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# A TOTAL FINITE-DIMENSIONAL SELECTION THEOREM†)

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## § 1. Statement of the Theorem

DEFINITION 1.1. A Dim-*filtration* of a $(p+1)$-dimensional paracompact space $X$ is a nested sequence $X = X_p \supset X_{p-1} \supset \dots \supset X_0 \supset X_{-1} \supset X_{-2} = \varnothing$ of subspaces of $X$ such that $\dim_{X_t}(X_{t-1}) \leq t$ for all $0 \leq t \leq p$ (which means that the Lebesgue dimension of every closed subset $E$ of $X_t$ lying in $X_{t-1}$ is at most $t$, $\dim E \leq t$).

In a metric space $X$, the above condition on dimension amounts to the inequality $\dim(Y_t) \leq (t+1)$ for every stratum $Y_t = X_t \setminus X_{t-1}$, $-1 \leq t \leq p$, of the dim-filtration of $X$. The question arises: In which case does the fulfillment of the conditions of Michael's $(t+1)$-dimensional selection theorem [1, 2] on every stratum $Y_t$ imply the existence of a global (local) selection? Our Theorem A answers this question.

We further assume that some (possibly empty) equi-locally-$t$-connected families $\mathfrak{S}_t$, $-1 \leq t < \infty$, of closed subsets of $Z$ are fixed in a metric space $(Z, \rho)$. Also, we assume that $\cup\mathfrak{S}_t$, the carrier or underlying set of each family $\mathfrak{S}_t$ with $0 \leq t < \infty$, is closed in the union $\cup\{\cup\mathfrak{S}_t \mid 0 \leq t < \infty\}$ of the carriers of all families.

**Theorem A.** *Assume given a $(p+1)$-dimensional paracompact space $X$, a dim-filtration $X = X_p \supset X_{p-1} \supset \dots \supset X_0 \supset X_{-1} \supset X_{-2} = \varnothing$ of $X$, and a lower semicontinuous multi-valued mapping $\Phi : X \to Z$ with complete (with respect to the metric $\rho$ on $Z$) values $\Phi(x)$, $x \in X$, for which*

(a) *the image $\Phi(x)$ of every point $x$ of the stratum $Y_t$, $-1 \leq t \leq p$, belongs to the family $\mathfrak{S}_t$.*

*Then, for every closed $A \subset X$ and every continuous selection $r : A \to L$ of the mapping $\Phi \restriction A$, there are a neighborhood $O(A)$ of $A$ and a continuous extension $r' : O(A) \to Z$ of $r$ that is a selection of the mapping $\Phi \restriction O(A)$. If we additionally know that the families $\mathfrak{S}_t$ consist of $t$-connected sets then we may assume that the neighborhood $O(A)$ equals $X$ (i.e., the local selection $r$ extends to some global selection).*

REMARK 1. If $X = X_{-1}$ then Theorem A coincides with the zero-dimensional selection theorem.

REMARK 2. If $X = X_t$ and $X_{t-1} = \varnothing$ then Theorem A coincides with the finite-dimensional selection theorem.

REMARK 3. If the set $\{t \leq p \mid Y_t \neq \varnothing\}$ consists of two elements then Theorem A is exactly Theorem 3 of [3].

REMARK 4. The following simple example shows that the requirement of closure of the carries $\cup\mathfrak{S}_t$ in the union $\cup\{\cup\mathfrak{S}_t \mid 0 \leq t < \infty\}$ is essential.

Let $a_n = 1/n$, let $\Delta_n = [a_{n+1}, a_n]$ be the closed interval, and let $S_n$ be the boundary of $\Delta_n$. Denote $X = \{0\} \cup \bigcup_n \Delta_n \subset \mathbb{R}^1$; $Z = \mathbb{R}^2$; $\mathfrak{S}_1$ consists of the cones $\operatorname{Con}(S_n)$, $n \geq 1$, in $\mathbb{R}^2$ over $S_n$ with vertex the point $(0,1)$; $\mathfrak{S}_0 = \{z \mid z \in \mathbb{R}^2\}$, $\mathfrak{S}_{-1} = \varnothing$; $X_1 = X$, $X_0 = \{0\} \cup \bigcup_n S_n$, and $X_{-1} = \varnothing$. Clearly, $\mathfrak{S}_1 \in$ equi-$LC^\infty \cap C^\infty$, $\mathfrak{S}_0$ consists of convex subsets, $\dim X_1 = 1$, and $\dim X_0 = 0$. It is easy to verify that the multi-valued mapping $F : X \to Z$, given by the formula $F(x) = \{x\}$ if $x = 0$

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or $x \in S_n$ and by the formula $F(x) = \operatorname{Con} S_n$ if $x \in \Delta_n \setminus S_n$, is lower semicontinuous and satisfies all (but one) conditions of the theorem: the carrier $\cup\mathfrak{S}_1$ of the system $\mathfrak{S}_1$ is not closed in $Z$. From the existence of a single-valued selection $f : X \to Z$ of $F$ we can easily deduce the possibility of retracting the interval $\Delta_n$ to its boundary $S_n$ for $n$ large enough, which is impossible.

REMARK 5. In § 3 we demonstrate that it suffices to prove Theorem A, as well as the forthcoming Theorems B–D, under the following simplifying assumption: $(Z, \rho)$ is a normed vector space $(L, \| * \|)$ and the carriers $\cup\mathfrak{S}_t$ with $t \geq -1$ coincide with $L$.

DEFINITION 1.2. Each lower semicontinuous mapping $\Phi$ mentioned in Theorem A is referred to as a multi-valued mapping *consistent with* the dim-filtration $X = X_p \supset X_{p-1} \supset \cdots \supset X_0 \supset X_{-1}$ of the paracompact space $X$ and the families of sets $\{\mathfrak{S}_t\}$.

We deduce Theorem A from the following theorem on approximation of $\delta$-selections.

**Theorem B.** *To each covering $\varepsilon \in \operatorname{cov} L$, there exists a covering $\delta \in \operatorname{cov} L$ with the following property: Each $\delta$-selection $r : X \to L$ of a multi-valued mapping $\Phi : X \to L$ consistent with the* dim-*filtration of the paracompact space $X$ and the families of sets $\{\mathfrak{S}_t\}$ can be $\varepsilon$-approximated by a selection $r' : X \to L$ of $\Phi$.*

By a method familiar from the selection theory, Theorem B is in turn reduced to the following theorems.

**Theorem C.** *To each covering $\varepsilon \in \operatorname{cov} L$, there exists a covering $\delta \in \operatorname{cov} L$ such that the following property holds for every covering $\mu \in \operatorname{cov} L$: Each $\delta$-selection $r : X \to L$ of a multi-valued mapping $\Phi : X \to L$ consistent with the* dim-*filtration of the paracompact space $X$ and the families of sets $\{\mathfrak{S}_t\}$ can be $\varepsilon$-approximated by a $\mu$-selection $r' : X \to L$ of $\Phi$.*

**Theorem D.** *Suppose that we additionally know that the families $\mathfrak{S}_t$ consist of $t$-connected sets. Then, for each covering $\mu \in \operatorname{cov} L$ and each multi-valued mapping $\Phi : X \to L$ consistent with the* dim-*filtration of the paracompact space $X$ and the families of sets $\{\mathfrak{S}_t\}$, there exists a $\mu$-selection $r' : X \to L$ of $\Phi$.*

## § 2. Preliminaries

We denote the set of all open coverings of $X$ by $\operatorname{cov} X$ and denote by $\omega \in \operatorname{cov} X$ some open covering. We denote $\sup\{\operatorname{diam} U \mid U \in \omega\}$ by $\operatorname{cal}(\omega)$ or $\operatorname{mesh}(\omega)$. The *star* (or the *neighborhood*) of a set $A \subset X$ with respect to $\omega \in \operatorname{cov} X$ is the set $\cup\{U | U \in \omega, U \cap A \neq \varnothing\}$, denoted by $\mathrm{N}(A, \omega)$ or $\mathrm{St}(A, \omega)$. The *star of a covering $\omega$ with respect to another covering $\omega'$* is the covering $\mathrm{St}(\omega, \omega') = \{\mathrm{St}(U, \omega') \mid U \in \omega\}$. For brevity, the iterated stars $\mathrm{St}(\omega_1, \mathrm{St}(\omega_2, \ldots, \omega_n) \ldots)$ are denoted by $\omega_n \circ \cdots \circ \omega_2 \circ \omega_1$ and, in case all $\omega_i$ are equal, by $(\omega_1)^k$. The *carrier* of a system $\omega$ of open sets is the set $\cup\{U \mid U \in \omega\}$ denoted by $\cup\omega$. The *intersection* of finitely many coverings $\omega_i$ is the covering composed of the intersections of the elements of $\omega_i$'s; it is denoted by $\bigwedge_{i=1}^{m} \omega_i$.

The record $\omega \succ \omega_1$ means as usual that the covering $\omega$ refines $\omega_1$. If $f, g : X \to Y$ are mappings, $A$ is a subset of $Y$, and $\omega \in \operatorname{cov} Y$ then the $\omega$-proximity of $f$ and $g$ is designated as $(f, g) \prec \omega$. The inclusion of a set $A$ in an element of a covering $\omega$ is designated as $A \prec \omega$.

The *nerve* of a covering $\omega = \{U_\beta \mid \beta \in B\}$ is the polytope $\mathfrak{N}\langle\omega\rangle$, with the weak Whitehead topology, whose vertices $\langle U_\beta \rangle$ are in a one-to-one correspondence with the index set $B$ and where $\omega = \langle U_{\beta_0}, \ldots, U_{\beta_s}\rangle$ is an $s$-dimensional simplex of $\mathfrak{N}\langle\omega\rangle$ with vertices $\langle U_{\beta_i}\rangle$ if and only if $\cap U_{\beta_i} \neq \varnothing$. The *$k$-dimensional skeleton* $\mathfrak{N}\langle\omega\rangle^{(k)}$ is the subpolytope of $\mathfrak{N}\langle\omega\rangle$ consisting of at most $k$-dimensional simplices; $\mathfrak{N}\langle\omega\rangle^{(-1)} = \varnothing$. The *open star* $\mathrm{St}_0(\langle U_{\beta_0}\rangle)$ of the vertex $\langle U_{\beta_0}\rangle$ is the set $\{\sum \alpha_\beta \cdot \langle U_\beta \rangle \in \mathfrak{N}\langle\omega\rangle \mid \alpha_{\beta_0} \neq 0\}$.

If a covering $\sigma$ refines a covering $\omega$, $\sigma \succ \omega$; then the simplicial mapping $\pi(\sigma, \omega) : \mathfrak{N}\langle\sigma\rangle \to \mathfrak{N}\langle\omega\rangle$ is defined that takes each vertex $\langle H \rangle \in \mathfrak{N}\langle\sigma\rangle$ into the vertex $\langle U \rangle \in \mathfrak{N}\langle\omega\rangle$ such that $H \subset U$. We say that the mapping $\pi$ is *generated* by the refinement $\sigma$ of $\omega$.

A mapping $\theta : X \to \mathfrak{N}\langle\omega\rangle$ is called *canonical* if the inverse image $\theta^{-1}(\mathrm{St}_0\langle U_\omega \rangle))$ of the open star of each vertex $\langle U_\omega \rangle$ lies in $U_\omega$. It is well known [4] that a canonical mapping exists for every open covering $\omega$ of a paracompact space $X$. We present the following fact without proof.

**Proposition 2.1.** *For every covering $\omega \in \operatorname{cov} X$ of a $(p+1)$-paracompact space $X$, there exists a canonical mapping $\theta : X \to \mathfrak{N}\langle\omega\rangle$ whose range $\theta(X)$ lies in the $(p+1)$-dimensional skeleton $\mathfrak{N}\langle\omega\rangle^{(p+1)}$.*

We denote the restriction of a mapping $f$ to a set $A$ by $f \restriction A$. All single-valued mappings are assumed continuous unless they result from some constructions.

A multi-valued mapping $\Phi : X \to Z$ is called *closed-valued* (*compact-valued, complete-valued*) if the image $\Phi(x)$ of each point $x$ is closed (compact, complete).

We recall that a multi-valued mapping $\Phi : X \to Z$ is said to be *lower semicontinuous* if the set $\{x \in X \mid \Phi(x) \cap U \neq \varnothing\}$ is open in $X$ for every open set $U$ in $Z$. A continuous single-valued mapping $f : X \to Z$ is called a *selection* (*$\delta$-selection* with $\delta \in \operatorname{cov} Z$) of a mapping $\Phi$ if $f(x) \in \Phi(x)$ ($f(x) \in \mathrm{N}(\Phi(x), \delta)$) for all $x \in X$.

DEFINITION 2.2. A family $\mathfrak{S} = \{Z_\alpha\}$ of closed subsets of a metric space $Z$ is called *equi-locally-$(n-1)$-connected* ($\mathfrak{S} \in \text{equi-LC}^{n-1}$), $1 \leq n \leq \infty$, if for every point $x \in \cup Z_\alpha$ and every neighborhood $U$ of $x$ there is a smaller neighborhood $V$ of $x$ such that every mapping $f : S^{k-1} \to V \cap Z_\alpha$, $k < n$, defined on the sphere $S^{k-1}$, extends to a mapping $\hat{f} : B^k \to U \cap Z_\alpha$ of the ball $B^{k-1}$ with boundary $S^{k-1}$.

If $\mathfrak{S}$ consists of a single set $Z_0$ then we say that $Z_0$ is *locally-$(n-1)$-connected* ($Z_0 \in \mathrm{LC}^{n-1}$). If $U = V = Z$ then we say that $\mathfrak{S}$ is *$(n-1)$-connected* ($\mathfrak{S} \in \mathrm{C}^{n-1}$). If $U = V = Z$ and $\mathfrak{S}$ consists of a single set $Z_0$ then we say that $Z_0$ is $(n-1)$-connected ($Z_0 \in \mathrm{C}^{n-1}$). It is well known [4] that the absolute extensors in dimension $n$ ($\mathrm{AE}(n)$) are exactly the spaces that are both locally $(n-1)$-connected and $(n-1)$-connected.

We now introduce some important notions that are connected with the name of S. Lefschetz.

DEFINITION 2.3. Let $\alpha$ be a system of open sets in $Z$ and let $\mathfrak{N}_0$ be a subpolytope of a polytope $\mathfrak{N}$ which contains all vertices. A *partial $\alpha$-realization* of $\mathfrak{N}$ is a mapping $\mathfrak{N}_0 \xrightarrow{f} Z$ such that the set $f(\Delta \cap \mathfrak{N}_0)$ lies in some element $V \in \alpha$ for every simplex $\Delta \in \mathfrak{N}$.

DEFINITION 2.4. A family $\mathfrak{S} = \{Z_\alpha\}$ of closed subsets of a metric space $Z$ possesses the *equi-Lefschetz property* equi-$\mathrm{Lf}^p$, $p \leq \infty$, if, for every system $\gamma$, $\cup\gamma \supset \bigcup_\alpha Z_\alpha$, of open sets, there exists a system $\delta$, $\cup\delta \supset \bigcup_\alpha Z_\alpha$, of open sets such that, for every set $Z_\alpha$ and every partial $\delta$-realization $\mathfrak{N} \hookleftarrow \mathfrak{N}_0 \xrightarrow{f} Z_\alpha$, $\dim(\mathfrak{N}) \leq (p+1)$, there exists a $\gamma$-realization $\hat{f} : \mathfrak{N} \to Z_\alpha$.

Dependence of a family $\delta$ on $\gamma$ is expressed as follows: $\delta = \text{equi-}\mathrm{Lf}^p_{\mathfrak{S}}(\gamma)$. If $\bigcup_\alpha Z_\alpha = Z$ then instead of systems we should consider coverings of $Z$.

It is well known [4, p. 156] that $Z \in \mathrm{ANE}(p+1) \equiv \mathrm{LC}^p$, $1 \leq p < \infty$, if and only if, for every $\gamma \in \operatorname{cov} Z$, there exists $\delta \in \operatorname{cov} Z$, $\gamma = \mathrm{Lf}^p_Z(\delta)$. By analogy, we establish the following

**Proposition 2.5.** *A family $\{Z_\alpha\}$ of closed subsets of a metric space $Z$ belongs to the class* equi-$\mathrm{LC}^p$, $p < \infty$, *if and only if $\{Z_\alpha\}$ possesses the equi-Lefschetz property* equi-$\mathrm{Lf}^p$.

## § 3. Derivation of Theorem A from Theorems B–D

First of all, we demonstrate how Theorem A is reduced to a situation in which the metric space $(Z, \rho)$ is a normed vector space $(L, \| * \|)$ and the carrier of the system $\cup\mathfrak{S}_t$ coincides with the whole space $L$. To this end, we embed the subspace $Z_0 = \cup\{\cup\mathfrak{S}_t \mid 0 \leq t < \infty\}$ isometrically in some normed vector space $L$, $e : Z_0 \to L$, so that the image of $Z_0$ be closed [5, p. 49]. From the closure of $\cup\mathfrak{S}_t$ in $Z_0$ and $e(Z_0)$ in $L$ we can easily infer that the families $\mathfrak{S}'_t = e(\mathfrak{S}_t) \cup \{L\}$ of closed subsets belong to the class equi-$\mathrm{LC}^t \cap \mathrm{C}^t$ and that their carriers coincide with $L$. The lower semicontinuous complete-valued mapping $\widetilde{\Phi} = e \circ \Phi : X \to L$ is consistent with the dim-filtration of $X$ and the families of sets $\{\mathfrak{S}'_t\}$, and by hypothesis there exists a selection $\tilde{r} : X \to L$ for $\widetilde{\Phi}$. Then $r : X \to Z$, $r(x) = e^{-1}(\tilde{r}(x))$, is a sought selection for $\Phi$.

From now on, we assume that in a normed vector space $L$ there are fixed equi-locally-$t$-connected families $\mathfrak{S}_t$, $-1 \leq t < \infty$, of closed $t$-connected sets; moreover, the carriers $\cup\mathfrak{S}_t$, $-1 \leq t$, of the families coincide with the whole $L$.

Forty years ago. E. Michael discovered first selection theorems whose proofs are reduced to construction of approximate selections and the subsequent procedure of finding new, more precise approximate selections in their neighborhoods. In broad outline, this scheme applies to the situation under consideration.

**Proposition 3.1.** *Theorem* C *implies Theorem* B.

PROOF. Assume that a covering $\varepsilon \in \operatorname{cov} L$ is fixed and let $\varepsilon_n \in \operatorname{cov} L$, $\varepsilon_1 = \varepsilon$, be a sequence of coverings of $L$ such that $(\varepsilon_{n+1})^3 \succ \varepsilon_n$ for $n \geq 1$ and $\lim \operatorname{mesh} \varepsilon_n \to 0$. An arbitrary Cauchy sequence $\{v_n \in L\}$ has a limit in $v$, provided that $\lim \rho(v_n, \Phi(x)) = 0$ for some point $x \in X$. Clearly, $v$ will belong to $\Phi(x)$.

Let $\delta_i \in \operatorname{cov} L$ be chosen so that the pair $(\varepsilon_i, \delta_i)$ satisfies Theorem C. Also, fix a sequence of coverings $\{\mu_i\}$ such that $\mu_i \succ \delta_i$ and $\operatorname{mesh}\{\mu_i\} \to 0$.

We take $\delta_2$ as a sought covering $\delta \in \operatorname{cov} L$. Indeed, consider a $\delta$-selection $r : X \to L$ of $\Phi$. Applying Theorem C repeatedly, we successively obtain mappings $r_n : X \to L$, $n \geq 2$, such that $(r_{n-1}, r_n) \prec \varepsilon_{n+2}$ and $r_n(x) \in \mathrm{N}(\Phi(x), \mu_n) \subset \mathrm{N}(\Phi(x), \delta_n)$, $x \in X$. Then $\{r_n\}$ is a Cauchy sequence, and, by virtue of the equality $\lim \rho(r_n(x), \Phi(x)) = 0$, it converges to some mapping $r' : X \to L$, $r'(x) \in \Phi(x)$. Clearly, $r'$ is a sought section of $\Phi$ $\varepsilon$-close to $r$.

**Proposition 3.2.** *Suppose that $\delta \in \operatorname{cov} L$. Then every partial selection $X \hookleftarrow A \xrightarrow{r} L$ of $\Phi$ extends to a $\delta$-selection $O(A) \xrightarrow{r'} L$ of $\Phi$ over some neighborhood $O(A)$.*

**Proposition 3.3.** *The global Theorem* A (*local Theorem* A) *is a consequence of Theorems* B *and* D (B *and Proposition* 3.2).

PROOF. To save room, we only settle the global case, leaving the case of the local extension of a selection to the reader.

Let $\varepsilon$ be a covering of $L$ consisting of one element and let the pair $(\varepsilon, \delta)$ satisfy Theorem B. Using Theorem D, find a $\delta$-selection $r$ of $\Phi$. An $\varepsilon$-approximation of $r$ is a sought selection of $\Phi$.

## § 4. Michael's λ-Condition

The rest of the article is devoted to proving Theorems C and D. A $\mu$-approximation $r' : X \to L$ in Theorem C is sought as a rule in the form of the composition of a canonical mapping $\theta : X \to \mathfrak{N}\langle\omega\rangle$, generated by some covering $\omega \in \operatorname{cov} X$, and a continuous mapping $m : \mathfrak{N}\langle\omega\rangle \to L$. We address the question: "Which conditions should be imposed on a mapping $m$ for the composition $m \circ \theta$ to satisfy Theorem C?" Answering this question, E. Michael introduced the following definition (in terms differing from those here) and proved a proposition which explains the value of this notion in many respects.

DEFINITION 4.1. Suppose that $\omega = \{V_\alpha\} \in \operatorname{cov} X$, $\mathfrak{N}_0$ is a subpolytope of the nerve $\mathfrak{N}\langle\omega\rangle$, and $\lambda \in \operatorname{cov} X$. We say that a mapping $m : \mathfrak{N}_0 \to L$ satisfies *Michael's $\lambda$-condition* if the inclusion $m(\Delta) \subset \mathrm{N}(\Phi(x), \lambda)$ holds for every simplex $\Delta = \langle V_0, \dots, V_k\rangle \in \mathfrak{N}_0$ and every point $x \in \cap V_i$.

**Proposition 4.2.** *Assume given*: (1) *coverings $\gamma, \lambda \in \operatorname{cov} L$*; (2) *a covering $\omega \in \operatorname{cov} X$*; (3) *a partial $\gamma$-realization $\mathfrak{N}\langle\omega\rangle \hookleftarrow \mathfrak{N}_0 \xrightarrow{m} L$ satisfying Michael's $\lambda$-condition*; (4) *a mapping $k : X \to L$ such that $k(U) \subset \mathrm{N}(m(\langle U\rangle), \gamma)$ for all $U \in \omega$.*

*Then*: (5) *the composition $g : X_0 \xrightarrow{\theta \restriction X_0} \mathfrak{N}_0 \xrightarrow{m} X$, where $X_0 = \theta^{-1}(\mathfrak{N}_0)$, of $m$ with every canonical mapping $\theta : X \to \mathfrak{N}\langle\omega\rangle$ is $(\gamma \circ \gamma)$-close to $k \restriction X$*; (6) *the mapping $g$ is a $\lambda$-selection of the multi-valued mapping $\Phi \restriction X_0$.*

REMARK. If $\theta(X)$ is in $\mathfrak{N}_0$ and if $\gamma^2 \succ \varepsilon$ and $\lambda = \mu$ then $g = m \circ \theta$ is a sought $\varepsilon$-approximation of $k$ and a $\mu$-approximation of $\Phi$.

PROOF OF THE PROPOSITION. Suppose that $x \in \bigcap_{i=0}^{k} U_i$ and $\theta(x) = \sum \alpha_i \langle U_i\rangle \in \Delta \subset \mathfrak{N}_0$. Since $m$ is a $\gamma$-realization, we have $m(\theta(x)) \in \mathrm{N}(m\langle U_i\rangle, \gamma)$. Moreover, by hypothesis $k(x) \in k(U_i) \subset \mathrm{N}(m\langle U_i\rangle, \gamma)$. Therefore, the pairs $m(\theta(x))$, $m(\langle U_i\rangle)$ and $k(x)$, $m(\langle U_i\rangle)$ belong to some elements of $\gamma$, whereas $g(x) = m(\theta(x))$ and $k(x)$ belong to some element of the covering $\gamma \circ \gamma$.

Finally, the inclusion $g(x) \in \mathrm{N}(\Phi(x), \lambda)$ follows from Michael s $\lambda$-condition: $g(x) = m(\theta(x)) \in m(\Delta) \subset \mathrm{N}(\Phi(x), \lambda)$.

The next proposition shows that an arbitrary $\beta$-selection $k : X \to L$ of $\Phi$ and an arbitrary covering $\lambda \in \operatorname{cov} L$ generate a covering $\omega \in \operatorname{cov} X$ and a mapping $\mathfrak{N}\langle\omega\rangle^{(0)} \xrightarrow{m} L$ of the zero-dimensional skeleton of the nerve of $\omega$ which satisfy the conditions of Proposition 4.2 for the coverings $\gamma = (\beta \circ \lambda^2 \circ \beta)$ and $\lambda$. We have thus accomplished all preliminaries for the base of the induction to be carried out in the sequel.

**Proposition 4.3.** *Assume given a $\beta$-selection $k : X \to L$ of $\Phi$ and a covering $\lambda \in \operatorname{cov} L$. Then there exist a covering $\omega \in \operatorname{cov} X$ and a partial mapping $\mathfrak{N}\langle\omega\rangle \hookleftarrow \mathfrak{N}\langle\omega\rangle^{(0)} \xrightarrow{m} L$ of the zero-dimensional skeleton which, together with the $\beta$-selection $k$, satisfy the conditions of Proposition 4.2 for the coverings $\gamma = (\beta \circ \lambda^2 \circ \beta)$ and $\lambda$ (i.e., $m$ is a $\gamma$-realization satisfying Michael's $\lambda$-condition and the inclusion $k(U) \subset \mathrm{N}(m(\langle U\rangle), \gamma)$ holds for all $U \in \omega$).*

REMARK. Proposition 4.2 implies that the composition $m \circ \theta : X_0 \to L$ of $m$ and every canonical mapping $\theta$ $\gamma^2$-approximates the mapping $k \restriction X_0$ and $\lambda$-approximates $\Phi \restriction X_0$, where $X_0 = \theta^{-1}(\mathfrak{N}\langle\omega\rangle^{(0)})$.

We precede the proof of Proposition 4.3 with some simple observation which is an easy corollary to the definition of lower semicontinuity.

**Lemma 4.4.** *If $\Phi$ is a lower semicontinuous multi-valued mapping then, for every covering $\lambda \in \operatorname{cov} L$ and every compact set $K \subset L$, the set $\{x \in X \mid K \subset \mathrm{N}(\Phi(x), \lambda)\}$ is open (possibly empty).*

PROOF OF PROPOSITION 4.3. Associate some point $m_x$ in the nonempty intersection $\mathrm{N}(k(x), \beta) \cap \Phi(x)$ with each point $x \in X$. Define the sought covering $\omega \in \operatorname{cov} X$ by the formula

$$\omega = \{U\langle x\rangle \mid k(U\langle x\rangle) \subset \mathrm{N}(k(x), \lambda),\ m_x \in \mathrm{N}(\Phi(x'), \lambda) \text{ for all } x' \in U\langle x\rangle\}$$

and define the partial realization $m$ by the formula $m\langle U\langle x\rangle\rangle = m_x$.

If $U\langle x\rangle \cap U\langle y\rangle \neq \varnothing$ then there exists a point $z \in k(U\langle x\rangle) \cap k(U\langle y\rangle)$. Since $(m_x, k(x)) \prec \beta$, $(k(x), z) \prec \lambda$, $(z, k(y)) \prec \lambda$, and $(k(y), m_y) \prec \beta$, it follows that $(m_x, m_y) \prec (\beta \circ \lambda^2 \circ \beta)$ and so the mapping $m$ is a partial $\gamma$-realization.

By the definition of $\omega$, for every point $y \in U\langle x\rangle$ we have $m(\langle U\langle x\rangle\rangle) = m_x \in \mathrm{N}(\Phi(y), \lambda)$ for all $y \in U\langle x\rangle$. We have so verified Michael's $\lambda$-condition for $m$.

Finally, the required inclusion $k(U\langle x\rangle) \subset \mathrm{N}(m\langle U\langle x\rangle\rangle, \gamma)$ holds because $k(U\langle x\rangle) \subset \mathrm{N}(k(x), \lambda)$ and $(k(x), m_x) \prec \beta$.

## § 5. Proof of Theorem C

The proof of Theorem C is based on two Propositions $\mathfrak{A}$ and $\mathfrak{B}$. Proposition $\mathfrak{A}$ shows that, in a certain situation, a mapping of a polytope can be approximated by a mapping whose range lies in a fiber $\Phi(x)$. We note that Proposition $\mathfrak{A}$ was proven by E. Michael in another form.

**Proposition $\mathfrak{A}$.** *To each covering $\xi \in \operatorname{cov} L$, there exists a covering $\zeta \in \operatorname{cov} L$ such that, for every continuous mapping $\phi : P \to \mathrm{N}(S, \zeta)$, $S \in \mathfrak{S}_t$, of a $(t+1)$-dimensional polytope $P$, $t < \infty$, there is a $\xi$-homotopy $\Phi : (P \times I) \to \mathrm{N}(S, \xi)$ in $L$ such that $\Phi_0 = \phi$ and $\mathrm{Im}(\Phi_1) \subset S$.*

We designate dependence of $\zeta$ on $\xi$ as follows: $\zeta = (\mathfrak{A})(\xi)$.

PROOF. Inscribe an open convex covering $\zeta_0 \in \operatorname{cov} L$ in $\xi$. Also, choose coverings $\zeta_1, \zeta_2, \zeta \in \operatorname{cov} L$ such that $(\zeta_1)^3 \succ \zeta_0$, $\zeta_2 = \text{equi-Lf}_{\mathfrak{S}^t}(\zeta_1)$ for all $t \leq p$, and $(\zeta)^3 \succ \zeta_2$. Without loss of generality we assume that $\xi \prec \zeta_0 \prec \zeta_1 \prec \zeta_2 \prec \zeta$.

Let $K$ be a triangulation of $P$ such that $\{\phi(\Delta) \mid \Delta \in K\} \succ \zeta$ and let $\phi_0 : P^{(0)} \to S$ be a mapping $\zeta$-close to $\phi \restriction P^{(0)}$. Then $\{\phi_0(\Delta^{(0)}) \mid \Delta \in K\} \succ \zeta^3 \succ \zeta_2$ and so the partial mapping $P \hookleftarrow P^{(0)} \xrightarrow{\phi_0} S$ is a $\zeta_2$-realization. Therefore, there exists a $\zeta_1$-realization $\tilde{\phi} : P \to S$, $\tilde{\phi} \restriction (P^{(0)}) = \phi$.

It is easy to verify that $(\phi, \tilde{\phi}) \prec \zeta \circ \zeta \circ \zeta_1$. Since the last covering refines $\zeta_0$, we infer that $(\phi, \tilde{\phi}) \prec \zeta_0$. As a $\xi$-homotopy $\Phi$, we can take the linear homotopy between $\tilde{\phi}$ and $\phi$.

The key point in the proof of Theorem C, and Theorem A therewith, is the following

**Proposition $\mathfrak{B}$.** *To each covering $\alpha \in \operatorname{cov} L$, there exists a covering $\rho \in \operatorname{cov} L$ that is a singleton if so is $\alpha$ and that possesses the following property: Given an arbitrary covering $\kappa \in \operatorname{cov} L$, there is a covering $\lambda \in \operatorname{cov} L$ such that*

(1) *for every multi-valued mapping consistent with the dim-filtration of $X$ and the families of sets $\{\mathfrak{S}_t\}$;*

(2) *for every closed embedding of $X$ in a paracompact space $\widetilde{X}$ and every extension of $\Phi$ to a lower semicontinuous mapping $\widetilde{\Phi} : X \to L$, $\widetilde{\Phi} \restriction X = \Phi$;*

(3) *for every locally finite system $\omega = \{U_\omega\}$, $\cup\omega \supset X$ of open subsets of $\widetilde{X}$;*

(4) *for every partial $\rho$-realization $\mathfrak{N}\langle\omega\rangle \hookleftarrow \mathfrak{N}\langle\omega\rangle^{(k)} \xrightarrow{m} L$, $k \geq 0$, satisfying Michael's $\lambda$-condition there exist a locally finite system $\sigma = \{H_\sigma\}$, $\cup\sigma \supset X$ of open sets in $\widetilde{X}$ which refines $\omega$ and a partial $\alpha$-realization $\mathfrak{N}\langle\sigma\rangle \hookleftarrow \mathfrak{N}\langle\sigma\rangle^{(k+1)} \xrightarrow{q} L$ satisfying Michael's $\kappa$-condition and coinciding on the $k$-dimensional skeleton $\mathfrak{N}\langle\sigma\rangle^{(k)}$ with the composition $m \circ q \restriction \mathfrak{N}\langle\sigma\rangle^{(k)} = m \circ (\pi \restriction \mathfrak{N}\langle\sigma\rangle^{(k)})$ of $m$ and the simplicial mapping $\pi = \pi(\sigma, \omega) : \mathfrak{N}\langle\sigma\rangle \to \mathfrak{N}\langle\omega\rangle$.*

We designate dependence of $\rho$ on $\alpha$, as well as $\lambda$ on $\alpha$, $\rho$, and $\kappa$, as follows: $\rho = (\mathfrak{B})(\alpha)$, $\lambda = (\mathfrak{B})(\alpha, \rho; \kappa)$.

Before proving Proposition $\mathfrak{B}$, we demonstrate how Theorems C and D are deduced from it and Propositions 4.2 and 4.3.

PROOF OF THEOREMS C AND D. Given a covering $\varepsilon$, we first construct a sought covering $\delta$. To this end, we consider a sequence of coverings $\delta_i \in \operatorname{cov} L$, $i = p+2, p+1, \ldots, 1$, such that

$(*)$ $\delta_i = (\mathfrak{B})(\delta_{i+1})$ for $i = p+1, p, \ldots, 1$ and $\delta_{p+2}^2 \succ \varepsilon$.

As $\delta$ we take a covering for which $\delta^8 \succ \delta_1$. It is clear that, for a one-element covering $\varepsilon$, the coverings $\delta_i = \varepsilon$ satisfy condition $(*)$, and so we can take $\delta$ to be a singleton.

Now, given a covering $\mu$, we construct one more covering $\lambda$. To this end, we consider a sequence of coverings $\lambda_i \in \operatorname{cov} L$, $i = p+2, p+1, \ldots, 1$, such that $(**)$ $\lambda_{p+2} = \mu$, $\lambda_i = (\mathfrak{B})(\delta_{i+1}, \delta_i; \lambda_{i+1})$ for $i = p+1, p, \ldots, 1$.

We put $\lambda = \lambda_1$.

Without loss of generality we may assume that $\delta_i \succ \delta_j$ and $\lambda_i \succ \lambda_j$ for $i < j$ and $\lambda_{p+1} \succ \delta$.

Let $k : X \to L$ be a $\delta$-approximation of $\Phi$. Applying Proposition 4.3 to the coverings $\lambda$ and $\beta = \delta \in \operatorname{cov} L$, we obtain a covering $\omega \in \operatorname{cov} X$ and a partial $\gamma$-realization $\mathfrak{N}\langle\omega\rangle \hookleftarrow \mathfrak{N}\langle\omega\rangle^{(0)} \xrightarrow{m} L$ which satisfies Michael's $\lambda$-condition, where $\gamma = (\beta \circ \lambda^2 \circ \beta)$. Moreover, $k(U) \subset \mathrm{N}(m\langle U\rangle, \gamma)$ for all $U \in \omega$. Since $\gamma^2 \succ \beta^8 = \delta^8 \succ \delta_1 \succ \delta_{p+2}$, the mapping $m$ is a $\beta_1$-realization.

Since $\lambda_i = (\mathfrak{B})(\delta_{i+1}, \delta_i; \lambda_{i+1})$; therefore, taking the paracompact spaces $X$ and $\widetilde{X}$ in Proposition $\mathfrak{B}$ coincident, we can successively construct coverings $\omega_i \in \operatorname{cov} X$ and $\omega_{i+1} \succ \omega_i$ and partial $\delta_i$-realizations $m_i : \mathfrak{N}\langle\omega_i\rangle^{(i)} \to L$ for $i = 1, 2, \ldots, p$ which satisfy Michael's $\lambda_i$-condition and meet the equalities $m \circ \pi(\omega_i, \omega) = m_i$ on the 0-dimensional skeleton $\mathfrak{N}\langle\omega_i\rangle^{(0)}$. Finally, we consider the partial $\delta_{p+2}$-realization $m_{p+2} : \mathfrak{N}\langle\omega_{p+2}\rangle^{(p+2)} \to L$. Since $\dim X \leq (p+1)$, by Proposition 2.1 there exists a canonical mapping $\theta : X \to \mathfrak{N}$, $\operatorname{Im}(\theta) \subset \mathfrak{N}\langle\omega_{p+2}\rangle^{(p+2)}$. In view of Proposition 4.2, the composition $m_{p+2} \circ \theta$ is $(\delta_{p+2})^2$-close to $k$ (and so $\varepsilon$-close to $k$) and $\lambda_{p+2}$-approximates $\Phi$ (and so $\mu$-approximates $\Phi$).

## § 6. Proof of Proposition $\mathfrak{B}$

We obtain the covering $\rho = (\mathfrak{B})(\alpha) \in \operatorname{cov} L$ on constructing a sequence $\rho_1 = \alpha, \rho_1'', \rho_1', \rho$ of coverings of $L$ such that

$(***)$ $(\rho_1'')^3 \succ \alpha$, $\rho_1' = \text{equi-Lf}^t_{\mathfrak{S}_t}(\rho_1'')$ for all $t \leq p$, $(\rho)^3 \succ \rho_1'$.

Without loss of generality we may assume that each successor in the sequence of coverings refines its predecessor.

Also, we construct the covering $\lambda = (\mathfrak{B})(\alpha, \rho; \kappa)$. It is convenient to represent the process of arranging the corresponding sequence $\lambda_1 = \kappa, \lambda_1', \lambda$ of convex coverings of $L$ as follows:

$$\lambda_1 = \kappa, (\lambda_1')^2 \succ \lambda_1, \lambda = (\mathfrak{A})(\beta \wedge \lambda_1').$$

We proceed further by inducting on $d = d(X) = \max\{t | Y_t = X_t \setminus X_{t-1} \neq \varnothing\}$. If $d = -1$ then $X = X_{-1}$ and $\dim(X) \leq 0$. In this case we easily validate Proposition $\mathfrak{B}$, establishing the induction base.

Suppose that Proposition $\mathfrak{B}$ is valid for all $X$ with $d(X) < p$ and examine the case of $d(X) = p$.

Consider the given partial $\rho$-realization $\mathfrak{N}\langle\omega\rangle \hookleftarrow \mathfrak{N}\langle\omega\rangle^{(k)} \xrightarrow{m} L$, $k \leq p$, satisfying Michael's $\lambda$-condition. Given a point $x \in X$, consider all elements $U_\omega \in \omega$ containing $x$ (there are only finitely many of them!) and denote the corresponding simplex by $\Delta_x = \langle U_0, \ldots, U_m\rangle$. Now, the partial mapping $\Delta_x^{k+1} \hookleftarrow \Delta_x^k \xrightarrow{m} L$ is a $\rho$-realization and satisfies Michael's $\lambda$-condition.

First of all, observe that the number $k$ can be assumed not exceeding $p$. Indeed, if $k > p$ then we should take as $\sigma$ an arbitrary locally finite open system in $\widetilde{X}$ refining $\omega$ and having multiplicity at most $(p+2)$. In this case $\mathfrak{N}\langle\sigma\rangle^{(k+1)} = \mathfrak{N}\langle\sigma\rangle^{(k)}$ and the partial realization $q = m \circ \pi(\sigma, \omega) \restriction (\mathfrak{N}\langle\sigma\rangle^{(k+1)})$ is also a partial $\rho$-realization and satisfies Michael's $\lambda$-condition. The possibility of constructing a desired covering $\sigma$ is provided by the following lemma.

**Lemma 6.1.** *If a closed subset $Z_0$ of a paracompact space $Z$ has dimension at most $t$ then, for every locally finite open covering $\xi = \{F_\xi\}$ in $Z_0$ of multiplicity at most $(t+1)$, there exists a locally finite open system $\{E_\xi\}$ in $Z$ such that $E_\xi \cap Z_0 = F_\xi$ for all index elements $\xi$ and the multiplicity of the system is at most $(t+1)$.*

PROOF. The claim can be proven by a slight modification of the arguments in [4, p. 70].

We now demonstrate that, in the case of $k \leq p$, with each point $x \in (X \setminus X_{p-1})$ we can associate a neighborhood $V\langle x\rangle$ in $\widetilde{X}$ and a mapping $m_x : \Delta_x^{(k+1)} \to L$ so that

(b) $m_x$ is an $\alpha$-realization;

(c) $\operatorname{Im}(m_x) \subset \mathrm{N}(\Phi(x'), \kappa)$ for all points $x' \in V\langle x\rangle$;

(d) $m_x \restriction (\Delta_x^{(k)}) = m \restriction (\Delta_x^{(k)})$.

Indeed, $m(\Delta_x^{(k)})$ is in $\mathrm{N}(\Phi(x), \lambda)$, and $\lambda = (\mathfrak{A})(\rho \wedge \lambda_1')$ implies existence of a $(\rho \wedge \lambda_1')$-homotopy $F : \Delta_x^{(k)} \times I \to \mathrm{N}(\Phi(x), \rho \wedge \lambda_1')$ for which $F_0 = m$ and $F_1(\Delta_x^{(k)}) \subset \Phi(x)$. It is easy to see that the mapping $\Delta_x^{(k+1)} \hookleftarrow \Delta_x^{(k)} \xrightarrow{F_1} \Phi(x)$ is a $(\rho \circ \rho \circ \rho)$-realization and, in consequence, a $\rho'$-realization. In view of $\rho_1' = \text{equi-Lf}^p_{\mathfrak{S}_p}(\rho_1'')$, there exists a $\rho_1''$-realization $m_x' : \Delta_x^{(k+1)} \to \Phi(x)$, $m_x' \restriction (\Delta_x^{(k)}) = F_1$. The mapping $m_x'$ and the $(\rho \wedge \lambda_1')$-homotopy $F$ agree on the common domain of definition $\Delta_x^{(k)} \times \{1\}$; therefore, the mapping $F \cup (m_x') : (\Delta_x^{(k)} \times [0,1]) \cup (\Delta_x^{(k+1)} \times \{1\}) \to L$ is well defined.

**Lemma 6.2.** *There exists a mapping $v : \Delta_x^{(k+1)} \to (\Delta_x^{(k)} \times [0,1]) \cup (\Delta_x^{(k+1)} \times \{1\})$ (the so-called "stamping" mapping) possessing the following properties:*

(i) *$v(a) = (a, 0)$ for all $a \in \Delta_x^{(k)}$;*

(ii) *$v(\delta) \subset \delta \times [0,1]$ for every simplex $\delta$ in $\Delta_x^{(k+1)}$.*

PROOF. The claim is proven by induction on the skeletons of the polyhedra $\Delta_x^{(k+1)}$.

Denote the composition $\{F \cup m_x'\} \circ v : \Delta_x^{(k+1)} \xrightarrow{v} (\Delta_x^{(k)} \times [0,1]) \cup (\Delta_x^{(k+1)} \times \{1\}) \xrightarrow{F \cup m_x'} L$ by $m_x : \Delta_x^{(k+1)} \to L$. Since $(\lambda_1' \wedge \rho) \succ \kappa$, we have $\operatorname{Im} F \subset \mathrm{N}(\Phi(x), \kappa)$. Therefore, $\operatorname{Im}(m_x) \subset \mathrm{N}(\Phi(x), \kappa)$. By Lemma 4.4 and compactness of $\operatorname{Im}(m_x)$, we can choose a neighborhood $V\langle x\rangle$ of $x$ in $\widetilde{X}$ such that $m_x(\Delta_x^{(k+1)}) \subset \mathrm{N}(\Phi(x'), \kappa)$ for every point $x' \in V\langle x\rangle$.

Since $F_0 = m$, we have $m_x \restriction (\Delta_x^{(k)}) = m$. It is also clear that $m_x$ is an $\alpha$-realization since $(\lambda_1' \wedge \rho) \circ \rho' \circ (\lambda_1' \wedge \rho) \succ (\rho_1'')^3 \succ \alpha$. Without loss of generality we may assume that $\{V\langle x\rangle \mid x \in (X \setminus X_{p-1})\} \succ \omega$. We obtain the system $\{V\langle x\rangle \mid x \in X\}$ of open sets in $\widetilde{X}$ by redefining $V\langle x\rangle = \widetilde{X}$ for $x \in X_{p-1}$.

Considering the intersection $\{V\langle x\rangle | x \in X\} \wedge \omega$ of the systems, we choose a locally finite refinement that consists of open subsets $\theta = W_\theta$, $\cup\theta \supset X$ ($X$ is closed in $\widetilde{X}$!), of $\widetilde{X}$. Without loss of generality we may assume that the system $\{\mathrm{cl}\, W_\theta\}$ also refines the same intersection. This guarantees the membership of the point $x$ in the set

$$C_x = V\langle x\rangle \setminus (\cup\{\mathrm{cl}\, W_\theta | x \notin \mathrm{cl}\, W_\theta\})$$

which is open in $\widetilde{X}$. Observe without proof the following obvious property:

(e) if $y \in C_x$ and $y \in \mathrm{cl}\, W_\theta$ then $x \in \mathrm{cl}\, W_\theta$.

Since $x \in C_x$, the set $C = \cup\{C_x \mid x \in X \setminus X_{p-1}\}$ is open in $\widetilde{X}$ and includes $X \setminus X_{p-1}$, whereas the set $E = X \setminus C$ is closed in $\widetilde{X}$ and is included in $X_{p-1}$. Hence, $\dim E \leq p$.

Consider the filtration $E_t = X_t \cap E$, $t \leq (p-1)$, of the $p$-dimensional paracompact space $E$. In view of the inequalities $\dim_{E_t}(E_{t-1}) < \dim_{X_t}(X_{t-1}) \leq t$, this filtration is a dim-filtration. Since $E_p \setminus E_{p-1} = (X_p \setminus X_{p-1}) \cap E = \varnothing$, it is obvious that $d(E) < d(X)$. By virtue of the inclusions $E_t \setminus E_{t-1} \subset X_p \setminus X_{p-1} \subset \{x \mid \Phi(x) \in \mathfrak{S}_t\}$, the restriction of the multi-valued mapping $\Phi$ to $E$ gives us a mapping consistent with the above dim-filtration of $E$ and the families $\{\mathfrak{S}_t\}$.

If we consider the locally finite system $\theta_1 = \{W \mid W \cap E \neq \varnothing\} \succ \theta$ of open sets in $\widetilde{X}$ and the partial realization $\mathfrak{N}\langle\theta_1\rangle \hookleftarrow \mathfrak{N}\langle\theta_1\rangle^{(k)} \xrightarrow{m\circ\pi(\theta,\omega)} L$, where $\pi(\theta,\omega) = \pi(\tilde\theta,\omega)\circ\pi(\theta,\tilde\theta)$, $\tilde\theta = \{\mathrm{cl}(W_\theta)\}$, then, for obvious reasons, $m\circ\pi(\theta,\omega)$ is a $\rho$-realization and satisfies Michael's $\lambda$-condition. By the induction hypothesis ($d(E) < d(X)$!), there exist a locally finite system $\sigma_2$, $\cup\sigma_2 \supset E$, of open sets in $\widetilde{X}$ which refines $\theta_1$ and a partial $\alpha$-realization $\mathfrak{N}\langle\sigma_2\rangle \hookleftarrow \mathfrak{N}\langle\sigma_2\rangle^{(k+1)} \xrightarrow{q_1} L$ satisfying Michael's $\kappa$-condition and agreeing with $m\circ\pi(\sigma_2,\theta_1)$ on $\mathfrak{N}\langle\sigma_2\rangle^{(k)}$. On applying Lemma 6.1 to $\sigma_2$, we may assume without loss of generality that the multiplicity of $\sigma_2$ is at most $(p+1)$ ($\dim E \leq p$).

To obtain a sought system $\sigma$, we must add the system $\sigma_1 = \{C_x \cap W_\theta \mid x \in X \setminus X_{p-1}\} : \sigma = \{H_\sigma\} = \sigma_1 \cup \sigma_2$ to $\sigma_2$.

We turn to constructing the partial $\alpha$-realization $q$. Consider the mapping $q_2 : \mathfrak{N}_0 \to L$, $q \restriction \mathfrak{N}\langle\sigma\rangle^{(k)} = m\circ\pi(\sigma,\omega)$, $q_2 \restriction \mathfrak{N}\langle\sigma_2\rangle^{(k+1)} = q_1$ on the subpolytope $\mathfrak{N}_0 = \mathfrak{N}\langle\sigma\rangle^{(k)} \cup \mathfrak{N}\langle\sigma_2\rangle^{(k+1)} \subset \mathfrak{N}\langle\sigma\rangle^{(k+1)}$ (we assume $\pi(\sigma,\theta)$ equal to $\pi(\tilde\theta,\omega)\circ\pi(\theta,\tilde\theta)\circ\pi(\sigma,\theta)$). Since $q_2$ and $m\circ\pi(\sigma,\omega)$ satisfy Michael's $\kappa$-condition and are $\alpha$-realizations, so is $q_2$. We now construct an extension of $q_2$ to $\mathfrak{N}\langle\sigma\rangle^{(k+1)}$, preserving these properties.

Let $\Delta = \langle H_{\sigma_0}, \dots, H_{\sigma_{k+1}}\rangle$ be an arbitrary simplex not lying in $\mathfrak{N}_0$. Among its vertices, there is at least one (say, $H_{\sigma_0}$) not belonging to $\mathfrak{N}\langle\sigma_2\rangle$. We represent $H_{\sigma_0}$ as $C_x \cap W_\theta$, $x \in X \setminus X_{p-1}$.

Denote the images of the vertices $\langle H_{\sigma_i}\rangle$ under the simplicial mapping $\pi(\sigma,\omega)$ by $\langle U_{\omega_i}\rangle \in \mathfrak{N}\langle\omega\rangle$ and demonstrate that

(f) $x \in U_{\omega_i}$ for all $i = 0, 1, \dots, k+1$.

Since $\sigma \succ \{W_\theta\} \succ \tilde\theta \succ \omega$, where $\tilde\theta = \{\mathrm{cl}\, W_\theta\}$, and $\pi(\sigma,\omega) = \pi(\tilde\theta,\omega)\circ\pi(\tilde\theta,\theta)\circ\pi(\sigma,\theta)$; therefore, we can insert the chain $H_{\sigma_i} \subset W_{\theta_i} \subset \mathrm{cl}\, W_{\theta_i} \subset U_{\omega_i}$ between $H_{\sigma_i}$ and $U_{\omega_i}$. On the other hand, $\varnothing \neq H_{\sigma_0} \cap H_{\sigma_i} \subset C_x \cap W_{\theta_i} \subset V\langle x\rangle \cap W_{\theta_i}$, and by (e) $x \in \mathrm{cl}\, W_{\theta_i}$ while $\mathrm{cl}\, W_{\theta_i} \subset U_{\omega_i}$.

It follows from (f) that $\pi(\sigma,\omega)(\Delta) \in \Delta_x^{(k+1)}$. Since $x \in X \setminus X_{p-1}$ and $k \leq p$, the mapping $m_x : \Delta_x^{(k+1)} \to L$ is thus defined which satisfies (b)–(d).

Extending the mapping $q_2$ to $\Delta$ by means of the formula $m_x\circ\pi(\sigma,\omega) \restriction \Delta$, we construct the mapping $q : \mathfrak{N}\langle\sigma\rangle^{(k+1)} \to L$. Let us check that

(g) $q$ is an $\alpha$-realization;

(h) $q$ satisfies Michael's $\kappa$-condition.

It is obvious that every point $z \in \cap(H_{\sigma_i})$ also lies in $C_x \cap W_\theta \subset V\langle x\rangle$. By (c) we thus have $m_x(\Delta_x^{(k+1)}) \subset N(\Phi(z), \kappa)$.

Since $m_x$ is a partial $\alpha$-realization and $\pi(\sigma,\omega)$ is a simplicial mapping, $q \restriction \Delta$ is an $\alpha$-realization.

We have constructed an $\alpha$-realization $q : \mathfrak{N}\langle\sigma\rangle^{(k+1)} \to L$ that satisfies Michael's $\kappa$-condition and meets the equality $q \restriction \mathfrak{N}\langle\sigma\rangle^{(k)} = q \restriction \mathfrak{N}\langle\sigma\rangle^{(k)} = m\circ\pi(\sigma,\omega)$, which was required for the proof of Proposition $\mathfrak{B}$.

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