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Division and k-th root theorems for Q-manifolds

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Abstract We prove that a locally compact ANR-space X is a Q-manifold if and only if it has the Disjoint Disk Property (DDP), all points of X are homological Z_∞-points and X has the countable-dimensional approximation property (cd-AP), which means that each map $f : K \to X$ of a compact polyhedron can be approximated by a map with the countable-dimensional image. As an application we prove that a space X with DDP and cd-AP is a Q-manifold if some finite power of X is a Q-manifold. If some finite power of a space X with cd-AP is a Q-manifold, then X^2 and $X \times [0, 1]$ are Q-manifolds as well. We construct a countable family $\mathcal{X}$ of spaces with DDP and cd-AP such that no space $X \in \mathcal{X}$ is homeomorphic to the Hilbert cube Q whereas the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q. We also show that no uncountable family $\mathcal{X}$ with such properties exists.

Keywords: Hilbert cube, Cantor cube, Tychonov cube, ANR, infinite-dimensional manifold, Disjoint Disk Property, cell-like map.

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1 Division and k-th root Theorems for Cantor and Tychonov cubes

It is obvious that the powers X^k, M^k of two homeomorphic topological spaces X, M are homeomorphic as well. For a certain "nice" space M the converse implication is also true: a space X is homeomorphic to M if for some finite number k the powers X^k and M^k are homeomorphic. Results of this type will be referred to as k-th root theorems. A typical example of such a theorem is the k-th root theorem for Cantor and Tychonov cubes.

Theorem 1 (k-th Root theorem for Cantor and Tychonov cubes). *Let M be either a Cantor cube $\{0, 1\}^\tau$ with $\tau \geqslant 1$ or a Tychonov cube $[0, 1]^\kappa$ with $\kappa \geqslant \aleph_1$. A topological space X is homeomorphic to M if and only if for some finite number $k \in \mathbb{N}$ the powers X^k and M^k are homeomorphic.*

By induction this theorem can be easily derived from the following

Theorem 2 (Division theorem for Cantor and Tychonov cubes). *Let M be either a Cantor cube $\{0, 1\}^\tau$ with $\tau \geqslant \aleph_0$ or a Tychonov cube $[0, 1]^\kappa$ with $\kappa \geqslant \aleph_1$. If the product $X \times Y$ of two spaces X, Y is homeomorphic to M, then X or Y is homeomorphic to M.*

This theorem can be easily derived from the famous topological characterizations of Cantor and Tychonov cubes, due to Ščepin (see [1] and [2]).

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Theorem 3 (Ščepin). *Let X be a compact Hausdorff space.*

1) X is homeomorphic to a Cantor cube $\{0,1\}^\tau$ of weight $\tau \geqslant \aleph_0$ if and only if X is a uniform-by-character zero-dimensional AE(0)*-space of weight $w(X) = \tau$;*

2) X is homeomorphic to a Tychonov cube $[0,1]^\tau$ of weight $\tau \geqslant \aleph_1$ if and only if X is a uniform-by-character AE*-space of weight $w(X) = \tau$.*

We recall that a topological space X is called an AE-space (resp. AE(0)-space) if any continuous map $f : B \to X$ defined on a closed subset B of a (zero-dimensional) compact Hausdorff space A can be extended to a continuous map $\bar{f} : A \to X$.

A topological space X is called the uniform-by-character if the character of X at each point $x \in X$ is equal to some fixed cardinal κ. We recall that the character of X at a point $x \in X$ is the smallest size $|\mathcal{B}|$ of a neighborhood base at x.

For $\tau = \aleph_0$ Ščepin's characterization of the Cantor cubes turns into the classical Brouwer characterization of the Cantor set: a space X is homeomorphic to the Cantor set $\{0,1\}^\omega$ if and only if X is a zero-dimensional compact metrizable space without isolated points.

Now we are able to derive Theorem 2 from Ščepin's characterization theorem. Assume that $M = \{0,1\}^\tau$ is a Cantor cube with $\tau \geqslant \aleph_0$ and let X, Y be two spaces whose product $X \times Y$ is homeomorphic to M.

Then X, Y are compact zero-dimensional AE(0)-spaces, being the retracts of the product $X \times Y$. Since all points of the product $X \times Y$ have character τ, either X or Y contains no point of character $< \tau$. We loose no generality assuming that X is such a space. Then X, being a uniform-by-character compact zero-dimensional AE(0)-space of weight τ, is homeomorphic to the Cantor cube $\{0,1\}^\tau = M$, according to the Ščepin characterization of $\{0,1\}^\tau$. This completes the proof of the division theorem for the Cantor cubes.

By analogy, the division theorem for the Tychonov cubes can be derived from Ščepin's characterization of the Tychonov cubes.

2 Division and k-th root theorems for the Hilbert cube

From now on all topological spaces are separable and metrizable. Observe that Theorems 1 and 2 do not cover the case of the Hilbert cube $Q = I^\omega$, where $I = [0,1]$. This is not incidental because without any restrictions the k-th root and division theorems for the Hilbert cube are not true. A suitable counterexample is due to Singh[3] who constructed a compact absolute retract S such that $S \times S$ and $S \times [0,1]$ are homeomorphic to Q, but S is not homeomorphic to Q. Singh's space S contains no topological copy of the 2-disk I^2 and hence does not possess the Disjoint Disks Property.

We recall that a space X has the Disjoint Disks Property (briefly, DDP) if any two maps $f, g : I^2 \to X$ from a 2-dimensional cube can be uniformly approximated by the maps with disjoint images. This property was introduced by Cannon[4] and is crucial in the topological characterization of finite-dimensional manifolds (see [5]).

In spite of Singh's counterexample, some restricted forms of the k-th root and division theorems still hold for the Hilbert cube. The restrictions involve the Disjoint Disk Property and the countable-dimensional approximation property, which is a special case of $\mathcal{P}$-approximation property.

We shall say that a topological space X has the $\mathcal{P}$-approximation property (briefly, $\mathcal{P}$-AP) where $\mathcal{P}$ is a family of subsets of a space X, if for each map $f : K \to X$ defined on a compact polyhedron K and each open cover $\mathcal{U}$ of X there is a map $f' : K \to X$ such that $f'(K) \in \mathcal{P}$ and f' is $\mathcal{U}$-near f in the sense that for each point $x \in K$ the set $\{f(x), f'(x)\}$ lies in some set $U \in \mathcal{U}$.

If $\mathcal{P}$ is the family of finite-dimensional (resp. countable-dimensional, weakly infinite-dimensional) subspaces of X, then we shall refer to the $\mathcal{P}$-AP as fd-AP (resp. cd-AP, wid-AP). We recall that a space X is countable-dimensional if X is the countable union of finite-dimensional subspaces.

All these approximation properties follow from Borsuk's property (Δ) (see [6, sec. VII]). We recall that a space X has the property (Δ) if for any point $x \in X$ and a neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that any compact subset $K \subset V$ is contractible in a subset $H \subset U$ having the dimension $\dim(H) \leqslant \dim(K) + 1$. It follows from (the proof of) Theorem VII.2.1 of [6] that each metrizable space with the property (Δ) has fd-AP. Therefore these properties are related as follows:

$$(\Delta) \Rightarrow \text{(fd-AP)} \Rightarrow \text{(cd-AP)} \Rightarrow \text{(wid-AP)}.$$

It is clear that the Hilbert cube Q has the property (Δ) and consequently all weaker approximation properties.

On the other hand, it is easy to construct a compact AR without wid-AP: just consider the space $I^2 \cup_\varphi Q$ obtained by gluing the 2-disk to the Hilbert cube Q along a surjective map $\varphi : J \to Q$ of an arc $J \subset I^2 \setminus \partial I^2$. Replacing the Hilbert cube by the 4-dimensional cube I^4 we can construct a 4-dimensional absolute retract without property (Δ). Replacing Q by a countable-dimensional infinite-dimensional (resp. weakly infinite-dimensional uncountable-dimensional) absolute retract we can construct a compact absolute retract having cd-AP but not fd-AP (resp. wid-AP but not cd-AP).

It turns out that the k-th root theorem for the Hilbert cube holds for spaces possessing DDP and cd-AP. The following four theorems are the main results of the paper and will be proved in Sections 6 and 7.

Theorem 4 (k-th root theorem for the Hilbert cube)**.** *A topological space X with* DDP *and* cd-AP *is homeomorphic to the Hilbert cube Q if some finite power X^k of X is homeomorphic to Q.*

This theorem will be applied to proving another

Theorem 5. *Let X be a space having* cd-AP*. If some finite power X^k of X is homeomorphic to Q, then both X^2 and $X \times I$ are homeomorphic to Q.*

The situation with the division theorem for the Hilbert cube is more delicate. On the one hand, we have a negative result.

Theorem 6. *There is a countable family $\mathcal{X}$ of spaces possessing* DDP *and fd-AP such that*

1) *the square $X \times X$ of any space $X \in \mathcal{X}$ is not homeomorphic to Q;*

2) *the product $X \times Y$ of any two different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q.*

On the other hand we have a positive result showing that the family $\mathcal{X}$ from the preceding theorem cannot be uncountable.

Theorem 7 (Collective division theorem for the Hilbert cube). *Let $\mathcal{X}$ be a family of pairwise non-homeomorphic topological spaces possessing* DDP *and* cd-AP*. The family $\mathcal{X}$ contains a topological copy of the Hilbert cube Q provided the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q.*

3 Homological characterizations of the Hilbert cube

The proofs of the collective division and k-th roots theorems for the Hilbert cube rely on the homological characterizations of Q, due to Daverman and Walsh[7]. First we recall some notations.

We use the singular homology $H_*(X;G)$ with coefficients in an abelian group G. By $\widetilde{H}_*(X;G)$ we shall denote the singular homology of X, reduced in dimension zero. If $G = \mathbb{Z}$, then we omit the symbol of the group and will write $H_*(X)$ in place of $H_*(X;\mathbb{Z})$.

A closed subset A of a space X is called

1) a Z_∞-set if every map $f : Q \to X$ can be approximated by maps into $X \setminus A$;

2) a homotopical Z_∞-set if for every open set $U \subset X$ the relative homotopy groups $\pi_k(U, U \setminus A)$ are trivial for all k;

3) a G-homological Z_∞-set if for every open set $U \subset X$ the relative homology groups $\widetilde{H}_k(U, U \setminus A; G)$ are trivial for all k;

4) a homological Z_∞-set if it is a $\mathbb{Z}$-homological Z_∞-set in X.

In [7] the homological Z_∞-sets are referred to as the closed sets of infinite codimension.

A point $x \in X$ is called a (homotopical, homological) Z_∞-point if the singleton $\{x\}$ is a (homotopical, homological) Z_∞-set in X. The excision axiom for singular homology[8, sec.2.20] implies that a point $x \in X$ is a G-homological Z_∞-point if and only if $H_k(X, X \setminus \{x\}; G) = 0$ for all k. It is well-known that each point of the Hilbert cube is a Z_∞-point and consequently, a G-homological Z_∞-point for any non-trivial abelian group G.

Theorem 2.3 of [9] implies that a closed subset A of an ANR-space is a Z_∞-set if and only if it is a homotopical Z_∞-set. Also each homotopical Z_∞-set is a homological Z_∞-set. Many examples of homological Z_∞-sets in Q, which are not homotopical Z_∞-sets, can be constructed using the following fact proved in Corollary 2.4 of [7]:

Proposition 1 (Daverman-Walsh). *Assume that X is a locally compact* ANR *whose any point is a homological Z_∞-point. Then each closed finite-dimensional subset of X is a homological Z_∞-set.*

The proof of this proposition follows from a characterization of G-homological Z_∞-sets proved in Proposition 2.3 of [7].

Proposition 2 (Daverman-Walsh). *A closed subset A of a locally compact* ANR*-space X is a G-homological Z_∞-set if each point $a \in A$ is a G-homological Z_∞-point in X and has arbitrarily small neighborhoods $U_a \subset A$ whose relative boundaries in A are G-homological Z_∞-sets in X.*

In fact, we shall derive a bit more from this proposition. Namely, each closed countable-dimensional subset of Q is a homological Z_∞-set. According to [10, 7.1.9] each completely-metrizable countable-dimensional space X has a transfinite inductive dimension $\mathrm{trind}\,(X) \neq \infty$ defined as follows. We set $\mathrm{trind}\,(X) = -1$ if and only if $X = \emptyset$. Given an ordinal α we set $\mathrm{trind}\,(X) \leqslant \alpha$ if X has a base of the topology consisting of open sets $U \subset X$ whose boundaries

∂U have the transfinite dimension trind$(\partial U) < \alpha$. The transfinite inductive dimension of a space X equals the smallest ordinal α with trind$(X) \leqslant \alpha$ if such ordinal α exists and trind$(X) = \infty$ otherwise.

Proposition 3. *Assume that X is a locally compact ANR whose points all are G-homological Z_∞-points for some non-trivial abelian group G. Then each closed countable-dimensional subset $A \subset X$ is a G-homological Z_∞-set in X.*

Proof. According to [10, 7.1.9], the space A, being completely-metrizable and countable-dimensional, has the transfinite inductive dimension trind$(A) \neq \infty$. So, we shall prove the proposition by transfinite induction on $\alpha =$ trind(A). For $\alpha = -1$ the proposition is trivial. Assume that for some ordinal α the assertion is true for all closed subsets $A \subset X$ with trind$(A) < \alpha$. Assuming that A is a closed subset of X with trind$(A) = \alpha$, we get that A has a base of the topology consisting of open sets $U \subset A$ whose relative boundaries $\partial_A U$ in A have the transfinite dimension trind$(\partial_A U) < \alpha$. By inductive hypothesis each set $\partial_A U$ is a G-homological Z_∞-set in X. Applying Proposition 2 we conclude that A is a G-homological Z_∞-set in X.

We shall say that a space X has Z-AP (resp. HZ-AP) if it has $\mathcal{P}$-AP for the class $\mathcal{P}$ of (homological) Z_∞-sets in X. The latter means that each map $f : K \to X$ from a compact polyhedron can be approximated by a map whose image is a (homological) Z_∞-set in X.

By a Q-manifold we understand a metrizable separable space M such that each point $x \in M$ has an open neighborhood $U \subset M$ homeomorphic to an open subset of Q. It is clear that each Q-manifold is a locally compact ANR. By Theorem 22.1 of [11], each compact contractible Q-manifold is homeomorphic to Q.

The following Z-AP characterization of Q-manifolds is due to Toruńczyk [12].

Theorem 8 (Toruńczyk)**.** *A space X is a Q-manifold if and only if X is a locally compact* ANR *possessing* Z-AP.

A homological version of this characterization was proved by Daverman and Walsh in [7].

Theorem 9 (Daverman-Walsh)**.** *A space X is a Q-manifold if and only if X is a locally compact* ANR *possessing* DDP *and* HZ-AP.

Combining this characterization theorem with Proposition 3 we get a local characterization of Q-manifolds whose fd-AP version can be found in Theorem 6.1 of [7].

Theorem 10. *A locally compact* ANR-*space X is a Q-manifold if and only if*
1) *X has* DDP; 2) *X has* cd-AP*; and* 3) *each point of X is a homological Z_∞-point.*

4 On Cell-like maps between Q-manifolds

In this section we shall apply Theorem 10 to obtaining some new characterizations of Q-manifolds, involving the cell-like maps. We recall that a map $\pi : X \to Y$ is called

1) proper if the preimage $\pi^{-1}(K)$ of every compact set is compact;
2) cell-like if π is proper and the preimage $\pi^{-1}(y)$ of every point $y \in Y$ has trivial shape;
3) countable-dimensional if the preimage $\pi^{-1}(y)$ of every point $y \in Y$ is countable-dimensional.

Theorem 11. *A locally compact* ANR-*space X is a Q-manifold if and only if X has* DDP, cd-AP, *and X is the image of a Q-manifold M under a countable-dimensional cell-like map*

$\pi : M \to X$.

This theorem can be easily derived from Theorem 10 and

Proposition 4. *Let $\pi : M \to X$ be a cell-like map between the locally compact* ANR*s and $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ be the nondegeneracy set of π. Then*

1) *a point $x \in X$ is a homological Z_∞-point in X if its preimage $\pi^{-1}(x)$ is a homological Z_∞-set in M;*

2) *a point $x \in X$ is a homological Z_∞-point in X if its preimage $\pi^{-1}(x)$ is countable-dimensional and each point $z \in \pi^{-1}(x)$ is a homological Z_∞-point in M;*

3) *If the space N_π is finite-dimensional (resp. countable-dimensional) and the space M has fd-AP (resp.* cd-AP*), then X has* fd-AP *(resp.* cd-AP*).*

Proof. 1) The first item is well-known and easily follows from the Approximate Lifting Theorem 16.7 in [6] for cell-like maps.

2) Assume that for some point $x \in X$ the preimage $\pi^{-1}(x)$ is countable-dimensional and each point $z \in \pi^{-1}(x)$ is a homological Z_∞-point. By Proposition 3, the set $\pi^{-1}(x)$ is a homological Z_∞-set in M and by the preceding item the point x is a homological Z_∞-point in X.

3) Assume that N_π is finite-dimensional and M has fd-AP. To show that X has the fd-AP, fix a map $f : K \to X$ of a compact polyhedron and an open cover $\mathcal{U}$ of X. Let $\mathcal{V}$ be an open cover of X whose star $\mathrm{St}(\mathcal{V})$ refines $\mathcal{U}$. By Approximate Lifting Theorem 16.7 in [5] for cell-like maps, there exists a map $g : K \to M$ such that $\pi \circ g$ is $\mathcal{V}$-near f. Since M has fd-AP, the map g can be approximated by a map $g' : K \to M$ such that $g'(K)$ is finite-dimensional and g' is $\pi^{-1}(\mathcal{V})$-near g. Then the map $f' = \pi \circ g'$ is $\mathrm{St}(\mathcal{V})$-near f. It remains to show that $f'(K)$ is finite-dimensional. Write $f'(K)$ as the union $f'(K) = \big(f'(K) \cap N_\pi\big) \cup \big(f'(K) \setminus N_\pi\big)$. The space $f'(K) \cap N_\pi$ is finite-dimensional. On the other hand, the restriction of $\pi|M \setminus \pi^{-1}(N_\pi) : M \setminus \pi^{-1}(N_\pi) \to X \setminus N_\pi$ of π, being a proper injective map, is an embedding and thus $\dim(f'(K) \setminus N_\pi) = \dim\big(\pi(g'(K)) \setminus \pi^{-1}(N_\pi)\big) = \dim(g'(K) \setminus \pi^{-1}(N_\pi)) \leqslant \dim(g'(K)) < \infty$. Then $f'(K)$ is finite-dimensional, being the union of two finite-dimensional subspaces (see Theorem 1.5.8 in [10]).

The cd-case of (3) can be proved analogously.

Theorem 10 combined with 3) in Proposition 4 implies another cell-like characterization of Q-manifolds.

Theorem 12. *A space X with* DDP *is a Q-manifold if and only if X is the image of a Q-manifold M under a countable-dimensional cell-like map $\pi : M \to X$ whose non-degeneracy set $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ is countable-dimensional.*

5 Disjoint Disk-Arc Property

Singh's example of a fake Hilbert cube[3] shows that HZ-AP does not imply DDP and hence DDP cannot be omitted from Theorems 9–12. Nevertheless, HZ-AP implies DDAP, a bit weaker property than DDP.

Following [5] we say that a space X has the disjoint disk-arc property (briefly DDAP) if any maps $f : I^2 \to X$, $g : I \to X$ can be approximated by maps $f' : I^2 \to X$ and $g' : I \to X$ with $f'(I^2) \cap g'(I) = \emptyset$. The following lemma linking DDP with DDAP can be proved by the argument of Proposition 26.6 of [5].

Proposition 5. *If an* ANR*-space X has* DDAP*, then for any* ANR*-space Y having no* *isolated point the product $X \times Y$ has* DDP.

Proposition 6. *Each space X with* HZ-AP *has* DDAP.

Proof. Take any maps $f : I^2 \to X$ and $g : I \to X$. Since X has HZ-AP, the map f can be approximated by a map $f' : I^2 \to X$ whose image $Z = f'(I^2)$ is a homological Z_∞-set in X. Next, we shall approximate the map g'. Given an open cover $\mathcal{U}$ of X we will construct a map $g' : I \to X \setminus Z$ which is $\mathcal{U}$-near g in the sense that for any point $t \in I$ the set $\{g(t), g'(t)\}$ lies in some $U \in \mathcal{U}$. By the compactness of the interval $[0, 1]$ there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for every $i \leqslant n$ the image $g([t_{i-1}, t_i])$ lies in some set $U_i \in \mathcal{U}$.

Since $H_0(U_i, U_i \setminus Z) = 0$, the path-connected component of U_i containing the point $g(t_i)$ meets the set $U_i \setminus Z$ at some point x_i. We claim that the points x_{i-1}, x_i lie in the same path-connected component of $U_i \setminus Z$. If the converse were true, then we would get a nontrivial 0-cycle $\alpha = x_{i-1} - x_i$ in $U_i \setminus Z$. On the other hand, this cycle is the boundary of an obvious 1-chain β in U_i and thus vanishes in the homology group $H_0(U_i)$. But this contradicts the exactness of the following sequence $0 = H_1(U_i, U_i \setminus Z) \to H_0(U_i \setminus Z) \to H_0(U_i)$ for the pair $(U_i, U_i \setminus Z)$.

Therefore x_{i-1}, x_i lie in the same path-connected component of $U_i \setminus Z$, ensuring the existence of a continuous map $g_i : [t_{i-1}, t_i] \to U_i \setminus Z$ with $g_i(t_{i-1}) = x_{i-1}$ and $g_i(t_i) = x_i$. The maps g_i, $i \leqslant m$, compose a single continuous map $g' : [0, 1] \to X \setminus Z = X \setminus f'(I^2)$ which is $\mathcal{U}$-near to g, confirming the DDAP of X.

Therefore for a locally compact ANR-space whose points are all homological Z_∞-points we have the following implications between different $\mathcal{P}$-approximation properties:

$$\begin{array}{ccccccc} \text{(wid-AP)} & \longleftarrow & \text{(cd-AP)} & \longrightarrow & \text{(HZ-AP)} & \longrightarrow & \text{(DDAP)} \\ & & \uparrow & & \uparrow & & \uparrow \\ & & (\Delta) & \longleftarrow & \text{(Z-AP)} & \longrightarrow & \text{(DDP).} \end{array}$$

6 Division and k-th Root Theorems for Q-manifolds

In this section we shall prove k-th root and collective division theorems for Q-manifolds, whose partial cases are Theorems 4 and 7. The proofs of these theorems essentially rely on the characterization Theorem 10 and the Künneth formula expressing the homology of the product $X \times Y$ of two spaces via the homology of the factors X, Y. We shall use the following relative version of the Künneth formula that can be found in Theorem 10 of [13, sec. 5.3].

Relative Künneth formula. *For open sets $U \subset X$, $V \subset Y$ in topological spaces X, Y and a non-negative integer n the following exact sequence holds*:

$$0 \to [H(X, U) \otimes H(Y, V)]_n \to H_n(X \times Y, X \times V \cup U \times Y) \to [H(X, U) * H(Y, V)]_{n-1} \to 0.$$

Here

$$[H(X, U) \otimes H(Y, V)]_n = \oplus_{i+j=n} H_i(X, U) \otimes H_j(Y, V),$$
$$[H(X, U) * H(Y, V)]_{n-1} = \oplus_{i+j=n-1} H_i(X, U) * H_j(Y, V),$$

where $G \otimes H$ and $G * H$ stand for the tensor and torsion products of abelian groups G, H, respectively (see [13]). We need three elementary properties of tensor and torsion products:

1) $G \otimes \mathbb{Z}$ is isomorphic to G;

2) $G \otimes H \neq 0$ if both G and H contain elements of infinite order; and

3) $G * H$ contains an element of finite order n if and only if both G and H contain such an element (see Exercise 6 in [8, p. 267]).

We shall apply the Künneth formula to prove:

Lemma 1. *If x is a homological Z_∞-point of a space X, then for any point y of a space Y the pair (x, y) is a homological Z_∞-point in $X \times Y$.*

Proof. We need to check that the groups $H_k(X \times Y, X \times Y \setminus \{(x, y)\})$ are trivial for all k. This trivially follows from the relative Künneth formula and the triviality of the homology groups $H_i(X, X \setminus \{x\})$.

Our next corollary of the Künneth formula is less trivial.

Proposition 7. *A closed subset A of a space X is a homological Z_∞-set in X provided that A^k is a homological Z_∞-set in X^k for some finite number k.*

Proof. First we show that the groups $H_n(U, U \setminus A)$ are the torsion groups for all $n \in \omega$ and all open sets $U \subset X$. Otherwise, for some n we can find an element $\alpha \in H_n(U, U \setminus A)$ of infinite order. Then $\alpha \otimes \alpha$ is a non-zero element of infinite order in $H_n(U, U \setminus A) \otimes H_n(U, U \setminus A)$. Now the Künneth formula implies that the homology group $H_{2n}(U^2, U^2 \setminus A^2)$ has a non-zero element of infinite order. Proceeding by induction we can show that for each $i \in \mathbb{N}$ the homology group $H_{in}(U^i, U^i \setminus A^i)$ contains a non-zero element of infinite order which is not possible as A^k is a homological Z_∞-set in X^k.

This proves that all the homology groups $H_n(U, U \setminus A)$ are the torsion groups. Assuming that A is not a homological Z_∞-point, we can find $n \in \omega$ and an open set $U \subset X$ such that $H_n(U, U \setminus A)$ is not trivial and thus contains an element of a prime order p. Then the torsion product $H_n(U, U \setminus A) * H_n(U, U \setminus A)$ also contains an element of order p. The exact sequence

$$0 \to [H(U, U \setminus A) \otimes H(U, U \setminus A)]_{2n+1} \to H_{2n+1}(U^2, U^2 \setminus A^2) \to [H(U, U \setminus A) * H(U, U \setminus A)]_{2n} \to 0$$

from the Künneth formula implies that the group $H_{2n+1}(U^2, U^2 \setminus A^2)$ contains an element of order p (here we also use the fact that the tensor product $[H(U, U \setminus A) \otimes H(U, U \setminus A)]_{2n+1}$ is a torsion group). Repeating this argument again, we can prove that the group $H_{3n+2}(U^3, U^3 \setminus A^3)$ contains an element of order p. Proceeding by induction we can prove that for any $i \in \mathbb{N}$ the group $H_{in+i-1}(U^i, U^i \setminus A^i)$ contains a non-zero element of order p which is not possible as A^k is a homological Z_∞-set in X^k.

Combining Theorem 10 with Proposition 7 we obtain the k-th root theorem for Q-manifolds.

Theorem 13. *A space X with* DDP *and* cd-AP *is a Q-manifold if and only if the power X^k is a Q-manifold for some finite k.*

Since each compact contractible Q-manifold is homeomorphic to Q, this theorem implies the k-th root theorem 4 for the Hilbert cube. For the same reason Theorem 5 follows from

Theorem 14. *If for some finite power a space X with* cd-AP *is a Q-manifold, then both X^2 and $X \times I$ are Q-manifolds.*

Proof. Assuming that X^k is a Q-manifold for some finite k, we conclude that X is a locally compact ANR and each point of X is a homological Z_∞-point, see Proposition 7. This property combined with cd-AP of X implies HZ-AP by Proposition 3. In its turn, HZ-AP of X implies DDAP for X by Proposition 6 while DDAP of X implies DDP for X^2 and $X \times I$ according to Proposition 5. By Lemma 1, all points in the spaces X^2 and $X \times I$ are homological Z_∞-points. Therefore X^2 and $X \times I$ are locally compact ANR-spaces possessing DDP, cd-AP, and having all points as the homological Z_∞-points. By Theorem 10, these spaces are Q-manifolds.

Since each compact contractible Q-manifold is homeomorphic to Q, the collective division theorem 7 for the Hilbert cube follows from

Proposition 8. *An uncountable family $\mathcal{X}$ of topologically distinct spaces contains a Q-manifold provided that*

1) *each space X has* DDP *and* cd-AP;

2) *the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is a Q-manifold.*

Proof. It follows from 2) that each space $X \in \mathcal{X}$ is a locally compact ANR. Suppose to the contrary that none of the spaces $X \in \mathcal{X}$ is a Q-manifold and apply the characterization theorem 10 to find a point $a_X \in X$ which fails to be a homological Z_∞-point in X. This means that the homology group $H_k(X, X \setminus \{a_X\})$ is not trivial for some $k = k(X)$. Since the family $\mathcal{X}$ is uncountable there are two different spaces $X, Y \in \mathcal{X}$ and two numbers k, n such that the groups $H_k(X, X \setminus \{a_X\})$ and $H_n(Y, Y \setminus \{a_Y\})$ contain the elements of the same order p, where either $p = \infty$ or p is a prime number. If $p = \infty$, then the tensor product $H_k(X, X \setminus \{a_X\}) \otimes H_n(Y, Y \setminus \{a_Y\})$ is not trivial and hence the group $H_{k+n}(X \times Y, X \times Y \setminus \{(a_X, a_Y)\})$ is not trivial by the Künneth formula, which is impossible since (a_X, a_Y) is a (homological) Z_∞-point in the Q-manifold $X \times Y$.

If p is a prime number, then the torsion product $H_k(X, X \setminus \{a_X\}) * H_n(Y, Y \setminus \{a_Y\})$ is not trivial and hence the group $H_{k+n+1}(X \times Y, X \times Y \setminus \{(a_X, a_Y)\})$ is not trivial by the Künneth formula, which is impossible since (a_X, a_Y) is a (homological) Z_∞-point in the Q-manifold $X \times Y$.

The obtained contradiction shows that some space $X \in \mathcal{X}$ must be a Q-manifold.

7 Examples of fake Hilbert cubes

In this section we survey some known examples of the fake Hilbert cubes. The first example is from [3].

Example 1 (Singh)**.** There exists a compact space S possessing the following properties:

1) S is a compact AR;

2) S is the image of Q under a cell-like map $\pi : Q \to S$ such that the set $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ is countable and the preimage $\pi^{-1}(y)$ of every point $y \in N_\pi$ is an arc;

3) Each compact ANR-subspace of dimension $\geqslant 2$ in S coincides with S;

4) Each point of S is a homological Z_∞-point and each point $x \in S \setminus N_\pi$ is a Z_∞-point;

5) S has fd-AP and consequently has HZ-AP and DDAP;

6) S fails to have DDP and hence is not homeomorphic to Q; and

7) S^2 and $S \times I$ are homeomorphic to Q.

The items 1)–4) and 6) were established by Singh in [3] while 5) and 7) follow from the preceding items and Propositions 4–5.

Another example of a fake Hilbert cube was constructed by Daverman and Walsh in [7, 9.3].

Example 2 (Daverman-Walsh). There exists a compact space X possessing the following properties:

1) X is a compact AR;

2) X is the image of Q under a cell-like map $\pi : Q \to X$ whose non-degeneracy set N_π is countable;

3) Each point of X is a Z_∞-point and thus a homological Z_∞-point;

4) X has fd-AP and consequently has HZ-AP and DDAP;

5) X fails to have DDP and hence is not homeomorphic to Q;

6) X^2 and $X \times I$ are homeomorphic to Q.

Our last example yields a bit more than that required in Theorem 6. We shall construct a series of compact absolute retracts (Λ_p) with fd-AP and DDP, parameterized by prime numbers p such that no finite power of Λ_p is homeomorphic to Q while all the products $\Lambda_p \times \Lambda_q$ for distinct $p \neq q$ are homeomorphic to the Hilbert cube.

Below $\mathbb{Z}_{p^\infty} = \{z \in \mathbb{C} : z^{p^k} = 1 \text{ for some } k \in \omega\}$ denotes the quasicyclic p-group.

Example 3. There is a family of pointed compact absolute retracts $(\Lambda_p, *_p)$ indexed by prime numbers p such that

1) $\Lambda_p \setminus *_p$ is a Q-manifold with the unique non-trivial homology group $H_1(\Lambda_p \setminus *_p) = \mathbb{Z}_{p^\infty}$;

2) the point $*_p$ is not a homological Z_∞-point in Λ_p;

3) no finite power Λ_p^k is a Q-manifold;

4) the space Λ_p has the property (Δ) and consequently has cd-AP;

5) the space Λ_p has DDAP and hence the square Λ_p^2 has DDP;

6) $\Lambda_p \times \Lambda_q$ is homeomorphic to the Hilbert cube for any prime numbers $p \neq q$.

Proof. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ stand for the unit circle in the complex plane. Given a prime number p consider the space $X_p = \omega \times [0,1] \times \mathbb{T}$ and its quotient space $Y_p = X_p/_\sim$ by the equivalence relation $\sim$ defined as follows: $(n,t,x) \sim (m,\tau,y)$ if and only if one of the following conditions holds:

1) $(n,t,x) = (m,\tau,y)$; 2) $m = n$, $t = \tau = 1$ and $x^p = y^p$; 3) $m = n+1$, $t = 1$, $\tau = 0$, and $x^p = y$; 4) $n = m+1$, $t = 0$, $\tau = 1$, and $y^p = x$.

Thus the space Y_p consists of an infinite sequence of cylinders of the map $z^p : \mathbb{T} \to \mathbb{T}$, glued together. It is routine to check that the higher homology groups $H_k(Y_p)$, $k > 1$, of the space Y_p are trivial while $H_1(Y_p) = \mathbb{Z}_{p^\infty}$. It is easy to see that the one-point compactification $\widetilde{Y}_p = Y_p \cup \{\infty\}$ is a two-dimensional absolute retract. Then the quotient space $\Lambda_p = \widetilde{Y}_p \times Q/\{\infty\} \times Q$ is an absolute retract, too. The point $*_p = \{\{\infty\} \times Q\}$ is the singular point of Λ_p.

We now check that the pointed spaces $(\Lambda_p, *_p)$ satisfy the following conditions 1)–6):

1) By Edwards' ANR-Theorem[11,44.1], the complement $\Lambda_p \setminus \{*_p\} = Y_p \times Q$ is a Q-manifold. Being homotopy equivalent to Y_p, it has a unique non-trivial homology group $H_1(\Lambda_p \setminus \{*_p\}) = H_1(Y_p) = \mathbb{Z}_{p^\infty}$.

2) The exact sequence of the pair $(\Lambda_p, \Lambda_p \setminus \{*_p\})$

$$0 = H_2(\Lambda_p) \to H_2(\Lambda_p, \Lambda_p \setminus \{*_p\}) \to H_1(\Lambda_p \setminus \{*_p\}) \to H_1(\Lambda_p) = 0$$

implies that $H_2(\Lambda_p, \Lambda_p \setminus \{*_p\}) = \mathbb{Z}_{p^\infty} \neq 0$ and thus $*_p$ fails to be a homological Z_∞-point in Λ_p.

3) By Proposition 7, the singleton $\{*_p\}^k$ fails to be a homological Z_∞-set in Λ_p^k for any finite

k, and hence Λ_p^k cannot be a Q-manifold.

4) The first item implies that Λ_p has the property (Δ) at each point $x \neq *_p$. To check that property at the point $*_p$, take any neighborhood $U \subset \Lambda_p$ of $*_p$ and find a neighborhood $V \subset U$ of $*_p$ that is contractible in U (such a neighborhood exists because Λ_p is an AR). Given any compact set $K \subset V$ let $h : K \times [0,1] \to U$ be a map contracting K to $*_p$ (in the sense that $h(x,0) = x$ and $h(x,1) = *_p$ for all $x \in K$). Consider the closed set $F = h^{-1}(*_p) \subset K \times [0,1]$ and the restriction $g = h|K \times I \setminus F$ of h, mapping the locally compact space $K \times I \setminus F$ into the Q-manifold $U \setminus \{*_p\}$. Applying Theorem 18.2 of [11], approximate g by an embedding $g' : K \times I \setminus F \to U \setminus \{*_p\}$ so near the map g that the map $\tilde{h} : K \times I \to U$ defined by $\tilde{h}|K \times I \setminus F = g'$ and $\tilde{h}|F = h|F$ is continuous. Then $\tilde{h} : K \times I \to U$ is a contraction of K in U such that $\dim(\tilde{h}(K \times I)) \leqslant \dim(K \times I) \leqslant \dim(K) + 1$, which means that Λ_p has the property (Δ). By Theorem VII.2.1 of [6] this space has fd-AP and hence cd-AP.

5) To prove the DDAP of Λ_p, fix an open cover $\mathcal{U}$ of Λ_p and two maps $f : I^2 \to \Lambda_p$ and $g : I \to \Lambda_p$. Repeating the argument of the preceding item, we can approximate f by a map $f' : I^2 \to \Lambda_p$ such that $f'(I^2) \setminus \{*_p\}$ is a Z_∞-set in the Q-manifold $\Lambda_p \setminus \{*_p\}$. Because the point $*_p$ is not locally separating in Λ_p the map g can be approximated by a map $g' : I \to \Lambda_p \setminus \{*_p\}$. Moreover, since $f'(I^2) \setminus \{*_p\}$ is a Z_∞-set in $\Lambda_p \setminus \{*_p\}$, we can additionally assume that $g'(I) \cap f'(I^2) = \emptyset$, which means that Λ_p has DDAP. By Proposition 5, the square Λ_p^2 has DDP.

6) Finally, we shall prove that for the distinct prime numbers $p \neq q$ the product $\Lambda_p \times \Lambda_q$ is homeomorphic to the Hilbert cube Q. Being the product of two spaces with cd-AP, this space has cd-AP. By Proposition 5 this product has DDP. According to Theorem 10 it remains to check that each point (x,y) of $\Lambda_p \times \Lambda_q$ is a homological Z_∞-point. This is trivial if $(x,y) \neq (*_p, *_q)$ (cf. Lemma 1). In case $(x,y) = (*_p, *_q)$ we may use item 1), the equality $\mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_{q^\infty} = 0 = \mathbb{Z}_{p^\infty} * \mathbb{Z}_{q^\infty}$, and the Künneth formula to show that $(*_p, *_q)$ is a homological Z_∞-point in $\Lambda_p \times \Lambda_q$.

8 Some open problems

Problem 1. Can cd-AP in the characterization theorem 10 be replaced by wid-AP?

Problem 1 is related to another

Problem 2. Is each closed weakly-infinite dimensional subset A of Q a homological Z_∞-set in Q?

Note that the k-th root and division theorems also hold for some non-locally compact spaces, for example for the Baire space $\mathbb{N}^\omega$.

Theorem 15 (Division Theorem for the Baire space)**.** *If the product $X \times Y$ of two spaces is homeomorphic to $\mathbb{N}^\omega$, then X or Y is homeomorphic to $\mathbb{N}^\omega$.*

This theorem easily follows from a topological characterization of the Baire space $\mathbb{N}^\omega$ due to Aleksandrov and Urysohn (see [14, 7.7]): A topological space X is homeomorphic to $\mathbb{N}^\omega$ if and only if X is a Polish zero-dimensional nowhere locally compact space.

In light of this result it is natural to ask if the k-th root and division theorems are true for the countable product $s = (0,1)^\omega$ of open intervals. As expected, the answer is negative.

Example 4. Take an arc $J \subset Q$ which is not a Z_∞-set in Q and consider the quotient map $\pi : Q \to Q/J$. Then $X = \pi(s)$ is not homomorphic to s but its square X^2 is homeomorphic to

s. This can be proved by the argument of [15].

Nonetheless it may happen that the k-th root and division theorems for s hold in some restricted form.

Problem 3. Find the conditions on a space X guaranteeing that X is homeomorphic to s if some finite power X^k is homeomorphic to s.

Observe that the finite power is an example of a normal functor on the category of the compact Hausdorff spaces, see [16]. Can the k-th root theorem for the Hilbert cube be extended to some functor distinct from the functor of finite power?

Problem 4. Let $F : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$ be a functor such that a compact space X with DDP and cd-AP is homeomorphic to Q if $F(X)$ is homeomorphic to Q. Is F isomorphic to a power functor?

Even for the functor $F = SP^2$ of a symmetric square the answer is unknown. Let us recall that the symmetric square of a compact space X is the quotient space $X^2/_\sim$ by the equivalence relation $(x, y) \sim (y, x)$.

Problem 5. Is a compact AR-space X with DDP and cd-AP homeomorphic to Q if its symmetric square $SP^2(X)$ is homeomorphic to Q?

In light of this problem let us mention that the quotient space $X = Q \times [-1, 1]/Q \times \{0\}$ is an AR-space with the property (Δ) whose symmetric square $SP^2(X)$ is homeomorphic to Q, see [17]. However the space X contains a separating point and hence fails to have DDP and be homeomorphic to Q.

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