dimensional polyhedron K, $p : \mathbb{R}^m \to \mathbb{R}^m$ a cell-like map, and $m \geqslant 3(n+1)/2$. Then for each $\varepsilon > 0$, there is an embedding $\varphi : K \to \mathbb{R}^m$ such that $p \circ \varphi$ is ε-close to f if and only if there exists an equivariant map $\Phi : \widetilde{K} \to \widetilde{\mathbb{R}}^m$ such that $\Phi(\widetilde{K}^f) \subset \widetilde{\mathbb{R}}^{mp}$ and $\tilde{f} \underset{\mathrm{eq}}{\simeq} \tilde{p} \circ \Phi|_{\widetilde{K}^f}$ (see diagram (1.2)).

$$\begin{array}{ccccc} \widetilde{K} & \supset & \widetilde{K}^f & \xrightarrow{\tilde{f}} & \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & S^{m-1} \\ \widetilde{R}^m & \supset & \widetilde{R}^{mp} & \xrightarrow{\tilde{p}} & \end{array} \qquad (1.2)$$

This generalization is proved analogously to Theorem 1.2. The following properties of the cell-like map $p : \mathbb{R}^m \to \mathbb{R}^m$ are used:

(1) for each $\varepsilon > 0$, there exists approximative ε-lifting $\varphi : K \to \mathbb{R}^m$; and

(2) for each $a \in \mathbb{R}^m$ and $\varepsilon > 0$, there exists a PL m-ball $B \subset \mathbb{R}^m$ such that $p^{-1}(a) \subset B \subset p^{-1}(O_\varepsilon a)$.

2. Idea of the proof

The proof of sufficiency in Theorem 1.2 may appear to be a trivial extension of that of Theorem 1.1. For the special case it was even claimed in [35, Theorem 3] and 'done' in [10, Corollary 4]. But in fact, the control requires additional general position arguments, for which the restriction $m \geqslant 3(n+1)/2$ is again necessary. This is the reason why the proofs of improvements of Theorem 1.1 beyond the metastable case $m \geqslant 3(n+1)/2$ [24,32] do not yield their controlled versions (which is false even for $m = 2n = 2$, cf. Examples 1.6 and 1.7).

To make a brief introduction into the rather technical Sections 3 and 4, let us sketch a proof of the sufficiency in Theorem 1.2 (in Theorem 1.4 it is proved analogously). Take a small triangulation T of K and approximate f by a general position map φ, linear on the simplices of T. Then the proof naturally splits into two steps. The first one (see Section 3) is a controlled version of the generalized Whitney trick [35, Proposition 6] (a controlled version of a similar theorem is [34, Theorem 3]). We modify φ by a homotopy for each $\sigma \times \tau \in \widetilde{T}$ and obtain $f(\sigma) \cap f(\tau) = \emptyset$, for each $\sigma \times \tau \in \widetilde{T}$, and preserve $f|_\sigma$ as an embedding for each $\sigma \in T$. By hypothesis (i.e., that $\tilde{f} \underset{\mathrm{eq}}{\simeq} \Phi$ on $\widetilde{T}^f$ and hence $\tilde{\varphi} \underset{\mathrm{eq}}{\simeq} \Phi$ on $\widetilde{T}^f$), and since T is small, each homotopy is small.

The second step (see Section 4) is a controlled version of the generalized van Kampen construction, cf. [35, Proposition 7]. Our proofs is a controlled version of [31, §3]. In fact, it is a new and short proof of [35, Theorem 3] (a stronger result was proved in [3]). We modify φ so as to obtain $\varphi(\sigma) \cap \varphi(\tau) = \varphi(\sigma \cap \tau)$, for each $\sigma \times \tau \in T \times T$, and $\varphi|_\sigma$ an embedding for each $\sigma \in T$. Hence φ becomes an embedding. We modify φ by a small homotopy for each $\sigma \times \tau \in T^2 \setminus \widetilde{T}$.

Although each of the above modifications is small, their number (depending on the number of simplices of T) can be arbitrary large. So without special care the resulting modification can be large (cf. [10, Proof of Corollary 4]). Example 1.6 illustrates this point. But using general position (which requires $m \geqslant 3(n+1)/2$) we can take the supports of the above modifications to be disjoint for the same $(\dim\sigma, \dim\tau)$. Hence we can make all modifications to be disjoint for the same $(\dim\sigma, \dim\tau)$ simultaneously. So the number of non-simultaneous modifications depends only on $n = \dim K$. Therefore the resulting modification of φ is small.

3. Elimination of distant double points

Given $\varepsilon > 0$, take a triangulation T of K such that

$$\operatorname{mesh} f(T) < \frac{\varepsilon}{9^{n^2-n} \cdot 7^{n^3+n^2+n}}$$

and f is linear on the simplices of T. Our Theorem 1.2 then follows by Proposition 3.1 (for $p = q = n$) and Proposition 4.1 (for $p = q = r = n$) below.

We fix some conventions and notations. Hereafter, the phrase "Since φ is a general position map, we may assume without loss of generality that ..." will be abbreviated to "By general position...". Since the approximability by embeddings is a topological property, we can choose a metric on $\mathbb{R}^m$ for which ε-neighborhoods of points are PL balls. For PL topology we follow the notation of [23]. We use the lexicographical order on pairs and triples of integers.

Proposition 3.1. *Suppose that K is an n-dimensional polyhedron with a triangulation T, $n \leqslant 2m/3 - 1$, $f: K \to \mathbb{R}^m$ a map, linear on simplices of T and $\Phi: \widetilde{T} \to S^{m-1}$ an equivariant map such that $\tilde{f} \underset{\text{eq}}{\simeq} \Phi|_{\widetilde{T}f}$. For each $0 \leqslant q \leqslant p \leqslant n$ let*

$$J_{pq} = \bigl\{\sigma \times \tau \in \widetilde{T} \mid \textit{either } (\dim\sigma, \dim\tau) \leqslant (p,q) \textit{ or } \operatorname{dist}(f(\sigma), f(\tau)) > 2 \cdot 9^{p^2+q} \operatorname{mesh} f(T)\bigr\}.$$

Then there exists a general position PL map $\varphi: K \to \mathbb{R}^m$ such that:
(3.1.1) *$\varphi|_\sigma$ is an embedding, for each $\sigma \in T$;*
(3.1.2) *$\varphi(\sigma) \cap \varphi(\tau) = \emptyset$, for each $\sigma \times \tau \in J_{pq}$;*
(3.1.3) *$\tilde{\varphi}|_{J_{pq}} \simeq \Phi|_{J_{pq}}$; and*
(3.1.4) *$\operatorname{dist}(\varphi, f) < 9^{p^2+q} \operatorname{mesh} f(T)$.*

Proof. By induction on (p,q). To begin the inductive argument, i.e., for $(p,q) = (0,0)$ take a map $\varphi: K \to \mathbb{R}^m$ to be linear on each simplex of T, in general position and sufficiently close to f. Inductive step for $q = 0$ follows by the inductive hypothesis. So assume that $q \geqslant 1$ and that φ satisfies (3.1.1)–(3.1.4). Let $\delta = \operatorname{mesh} f(T)$. If $p = q$, take an ordering '$<$' on p-simplices of T. Let

$$J^+ = \bigl\{\sigma \times \tau \in \widetilde{T} \mid (\dim\sigma, \dim\tau) = (p,q) \text{ and } \operatorname{dist}(f(\sigma), f(\tau)) \leqslant 2 \cdot 9^{p^2+q-1}\delta \text{ (and if } p = q,\ \sigma > \tau)\bigr\}.$$

Suppose that $p + q \geqslant m - 1$ (otherwise (3.1.2) and (3.1.3) hold by general position and (3.1.1) and (3.1.4) by the inductive hypothesis).

First Ball Lemma 3.2 (cf. [35, Lemme 2, p. 41]). *There exists a collection $\{B_{\sigma\tau}\}_{\sigma\times\tau\in J^+}$ of PL m-balls in $\mathbb{R}^m$ such that for each $\sigma \times \tau \in J^+$, the following assertions hold:*
(3.2.1) *$B_{\sigma\tau} \cap \varphi(\sigma) \subset \varphi(\mathring{\sigma})$ (respectively $B_{\sigma\tau} \cap \varphi(\tau) \subset \varphi(\mathring{\tau})$) is a PL p-ball (respectively q-ball), properly embedded in $B_{\sigma\tau}$;*
(3.2.2) *$\varphi(\sigma) \cap \varphi(\tau) \subset \mathring{B}_{\sigma\tau}$;*
(3.2.3) *$B_{\sigma\tau} \cap \varphi(P_\sigma) = \emptyset$, where $P_\sigma = \{\alpha \in T \mid \sigma \times \alpha \in J_{p,q-1}\}$;*
(3.2.4) *$\operatorname{diam} B_{\sigma\tau} < 8 \cdot 9^{p^2+q-1}\delta$; and*
(3.2.5) *$B_{\sigma\tau} \cap B_{\sigma'\tau'} = \emptyset$ provided that $\sigma \times \tau \neq \sigma' \times \tau'$.*

Proof of Proposition 3.1 modulo the First Ball Lemma. We have that

$$J_{pq} \setminus J_{p,q-1} \subset J^+ \cup \{\tau \times \sigma : \sigma \times \tau \in J^+\}.$$

Take a collection $\{B_{\sigma\tau}\}_{\sigma\times\tau\in J^+}$ given by the First Ball Lemma. Then we follow [35, Preuve de l'affirmation l'aide du Lemme 2]. For each $\sigma \times \tau \in J^+$ let $u: D^p \to B_{\sigma\tau} \cap$

$\varphi(\sigma)$ and $v: D^q \to B_{\sigma\tau} \cap \varphi(\tau)$ be a PL-homeomorphism. Define the map $a: \partial(D^p \times D^q) \to S^{m-1}$ by $a(x,y) = (u(x) - v(y))/(\|u(x) - v(y)\|)$. Since $m - p \geqslant 3$, $u(D^p)$ is unknotted in $B_{\sigma\tau}$, it follows that $B_{\sigma\tau} \setminus u(D^p) \simeq S^{m-p-1}$.

Define the coefficient of the intersection $I(v,u) \in \pi_{q-1}(S^{m-p-1})$ to be the homotopy class of the map $v|_{\partial D^q} : \partial D^q \to B_{\sigma\tau} \setminus u(D^p)$. Let

$$g: \partial(D^p \times D^q) \to \partial\big[(\sigma \cap \varphi^{-1}u(D^p)) \times (\tau \cap \varphi^{-1}v(D^q))\big] = X$$

be a PL-homeomorphism such that $a = \widetilde{\varphi} \circ g$. By (3.1.3), $\widetilde{\varphi}|_X$ is homotopic to $\Phi|_X$. Since Φ is defined over $\widetilde{T}$, $\Phi|_X$ is null-homotopic, hence $\widetilde{\varphi}|_X$ is null-homotopic and so a is null-homotopic. By [35, Proposition 1], the homotopy class of a is $\Sigma^q I(v;u)$, where Σ is the suspension. Since $p - 1 < 2(m - q - 1) - 1$, it follows by the Freudenthal Suspension Theorem that $I(v,u) = 0$. Therefore by [35, Proposition 3], there is a family of isotopies $\{h_{\sigma\tau} : (\sigma \cap \varphi^{-1}B_{\sigma\tau}) \times I \to B_{\sigma\tau} \mid \sigma \times \tau \in J^+\}$ such that:

$$\varphi(\tau) \cap h_{\sigma\tau,1}(\sigma \cap \varphi^{-1}B_{\sigma\tau}) = \emptyset \quad \text{and} \quad F_{\sigma\tau}|_{\sigma\times\tau} \simeq \Phi|_{\sigma\times\tau} \text{ rel } \partial(\sigma \times \tau), \tag{3.2.6}$$

where

$$F_{\sigma\tau}(x,y) = \frac{h_{\sigma\tau,1}(x) - \varphi(y)}{\|h_{\sigma\tau,1}(x) - \varphi(y)\|}.$$

Since $3(n+1)/2 \geqslant n + 3$, it follows by [23, 7.3] that the isotopies $h_{\sigma\tau}$ are ambient, i.e., there is a family of isotopies $\{h'_{\sigma\tau} : B_{\sigma\tau} \times I \to B_{\sigma\tau} \mid \sigma \times \tau \in J^+\}$ such that $h_{\sigma\tau,t} = h'_{\sigma\tau,t} \circ \varphi$. Define $\varphi^+ : K \to \mathbb{R}^m$ as follows

$$\varphi^+(x) = \begin{cases} h'_{\sigma\tau,1}(\varphi(x)) & \text{if } \varphi(x) \in B_{\sigma\tau} \text{ and } x \in \eta, \\ & \text{for some } \sigma \times \tau \in J^+ \text{ and } \eta \supset \sigma, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

By (3.2.5), φ^+ is well-defined. Since $B_{\sigma\tau} \cap \varphi(\sigma) \subset \varphi(\mathring{\sigma})$, the map φ^+ is continuous. Let us verify (3.1.1)–(3.1.4) for φ^+ and (p,q). Since $h'_{\sigma\tau}$ are isotopies, (3.1.1) follows. From (3.2.6) we get (3.1.2) and (3.1.3) for $(\dim\sigma, \dim\tau) = (p,q)$. By (3.2.3) we get (3.1.2) and (3.1.3) for $(\dim\sigma, \dim\tau) < (p,q)$. Since $\operatorname{dist}(\varphi^+, f) < \operatorname{dist}(\varphi, f) + \max_{\sigma\times\tau\in J^+} \operatorname{diam} B_{\sigma\tau} < 9^{p^2+q}\delta$ by (3.2.4), (3.1.4) follows. Since $\operatorname{dist}(\varphi^+(\sigma), \varphi^+(\tau)) > \operatorname{dist}(f(\sigma), f(\tau)) - 2\operatorname{dist}(\varphi^+, f) > 0$, (3.1.2) and (3.1.3) for $\operatorname{dist}(f(\sigma), f(\tau)) > 2 \cdot 9^{p^2+q}\delta$ follows. Therefore φ^+ is the required map. □

Proof of the First Ball Lemma. By (3.1.2), $\varphi(\sigma) \cap \varphi(\partial\tau) = \varphi(\partial\sigma) \cap \varphi(\tau) = \emptyset$. By general position, $\dim(\varphi(\sigma) \cap \varphi(\tau)) \leqslant p + q - m$. Let $C_{\sigma\tau}$ (respectively $C_{\tau\sigma}$) be the trail of $\varphi(\sigma) \cap \varphi(\tau)$ under a sequence of collapses $\varphi(\sigma) \searrow$ (a point in $\varphi(\mathring{\sigma})$) (respectively $\varphi(\tau) \searrow$ (a point in $\varphi(\tau)$)). Then $C_{\sigma\tau}$, $C_{\tau\sigma}$ are collapsible, $C_{\sigma\tau} \subset \varphi(\mathring{\sigma})$ and $C_{\tau\sigma} \subset \varphi(\mathring{\tau})$, $\varphi(\sigma) \cap \varphi(\tau) \subset C_{\sigma\tau}$, $C_{\tau\sigma}$ and $\dim C_{\sigma\tau}$, $\dim C_{\tau\sigma} \leqslant p + q - m + 1$.

By general position on $C_{\sigma\tau}$, it follows that $C_{\sigma\tau} \cap C_{\sigma\eta} = \emptyset$ when $\tau \neq \eta$ (since $2(p+q+1-m)+1 \leqslant p$). Because of that and since $C_{\sigma\tau} \cap C_{\sigma'\tau'} \subset \varphi(\sigma) \cap \varphi(\sigma') \subset C_{\sigma\sigma'}$, it follows that $C_{\sigma\tau} \cap C_{\sigma'\tau'} = \emptyset$, when $\sigma \times \tau \neq \sigma' \times \tau'$. By (3.1.2), $C_{\sigma\tau} \cap \varphi(P_{\sigma'}) = \emptyset$. By general position, $\dim(\varphi(P_\sigma) \cap \varphi(\tau)) \leqslant n + q - m$, hence again by general position on φ, $C_{\tau\sigma} \cap \varphi(P_\sigma) = \emptyset$ (since $n - q - m + p + q + 1 - m < q$).

Take any points $x_\sigma \in \varphi(\sigma)$. Since $\text{dist}(f(\sigma), f(\tau)) < 2 \cdot 9^{p^2+q-1}\delta$, we have by (3.1.4) that

$$\varphi(\sigma) \cup \varphi(\tau) \subset O_{4 \cdot 9^{p^2+q-1}\delta} x_\sigma.$$

Since $O_{4 \cdot 9^{p^2+q-1}\delta} x_\sigma$ is a PL ball, it collapses to some point in its interior. Let $G_{\sigma\tau}$ be the trail of $C_{\sigma\tau} \cup C_{\tau\sigma}$ under this sequence of collapses. Then $G_{\sigma\tau}$ is collapsible,

$$G_{\sigma\tau} \subset \text{Int}(O_{4 \cdot 9^{p^2+q-1}\delta} x_\sigma),$$
$$G_{\sigma\tau} \supset C_{\sigma\tau} \cup C_{\tau\sigma},$$
$$\dim G_{\sigma\tau} \leqslant p + q - m - 2.$$

By general position on $\{G_{\sigma\tau}\}$,

$$G_{\sigma\tau} \cap \varphi(\sigma) = C_{\sigma\tau}, \qquad G_{\sigma\tau} \cap \varphi(\tau) = C_{\tau\sigma}$$

(since $p + p + q - m + 2 < m$ and $q + p + q - m + 2 < m$), $G_{\sigma\tau} \cap G_{\sigma'\tau'} = \emptyset$ when $\sigma \times \tau \neq \sigma' \times \tau'$ (since $2(p + q - m + 2) + 1 \leqslant m$) and $G_{\sigma\tau} \cap \varphi(P_\sigma) = \emptyset$ (since $p + q + p - m + 2 < m$). Therefore, in some sufficiently small triangulations of $\mathbb{R}^m$, the regular neighborhoods of $G_{\sigma\tau}$ are the required balls $B_{\sigma\tau}$. □

4. Elimination of close double points

The upper index of a polyhedron shows its dimension (but if $a = b$, then X^a and X^b are distinct).

Proposition 4.1. *Suppose that K is an n-dimensional polyhedron with a triangulation T, $m \geqslant 3(n+1)/2$, and $f : K \to \mathbb{R}^m$ is a general position map such that $f|_\sigma$ is an embedding for each $\sigma \in T$, and $f\sigma \cap f\tau = \emptyset$, for every $\sigma \times \tau \in \widetilde{T}$. Then for every triple of integers p, q, r such that $-1 \leqslant r < q \leqslant p \leqslant n$, there exists a general position PL map $\varphi : K \to \mathbb{R}^m$ such that:*

(4.1.1) $\varphi(\alpha) \cap \varphi(\beta) = \emptyset$, *for each* $\alpha \times \beta \in \widetilde{T}$;
(4.1.2) $\varphi|_\alpha$ *is an embedding, for each* $\alpha \in T$;
(4.1.3) $\widetilde{\varphi}|_{\widetilde{T}}$ *is equivariantly homotopic to* $\widetilde{f}|_{\widetilde{T}}$;
(4.1.4) $\varphi(\sigma) \cap \varphi(\tau) = \varphi(\sigma \cap \tau)$, *for* $(\dim \sigma, \dim \tau, \dim(\sigma \cap \tau)) < (p, q, r)$; *and*
(4.1.5) $\text{dist}(f, \varphi) < 7^{p^3+q^2+r}$ mesh $f(T)$.

Proof (cf. [31, Proof of Proposition 2.1]). By induction on (p, q, r). To begin the inductive argument, i.e., when $(p, q, r) = (0, 0, -1)$ we take $\varphi = f$. The inductive step for $q = 0$ and $r = -1$ follows by the inductive hypothesis. So assume that $q > r \geqslant 0$ and that φ satisfies (4.1.1)–(4.1.5). If $p = q$, take an ordering '<' on the p-simplices of T. Let

$$J^+ = \bigl\{\sigma \times \tau \in T^2 \setminus \widetilde{T} \mid \bigl(\dim \sigma, \dim \tau, \dim(\sigma \cap \tau)\bigr) = (p, q, r), \tau^2 \not\subset \sigma^p, \text{ and } \sigma > \tau \text{ when } p = q\bigr\}.$$

Second Ball Lemma 4.2. *There exist collections $\{D^r_{\sigma\tau}, D^p_{\sigma\tau}, D^q_{\sigma\tau}, D^m_{\sigma\tau} \subset \mathbb{R}^m \mid \sigma\times\tau \in J^+\}$ of PL-balls such that for every $\sigma\times\tau \in J^+$:*

(4.2.1) $D^p_{\sigma\tau} \subset D^r_{\sigma\tau} \cup \sigma(\mathring{\sigma})$ *and* $D^q_{\sigma\tau} \subset D^r_{\sigma\tau} \cup \varphi(\mathring{\tau})$;

(4.2.2) $D^p_{\sigma\tau} = D^m_{\sigma\tau} \cap \varphi(\sigma)$ *and* $D^q_{\sigma\tau} = D^m_{\sigma\tau} \cap \varphi(\tau)$ *are properly embedded in* $D^m_{\sigma\tau}$;

(4.2.3) $D^r_{\sigma\tau} = \partial D^p_{\sigma\tau} \cap \partial D^q_{\sigma\tau}$;

(4.2.4) $D^r_{\sigma\tau}$ *is unknotted in* $\partial D^p_{\sigma\tau}$ *and in* $\partial D^q_{\sigma\tau}$;

(4.2.5) $\Sigma_{\sigma\tau} = \mathrm{Cl}((\varphi(\sigma) \cap \varphi(\tau)) - D^r_{\sigma\tau}) \subset \mathring{D}^m_{\sigma\tau} \cup D^r_{\sigma\tau}$;

(4.2.6) $D^m_{\sigma\tau} \cap X_\sigma \subset D^r_{\sigma\tau}$, *where* $X_\sigma = \bigcup \varphi\{\alpha \in T \mid \alpha \cap \sigma = \emptyset$ *or* $\dim\alpha \leqslant q\}$;

(4.2.7) $\operatorname{diam} D^m_{\sigma\tau} < 2 \operatorname{mesh} \varphi(T)$; *and*

(4.2.8) $\mathring{D}^m_{\sigma\tau} \cap \mathring{D}^m_{\sigma'\tau'} = \emptyset$, *whenever* $\sigma\times\tau \neq \sigma'\times\tau'$.

Proof of Proposition 4.1 modulo the Second Ball Lemma 4.2. Take collection of PL-balls $D^r_{\sigma\tau}, D^p_{\sigma\tau}, D^q_{\sigma\tau}, D^m_{\sigma\tau}$ given by the Second Ball lemma. Recall [17, Theorem 9] and discussion before its statement: If $m-3 \geqslant p, q$, $S^p, S^q \subset S^m$ and $S^p \cap S^q = D^r$, where D^r is unknotted in S^p and in S^q, then $S^p \cup S^q$ is unknotted in S^m. Hence we may assume that the embedding $\partial D^p_{\sigma\tau} \cup_{D^r_{\sigma\tau}} \partial D^q_{\sigma\tau} \subset \partial D^m_{\sigma\tau}$ is the standard one.

By the relative Unknotting Balls Theorem (which follows from [23, Theorems 7.1 and 3.22i] and [37, 1.2] we may assume that the embedding $(D^q_{\sigma\tau}, \partial D^q_{\sigma\tau}) \subset (D^m_{\sigma\tau}, \partial D^m_{\sigma\tau})$ is the standard one. Hence the embedding $\partial D^p_{\sigma\tau} \in \partial D^m_{\sigma\tau}$ can be extended to a new embedding of $D^p_{\sigma\tau}$ into $(\mathring{D}^m_{\sigma\tau} \setminus D^q_{\sigma\tau}) \cup \partial D^p_{\sigma\tau}$.

By the relative Unknotting Balls Theorem this new embedding is ambiently isotopic to $D^p_{\sigma\tau} \subset D^m_{\sigma\tau}$ rel $\partial D^m_{\sigma\tau}$. So there is a collection of isotopies $\{h_{\sigma\tau,t} : D^m_{\sigma\tau} \to D^m_{\sigma\tau}$ rel $\partial D^m_{\sigma\tau} \mid \sigma\times\tau \in J^+\}$ such that $D^q_{\sigma\tau} \cap h_1 D^p_{\sigma\tau} = D^r_{\sigma\tau}$. Define a map $\varphi^+ : K \to \mathbb{R}^m$ as

$$\varphi^+(x) = \begin{cases} h_{\sigma\tau,1}(\varphi(x)) & \text{if } \varphi(x) \in D^m_{\sigma\tau} \text{ and } x \in \gamma \text{ for some} \\ & \gamma \in T,\ \sigma\times\tau \in J^+,\ \gamma \supset \sigma, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

By (4.2.8), φ^+ is well-defined. Since $D^p_{\sigma\tau} \subset \partial D^m_{\sigma\tau} \cup \varphi(\mathring{\sigma})$, φ^+ is continuous. Evidently, φ^+ satisfies (4.1.1)–(4.1.3). By (4.1.2), σ and τ are not contained in the boundary of the same simplex of T. Therefore $D^q_{\sigma\tau} \cap h_{\sigma\tau 1} D^p_{\sigma\tau} = D^r_{\sigma\tau}$ and by (4.2.5) and (4.2.6), φ^+ satisfies also (4.1.4). From (4.1.5) it follows that mesh $\varphi(T) < 3\times 7^{p^3+q^2+r}$mesh $f(T)$. By (4.2.7), φ^+ is 2 mesh $\varphi(T)$-homotopic to φ and hence $7^{p^3+q^2+r+1}$ mesh $f(T)$-homotopic to f. Therefore φ^+ is the required map. The inductive step is thus completed. □

Proof of the Second Ball Lemma. Let us make two conventions concerning triangulations. First, for polyhedra $M \supset Z \supset Y$ the notation $R_M(Z, Y)$ shall mean 'a regular neighborhood of Z rel Y in M in some small triangulation of $\mathbb{R}^m$', when first appears, and 'the regular neighborhood of Z rel Y in M', when second or more appears. Second, regular neighborhoods, defining D^p, D^q and D^m below are in the restrictions of the same triangulation of $\mathbb{R}^m$. Also, $R_M(Z) = R_M(Z, \emptyset)$.

For every $\sigma\times\tau \in J^+$, make the following constructions.

Construction of $D^r_{\sigma\tau}$. If $r \leqslant q-2$, then let $D^r_{\sigma\tau} = \varphi(\sigma\cap\tau)$. If $r = q-1$, then $\dim\varphi(\sigma\cap\tau)\cap\Sigma_{\sigma\tau} \leqslant p+q-m-1$. Since $2(p+q-m-1) < r = q-1$, it follows that (as in the proof of the First Ball Lemma 3.2) for each $\gamma\in T$, such that $\dim\gamma = r$, there is a collection of disjoint PL r-balls $\{D^r_{\sigma\tau}\subset\varphi(\mathring{\gamma}) \mid \sigma\times\tau\in J^+ \text{ and } \sigma\cap\tau=\gamma\}$ such that $\mathring{D}^r_{\sigma\tau}\supset\varphi(\sigma\cap\tau)\cap\Sigma_{\sigma\tau}$. Since $D^r_{\sigma\tau}\searrow$ (some point in $\mathring{D}^r_{\sigma\tau}$), $D^r_{\sigma\tau}\in \operatorname{Int}\varphi(\sigma\cap\tau)$ and by [23, Theorem 4.11], we have $(\partial\varphi(\sigma), D^r_{\sigma\tau})\cong(\partial\sigma,\sigma\cap\tau)$. Hence $D^r_{\sigma\tau}$ is unknotted in $\partial\varphi(\sigma)$. Analogously, $D^r_{\sigma\tau}$ is unknotted in $\partial\varphi(\tau)$. Let $D^r_{\tau\sigma} = D^r_{\sigma\tau}$.

Construction of $S_{\sigma\tau}$ and $\delta_{\sigma\tau}$. Let $S_{\sigma\tau}$ be the link of some r-simplex from $\mathring{D}^r_{\sigma\tau}$ in some small triangulation of $\mathbb{R}^m$. Then $S_{\sigma\tau}$ is a PL $(m-r-1)$-sphere and $\delta_{\sigma\tau} = R_{\mathbb{R}^m}(D^r_{\sigma\tau},\partial D^r_{\sigma\tau})\cong S_{\sigma\tau} * D^r_{\sigma\tau}$ is a PL m-ball. By (4.1.2), $\delta_{\sigma\tau}\cap f(\alpha) = R_{f(\alpha)}(D^r_{\sigma\tau},\partial D^r_{\sigma\tau})$ goes to $(S_{\sigma\tau}\cap f(\alpha)) * D^r_{\sigma\tau}$ under this homeomorphism for each $\alpha\in T$ (for $\alpha\not\supset\sigma\cap\tau$ each of these three sets is empty). Also $S_{\sigma\tau}\cap f(\alpha)$ is a PL $((\dim\alpha)-r-1)$-ball for each $\alpha\in T$, $\alpha\supset\sigma\cap\tau$. If $r\leqslant q-2$ and $D^r_{\sigma\tau}=\varphi(\sigma\cap\tau)$ then we take these $S_{\sigma\tau}$ so that $S_{\sigma\tau}=S_{\sigma'\tau'}$, whenever $\sigma\cap\tau=\sigma'\cap\tau'$.

Construction of $\beta_{\sigma\tau}$. If $r\leqslant q-2$, then $\dim(S_{\sigma\tau}\cap\varphi(\sigma))$, $\dim(S_{\sigma\tau}\cap\varphi(\tau)) = q-r-1\geqslant 1$. If $r=q-1$ then $S_{\sigma\tau}$ are disjoint for distinct $\sigma\times\tau$. Because of this and since $\dim(S_{\sigma\tau}\cap X_\sigma)\leqslant n-r-1$, there are points $a_{\sigma\tau}\in(S_{\sigma\tau}\cap\varphi(\mathring{\sigma}))-X_\sigma$ and $a_{\sigma\tau}\in(S_{\sigma\tau}\cap\varphi(\mathring{\tau}))-X_\sigma$, distinct from each other for distinct $\sigma\times\tau$. Since $m-r-1\geqslant 2$ and $(n-r-1)+1<m-r-1$, it follows that there are arcs $l_{\sigma\tau}\subset S_{\sigma\tau}$, joining $a_{\sigma\tau}$ to $a_{\sigma\tau}$ such that $l_{\sigma\tau}\cap X_\sigma=\emptyset$, $l_{\sigma\tau}\cap\varphi(\sigma)=a_{\sigma\tau}$, $l_{\sigma\tau}\cap\varphi(\tau)=a_{\sigma\tau}$ and $l_{\sigma\tau}$ are disjoint for distinct $\sigma\times\tau$. Let $\beta_{\sigma\tau}=R_{S_{\sigma\tau}}(l_{\sigma\tau}) * D^r_{\sigma\tau}$. Then

$$\beta_{\sigma\tau}\cap\varphi(\sigma) = \bigl(R_{S_{\sigma\tau}}(l_{\sigma\tau})\cap\varphi(\sigma)\bigr) * D^r_{\sigma\tau} = R_{S_{\sigma\tau}\cap\varphi(\sigma)}(a_{\sigma\tau}) * D^r_{\sigma\tau}$$

is a PL p-ball. Analogously, $\beta_{\sigma\tau}\cap\varphi(\tau)$ is a PL q-ball. Also, $\beta_{\sigma\tau}\cap\varphi(\sigma\cap\tau)=D^r_{\sigma\tau}$. If $\alpha\not\supset\sigma\cap\tau$ then $\varphi(\alpha)\cap R_{\mathbb{R}^m}(D^r_{\sigma\tau},\partial D^r_{\sigma\tau})=\emptyset$. If $\alpha\supset\sigma\cap\tau$ then

$$\varphi(\alpha)\cap R_{\mathbb{R}^m}(D^r_{\sigma\tau},\partial D^r_{\sigma\tau}) = \bigl(S_{\sigma\tau}\cap\varphi(\alpha)\bigr) * D^r_{\sigma\tau}.$$

Therefore $\beta_{\sigma\tau}\cap X_\sigma = D^r_{\sigma\tau}$. If $\sigma\times\tau,\sigma'\times\tau'\in J^+$ then $\dim(\sigma\cap\tau)=\dim(\sigma'\cap\tau')=r$. Hence $\operatorname{Int}R_{\mathbb{R}^m}(D^r_{\sigma\tau},\partial D^r_{\sigma\tau})$ either coincide or do not intersect for distinct $\sigma\times\tau$, $\sigma'\times\tau'\in J^+$. Therefore $\mathring{\beta}_{\sigma\tau}\cap\mathring{\beta}_{\sigma'\tau'}=\emptyset$, for distinct $\sigma\times\tau$, $\sigma'\times\tau'\in J^+$.

Collapsing Lemma 4.3. *If A and F are regular neighborhoods of a polyhedron X in a PL-manifold M* rel Y *and $A\subset F$ then $F\searrow A$* rel Y.

Proof. Follows from [5, Theorem 3.1 and Addendum 3.4].

Construction of $D^p_{\sigma\tau}$ and $D^q_{\sigma\tau}$. By the inductive hypothesis, $\varphi(\sigma)\cap\varphi(\partial\tau)=\varphi(\partial\sigma)\cap\varphi(\tau)=D^r_{\sigma\tau}$. Hence $\Sigma_{\sigma\tau}\subset(\varphi(\mathring{\sigma})\cap\varphi(\mathring{\tau}))\cup D^r_{\sigma\tau}$. Both $\varphi(\sigma)$ and $\varphi(\sigma)\cap\delta_{\sigma\tau} = (S_{\sigma\tau}\cap\varphi(\sigma)) * D^r_{\sigma\tau}$ are regular neighborhoods of $D^r_{\sigma\tau}$ rel $\partial D^r_{\sigma\tau}$ in $\varphi(\sigma)$. Then by Collapsing Lemma 4.3, $\varphi(\sigma)\searrow(S_{\sigma\tau}\cap\varphi(\sigma)) * D^r_{\sigma\tau}$ rel $D^r_{\sigma\tau}$. Both $S_{\sigma\tau}\cap\varphi(\sigma)$ and

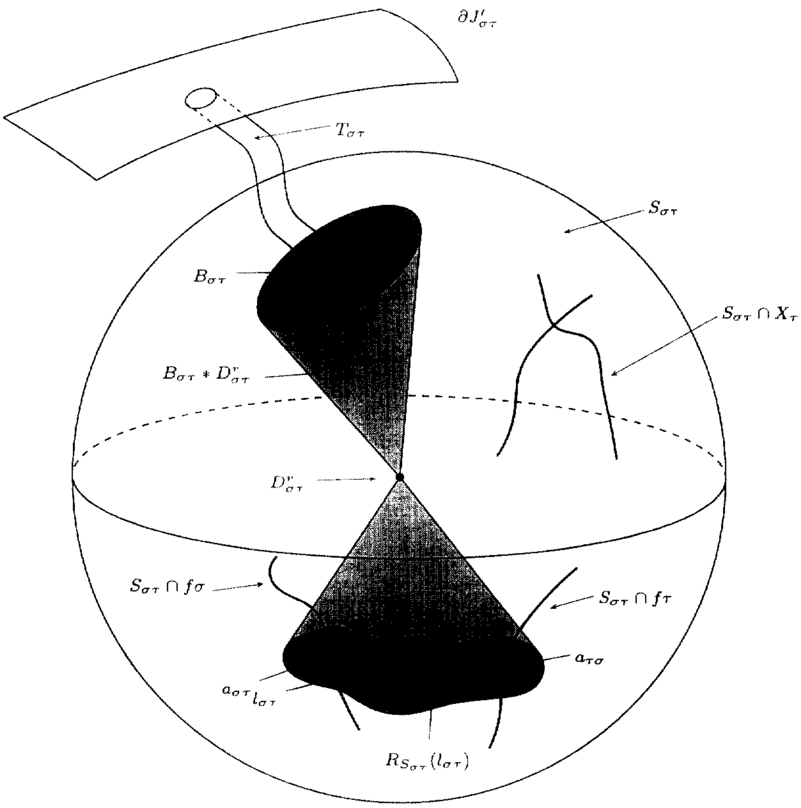

Fig. 2.

$R_{S_{\sigma\tau}\cap\varphi(\sigma)}(a_{\sigma\tau})$ are regular neighborhoods of $a_{\sigma\tau}$ in $S_{\sigma\tau} \cap \varphi(\sigma)$. Again by Collapsing Lemma 4.3, $S_{\sigma\tau} \cap \varphi(\sigma) \searrow R_{S_{\sigma\tau}\cap\varphi(\sigma)}(a_{\sigma\tau})$. Hence

$$\big(S_{\sigma\tau} \cap \varphi(\sigma)\big) * D^r_{\sigma\tau} \searrow R_{S\cap\varphi(\sigma)}(a_{\sigma\tau}) * D^r_{\sigma\tau} = \beta_{\sigma\tau} \cap \varphi(\sigma) \text{ rel } D^r_{\sigma\tau}.$$

Let $C_{\sigma\tau}$ be a trail of $\Sigma_{\sigma\tau}$ under the above sequence of collapses.

$$\varphi(\sigma) \searrow \big(S_{\sigma\tau} \cap \varphi(\sigma)\big) * D^r_{\sigma\tau} \searrow \beta_{\sigma\tau} \cap \varphi(\sigma) \text{ rel } D^r_{\sigma\tau}$$

that is in general position. Let $D^p_{\sigma\tau} = R_{\varphi(\sigma)}((\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}, D^r_{\sigma\tau})$. Then (4.2.1) and (4.2.4) are true for $D^p_{\sigma\tau}$ and we have that

(a) $C_{\sigma\tau} \subset \varphi(\sigma)$;
(b) $\Sigma_{\sigma\tau} \subset (\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$;
(c) $D^p_{\sigma\tau}$ is a PL p-ball;
(d) $C_{\sigma\tau} \cap X_\sigma = \emptyset$;
(e) $D^p_{\sigma\tau} \cap X_\sigma \subset D^r_{\sigma\tau}$;
(f) $C_{\sigma\tau} \cap \varphi(\tau) = \Sigma_{\sigma\tau}$; and
(g) $C_{\sigma\tau} \cap C_{\sigma'\tau'} \subset D^r_{\sigma\tau} \cap D^r_{\sigma'\tau'}$.

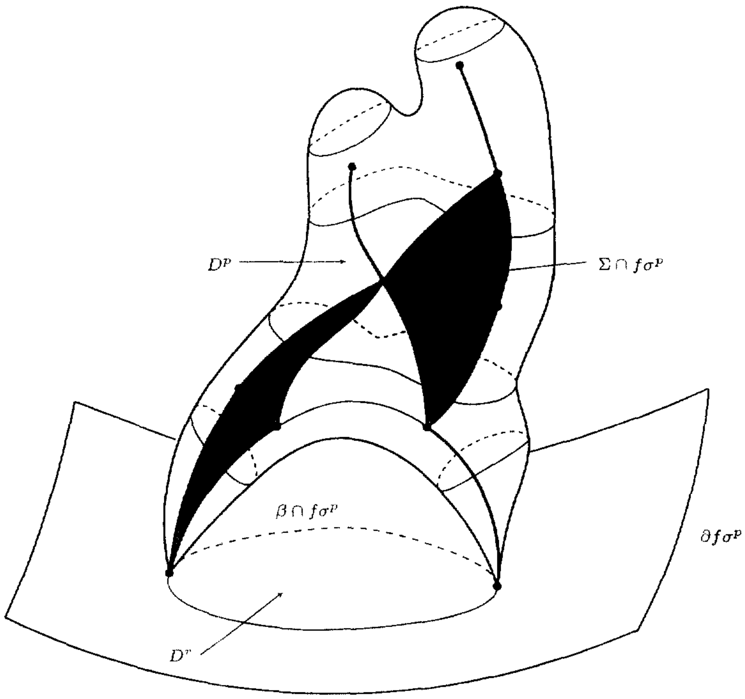

Fig. 3.

In fact, (a), (b) are obvious. Since $\Sigma_{\sigma\tau} \subset D^r_{\sigma\tau} \cup \varphi(\mathring{\sigma})$ it follows that $C_{\sigma\tau} \subset D^r_{\sigma\tau} \cup \varphi(\mathring{\sigma})$, hence (4.2.1) is true. Since $\varphi(\sigma)$ is a PL-manifold and $\varphi(\sigma) \searrow (\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$ rel $D^r_{\sigma\tau}$ then $\varphi(\sigma)$ is a regular neighborhood of $(\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$ in $\varphi(\sigma)$ rel $D^r_{\sigma\tau}$ [5, Theorem 9.1]. By [5, Theorem 3.1] there is an isotopy $H_t : \varphi(\sigma) \to \varphi(\sigma)$ rel $(\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$ between $H_0 = \mathrm{id}$ and a homeomorphism G_1 of $\varphi(\sigma)$ onto $D^r_{\sigma\tau}$ rel $(\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$. This implies (c).

Moreover, $H_{1|\partial\varphi(\sigma)}$ is a homeomorphism of $\partial\varphi(\sigma)$ onto $\partial D^p_{\sigma\tau}$ rel $D^r_{\sigma\tau}$. Since $D^r_{\sigma\tau}$ is unknotted in $\partial\varphi(\sigma)$, (4.2.4) is true for $D^p_{\sigma\tau}$. By general position, $\dim \Sigma_{\sigma\tau} \leqslant 2n - m$. Then $\dim C_{\sigma\tau} \leqslant 2n - m + 1$. By general position and since $n + (2n - m) < m$, $\Sigma_{\sigma\tau} \cap X_\sigma = \emptyset$. Again, general position and $n + (2n - m + 1) < m$ imply (d).

Since $l_{\sigma\tau} \cap X_\sigma = \emptyset$, it follows that $\beta_{\sigma\tau} \cap \varphi(\sigma) \cap X_\sigma = D^r_{\sigma\tau}$. This and (d) imply (e). By definition of relative collapse, $C_{\sigma\tau} \cap D^r_{\sigma\tau} = \Sigma_{\sigma\tau} \cap D^r_{\sigma\tau}$. Hence by general position, $n + (2n - m + 2) < m$, we have

$$C_{\sigma\tau} \cap \varphi(\tau) = (C_{\sigma\tau} \cap \varphi(\mathring{\tau})) \cup (C_{\sigma\tau} \cap D^r_{\sigma\tau}) = \Sigma_{\sigma\tau} \cup (C_{\sigma\tau} \cap D^r_{\sigma\tau}) = \Sigma_{\sigma\tau},$$

i.e., (f). By general position, (g) is true.

Analogously we can construct polyhedra $C_{\tau\sigma}$ and $D^q_{\sigma\tau}$ such that (4.2.1), (4.2.4) and (a)–(g) are true for $C_{\sigma\tau} \to C_{\tau\sigma}$ and $p \to q$.

Construction of $D^m_{\sigma\tau}$. By our assumption, $J'_{\sigma\tau} = O_{\text{mesh}\varphi(T)}$ (some point in $\overset{\circ}{D}{}^r_{\sigma\tau}$) is a PL m-ball. Then $J'_{\sigma\tau} \subset \delta_{\sigma\tau}$. Moreover, $(J'_{\sigma\tau}, \delta_{\sigma\tau}) \cong ([-2,2]^n, [-1,1]^n)$ (actually, by [23, 3.19], $J'_{\sigma\tau} - \overset{\circ}{\delta}_{\sigma\tau} \cong [-2,2]^n - \operatorname{Int}[-1,1]^n$, and applying the Alexander trick we can extend the homeomorphism $\partial\delta_{\sigma\tau} \cong \partial[-1,1]^n$ to a homeomorphism $\delta_{\sigma\tau} \cong [-1,1]^n$).

Take a PL $(m-r-1)$-ball $B_{\sigma\tau} \subset S_{\sigma\tau} \setminus (l_{\sigma\tau} \cup \varphi(\sigma \cup \tau))$. Since $J'_{\sigma\tau} - \overset{\circ}{\delta}_{\sigma\tau} \cong \partial\delta_{\sigma\tau} \times I$ it follows that $T_{\sigma\tau} = R_{\partial\delta_{\sigma\tau}}$ (a point in $\overset{\circ}{B}_{\sigma\tau}) \times I$ is a tube joining $\partial J'_{\sigma\tau}$ to $\partial\delta_{\sigma\tau}$. Since $B_{\sigma\tau} \cap \varphi(\sigma \cup \tau) = \emptyset$, we may assume that $T_{\sigma\tau} \cap \varphi(\sigma \cup \tau) = \emptyset$.

Since $(J'_{\sigma\tau}, \delta_{\sigma\tau}) \cong ([-2,2]^n, [-1,1]^n)$, we have that $J_{\sigma\tau} = J'_{\sigma\tau} \setminus T_{\sigma\tau} \setminus (B_{\sigma\tau} * D^r_{\sigma\tau})$ is a PL m-ball. Since $C_{\sigma\tau} \cup J'_{\sigma\tau}$, $C_{\sigma\tau} \cap (B_{\sigma\tau} * D^r_{\sigma\tau}) \subset \varphi(\sigma) \cap (B_{\sigma\tau} * D^r_{\sigma\tau}) = (\varphi(\sigma) \cap B_{\sigma\tau}) * D^r_{\sigma\tau} = D^r_{\sigma\tau}$ and $C_{\sigma\tau} \cap T_{\sigma\tau} \subset \varphi(\sigma) \cap T_{\sigma\tau} = \emptyset$, it follows that $C_{\sigma\tau} \subset \overset{\circ}{J}_{\sigma\tau} \cup D^r_{\sigma\tau}$. Analogously $C_{\tau\sigma} \subset \overset{\circ}{J}_{\sigma\tau} \cup D^r_{\sigma\tau}$. Then similarly to construction of $D^p_{\sigma\tau}$ and $D^q_{\sigma\tau}$, let $G_{\sigma\tau}$ be a trail of $C_{\sigma\tau} \cup C_{\tau\sigma}$ under a sequence of collapses

$$J_{\sigma\tau} \searrow J_{\sigma\tau} \cap \delta_{\sigma\tau} = (S_{\sigma\tau} - \overset{\circ}{B}_{\sigma\tau}) * D^r_{\sigma\tau} \searrow R_{S_{\sigma\tau}}(l_{\sigma\tau}) * D^r_{\sigma\tau} = \beta_{\sigma\tau} \text{ rel } D^r_{\sigma\tau}.$$

Analogously to (a)–(c), it is proved that

$$G_{\sigma\tau} \subset \overset{\circ}{J}_{\sigma\tau} \cup D^r_{\sigma\tau},$$
$$C_{\sigma\tau} \cup C_{\tau\sigma} \subset \beta_{\sigma\tau} \cup G_{\sigma\tau},$$
$$D^m_{\sigma\tau} = R_{J_{\sigma\tau}}(\beta_{\sigma\tau} \cup G_{\sigma\tau}, D^r_{\sigma\tau})$$

is a PL m-ball. Analogously to (d), using (d) and $n + (2n - m + 2) < m$ it is proved that $G_{\sigma\tau} \cap X_\sigma = \emptyset$. Then (4.2.6) is proved analogously to (e). (f) and general position imply

$$G_{\sigma\tau} \cap \varphi(\tau) = (C_{\sigma\tau} \cup C_{\tau\sigma}) \cap \varphi(\tau) = C_{\tau\sigma} \cup (C_{\sigma\tau} \cap \varphi(\tau)) = C_{\tau\sigma} \cup \Sigma_{\sigma\tau} = C_{\tau\sigma}.$$

Analogously, $G_{\sigma\tau} \cap \varphi(\sigma) = C_{\sigma\tau}$. Therefore, $(\beta_{\sigma\tau} \cup G_{\sigma\tau}) \cap \varphi(\sigma) = (\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$ and $(\beta_{\sigma\tau} \cup G_{\sigma\tau}) \cap \varphi(\tau) = (\beta_{\sigma\tau} \cap \varphi(\tau)) \cup C_{\tau\sigma}$. Because of that and since $D^p_{\sigma\tau}$, $D^q_{\sigma\tau}$ and $D^m_{\sigma\tau}$ are regular neighborhoods rel $D^r_{\sigma\tau}$ of $(\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau}$, $(\beta_{\sigma\tau} \cap \varphi(\tau)) \cup C_{\tau\sigma}$ and $\beta_{\sigma\tau} \cup G_{\sigma\tau}$ in restriction of the same triangulation of $\mathbb{R}^m$ to $\varphi(\sigma)$, $\varphi(\tau)$ and J, (4.2.2) follows. By (b) and definitions of $D^p_{\sigma\tau}$, $D^q_{\sigma\tau}$, $\Sigma_{\sigma\tau}$,

$$(\partial D^p_{\sigma\tau} - D^r_{\sigma\tau}) \cap (\partial D^q_{\sigma\tau} - D^r_{\sigma\tau}) \subset (\varphi(\overset{\circ}{\sigma}) - \Sigma_{\sigma\tau}) \cap (\varphi(\overset{\circ}{\tau}) - \Sigma_{\sigma\tau}) = \emptyset.$$

Hence (4.2.3) is true. By (a) we have

$$\Sigma_{\sigma\tau} \subset (\beta_{\sigma\tau} \cap \varphi(\sigma)) \cup C_{\sigma\tau} \subset \beta_{\sigma\tau} \cup G_{\sigma\tau} \subset \overset{\circ}{D}{}^m_{\sigma\tau} \cup D^r_{\sigma\tau}$$

so (4.2.5) is true. From $\operatorname{diam} D^m_{\sigma\tau} \leqslant \operatorname{mesh} f(T) + \operatorname{diam}(\beta_{\sigma\tau} \cup G_{\sigma\tau}) \leqslant \operatorname{mesh} f(T) + \operatorname{diam} \delta_{\sigma\tau} + \operatorname{diam} \beta_{\sigma\tau}$, we derive (4.2.7). Since $\beta_{\sigma\tau} \cap X_\sigma = \emptyset$ and $G_{\sigma\tau} \cap X_\sigma = \emptyset$, we have that $D^m_{\sigma\tau} \cap X_\sigma = \emptyset$. Since $\beta_{\sigma\tau} \subset J_{\sigma\tau} \subset J'_{\sigma\tau}$, it follows that $\operatorname{diam} D^m_{\sigma\tau} < 2 \operatorname{mesh} \varphi(T)$. Since $2(p + q - m + 2) < m$, we may assume by general position that $G_{\sigma\tau} \cap G_{\sigma'\tau'} \subset D^r_{\sigma\tau} \cap D^r_{\sigma'\tau'}$. From this and $\overset{\circ}{\beta}_{\sigma\tau} \cap \overset{\circ}{\beta}_{\sigma'\tau'} = \emptyset$, (4.2.8) follows. □

5. Maps of spheres, nonapproximable by embeddings

Recall the construction of Akhmetiev's example. For $k = 1, 3, 7$ there exists an immersion $\mathbb{R}P^k \looparrowright S^n$ with trivial normal bundle. Let $f : S^k \times D^{n-k} \to \mathbb{R}P^k \times D^{n-k} \looparrowright S^n$ be a composition of (the projection $S^k \to \mathbb{R}P^k) \times \operatorname{id} D^{n-k}$ and an immersion $\mathbb{R}P^k \times D^{n-k} \to S^n$, extending the immersion $\mathbb{R}P^k \looparrowright S^n$. We will prove that f is not embeddable in $\mathbb{R}^{n+k}$ through the standard embedding $S^n \subset \mathbb{R}^{n-k}$. Then any extension of f to S^n, in which $S^k \times D^{n-k}$ is standardly embedded, is the required map $S^n \to S^n$, non-embeddable in $\mathbb{R}^{n+k}$.

Recall that

$$D^p = \{(x_1 \cdots x_p) \in \mathbb{R}^p \mid x_1^2 + \cdots + x_p^2 \leqslant 1\}$$

and

$$S^{p-1} = \{(x_1 \cdots x_p) \in \mathbb{R}^p \mid x_1^2 + \cdots + x_p^2 = 1\}.$$

For each $x \in S^k$ take neighborhoods D_x^k of x in S^k and $B_x^n = D_x^k \times D^{n-k}$ of $x \times 0$ in $S^k \times D^{n-k}$, so small that $D_x^k \cap D_{-x}^k = \emptyset$ and $f|_{B_x^n}$ is an embedding. Since $k = 1, 3, 7$, S^k and $\mathbb{R}P^k$ are parallelizable. Therefore there is a family of homeomorphisms $h_x : D^k \to D_x^k$, continuously depending on $x \in S^k$, and such that $h_{-x} \equiv -h_x$, for each $x \in S^k$. Let $B^n = D^k \times D^{n-k}$ and take families of homeomorphisms $q_x = f \circ (h_x \times \operatorname{id} D^{n-k}) : B^n \to f(B_x^n)$ and $p_x = q_x \times \operatorname{id} D^k : B^n \times D^k \to f(B_x^n) \times D^k$.

Suppose to the contrary that f is ε-close to an embedding $F : S^k \times D^{n-k} \to S^n \subset S^n \times D^k \subset \mathbb{R}^{n+k}$, where ε is sufficiently small:

$$\varepsilon < \min_{x \in S^{n-1}} \operatorname{dist}\left(q_x\left(\frac{1}{2}B^n\right), q_x(\partial D^n)\right)$$

and

$$\varepsilon < \min_{(x,y) \in S^k \times \partial D^n} \frac{1}{2}\operatorname{dist}\left(q_x\left(\frac{y}{2}\right), q_x\left(-\frac{y}{2}\right)\right).$$

Since ε is small, $F(\frac{1}{2}B_x^n) \subset f(B_x^n) \times D^k$. Then $\{p_x^{-1}F(\frac{1}{2}B_x^n)\}_{x \in S^k}$ is a family of n-balls in $B^n \times D^k$ (Fig. 3). We shall prove that

$$p_x^{-1}F\left(\frac{1}{2}B_x^n\right) \cap p_{-x}^{-1}F\left(\frac{1}{2}B_{-x}^n\right) \neq \emptyset \quad \text{for some } x \in S^k. \tag{$*$}$$

Since $h_x \equiv -h_{-x}$, $q_x \equiv q_{-x}$ and hence $p_x \equiv p_{-x}$ for each $x \in S^k$. Therefore $F(\frac{1}{2}B_x^n) \cap F(\frac{1}{2}B_{-x}^n) \neq \emptyset$, which contradicts the assumption that F is an embedding.

To prove $(*)$, consider the collection $\{p_x^{-1}F(\frac{1}{2}B_x^n)\}_{x \in S^n}$ of n-balls and the map $\varphi : S^k \times B^n = S^k \times D^k \times D^{n-k} \to B^n \times D^n$, defined by $\varphi(x, y, z) = p_x^{-1}F(h_x(y)/2, z/2)$. Look at $S^k \times B^n$ as at a neighborhood of the standard $S^k \subset S^{n+k}$. We shall prove that some extension of φ onto S^{n+k} identify some pair of antipodes. These antipodes will actually lie in $S^k \times B^n$. Then $(*)$ is true by definition of φ. More precisely, consider

$$S^{n+k} \cong S^k * S^{n-1} = S^k \times S^{n-1} \times I / S^k \times \{y\} \times \{1\}, \qquad \{x\} \times S^{n-1} \times \{0\}.$$

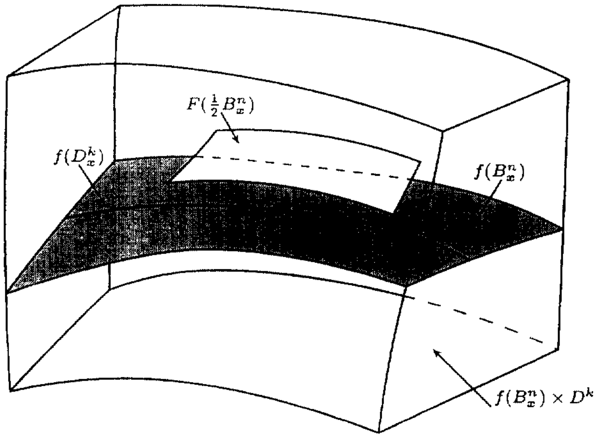

Fig. 4.

Embed $S^k \times B^n$ into such S^{n+k} by the formula

$$(x,y) \to \begin{cases} \operatorname{pr}\left(x, \dfrac{y}{\|y\|}, \dfrac{\|y\|}{2}\right) & \|y\| \neq 0, \\ \operatorname{pr}(\{x\} \times S^{n-1} \times \{0\}) & \|y\| = 0. \end{cases}$$

Extend φ to a map $\overline{\varphi}: S^{n+k} \to B^n \times D^k$ by

$$\overline{\varphi}\left(\operatorname{pr}\left(x, y, \frac{1+t}{2}\right)\right) = t\left(\frac{y}{2}, 0\right) + (1-t)\varphi(x,y), \quad \text{for } t \in [0,1].$$

Evidently, $\overline{\varphi}$ is well-defined. By the Borsuk–Ulam theorem, $\overline{\varphi}$ identifies some pair of antipodes $\operatorname{pr}(x,y,t)$ and $\operatorname{pr}(-x,-y,-t)$. But since ε is small, for each $x \in S^k$ and $y \in \partial B^n$, $\varphi(x,y) = \overline{\varphi}(x,y)$ is very close to $(\frac{y}{2}, 0)$. Again, since ε is small, $\overline{\varphi}(x,y) \neq \overline{\varphi}(-x,-y)$. Since the involution $\operatorname{pr}(x,y,t) \leftrightarrow \operatorname{pr}(-x,-y,-t)$ on $S^{n-k} \supset S^k \times B^n$ is an extension of the involution $(x,y) \leftrightarrow (-x,-y)$ on $S^k \times B^n$, it follows that φ actually identifies some pair of antipodes (x,y) and $(-x,-y)$.

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