

ON THE MENGER COVERING PROPERTY AND D -SPACES

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ABSTRACT. The main results of this paper are:

- It is consistent that every subparacompact space X of size ω_1 is a D -space.
- If there exists a Michael space, then all productively Lindelöf spaces have the Menger property and, therefore, are D -spaces.
- Every locally D -space which admits a σ -locally finite cover by Lindelöf spaces is a D -space.

1. INTRODUCTION

A *neighbourhood assignment* for a topological space X is a function N from X to the topology of X such that $x \in N(x)$ for all x . A topological space X is said to be a D -space [6] if for every neighbourhood assignment N for X there exists a closed and discrete subset $A \subset X$ such that $N(A) = \bigcup_{x \in A} N(x) = X$.

It is unknown whether paracompact (even Lindelöf) spaces are D -spaces. Our first result in this note answers [7, Problem 3.8] in the affirmative and may be thought of as a very partial solution to this problem.¹

Our second result shows that the affirmative answer to [19, Problem 2.6], which asks whether all productively Lindelöf spaces are D -spaces, is consistent. It is worth mentioning that our premises (i.e., the existence of a Michael space) are not known to be inconsistent.

Our third result is a common generalization of two theorems from [10].

Most of our proofs use either the recent important result of Aurichi [2] asserting that every topological space with the Menger property is a D -space or the ideas from its proof. We consider only regular topological spaces. For the definitions of small cardinals \mathfrak{d} and $\text{cov}(\mathcal{M})$ used in this paper we refer the reader to [22].

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¹While completing this manuscript, we learned that this result was independently obtained by Hang Zhang and Wei-Xue Shi; see [15].

2. SUBPARACOMPACT SPACES OF SIZE ω_1

Following [4] we say that a topological space X has the property E_ω^* if for every sequence $\langle u_n : n \in \omega \rangle$ of countable open covers of X there exists a sequence $\langle v_n : n \in \omega \rangle$ such that $v_n \in [u_n]^{<\omega}$ and $\bigcup_{n \in \omega} v_n = X$. In the realm of Lindelöf spaces the property E_ω^* is usually called the *Menger property* or $\bigcup_{fin}(\mathcal{O}, \mathcal{O})$; see [21] and references therein.

We say that a topological space X has property D_ω if for every neighbourhood assignment N there exists a countable collection $\{A_n : n \in \omega\}$ of closed discrete subsets of X such that $X = \bigcup_{n \in \omega} N(A_n)$. Observe that the property D_ω is inherited by all closed subsets.

The following theorem is the main result of this section.

Theorem 2.1. *Suppose that a topological space X has properties D_ω and E_ω^* . Then X is a D -space.*

The proof of Theorem 2.1 is analogous to the proof of [2, Proposition 2.6]. In particular, it uses the following game of length ω on a topological space X : On the n th move player I chooses a countable open cover $u_n = \{U_{n,k} : k \in \omega\}$ such that $U_{n,k} \subset U_{n,k+1}$ for all $k \in \omega$, and player II responds by choosing a natural number k_n . Player II wins the game if $\bigcup_{n \in \omega} U_{n,k_n} = X$. Otherwise, player I wins. We shall call this game an E_ω^* -game. In the realm of Lindelöf spaces this game is known under the name *Menger game*. It is well known that a Lindelöf space X has the property E_ω^* if and only if the first player has no winning strategy in the E_ω^* -game on X ; see [8, 14]. The proof of [14, Theorem 13] also works without any change for non-Lindelöf spaces.

Proposition 2.2. *A topological space X has the property E_ω^* if and only if the first player has no winning strategy in the E_ω^* -game.*

A strategy of the first player in the E_ω^* -game may be thought of as a map $\Upsilon : \omega^{<\omega} \rightarrow \mathcal{O}(X)$, where $\mathcal{O}(X)$ stands for the collection of all countable open covers of X . The strategy Υ is winning if $X \neq \bigcup_{n \in \omega} U_{z \upharpoonright n, z(n)}$ for all $z \in \omega^\omega$, where $\Upsilon(s) = \{U_{s,k} : k \in \omega\} \in \mathcal{O}(X)$.

We are in a position now to present the proof of Theorem 2.1.

Proof. We shall define a strategy $\Upsilon : X \rightarrow \mathcal{O}(X)$ of the player I in the E_ω^* -game on X as follows. Set $F_\emptyset = X$. The property D_ω yields an increasing sequence $\langle A_{\emptyset,k} : k \in \omega \rangle$ of closed discrete subsets of F_\emptyset such that $X = \bigcup_{k \in \omega} N(A_{\emptyset,k})$. Set $\Upsilon(\emptyset) = u_\emptyset = \{N(A_{\emptyset,k}) : k \in \omega\}$.

Suppose that for some $m \in \omega$ and all $s \in \omega^{\leq m}$ we have already defined a closed subset F_s of X , an increasing sequence $\langle A_{s,k} : k \in \omega \rangle$ of closed discrete subsets of F_s , and a countable open cover $\Upsilon(s) = u_s$ of X such that $u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\}$.

Fix $s \in \omega^{m+1}$. Since X has the property D_ω , so does its closed subspace $F_s := X \setminus \bigcup_{i < m+1} N(A_{s \upharpoonright i, s(i)})$, and hence there exists an increasing sequence $\langle A_{s,k} : k \in \omega \rangle$ of closed discrete subsets of F_s such that $F_s \subset \bigcup_{k \in \omega} N(A_{s,k})$. Set $\Upsilon(s) = u_s = \{(X \setminus F_s) \cup N(A_{s,k}) : k \in \omega\}$. This completes the definition of Υ .

Since X has the property E_ω^* , Υ is not winning. Thus there exists $z \in \omega^\omega$ such that $X = \bigcup_{n \in \omega} (X \setminus F_{z \upharpoonright n}) \cup N(A_{z \upharpoonright n, z(n)})$. By the inductive construction, $X \setminus F_\emptyset = \emptyset$ and $X \setminus F_{z \upharpoonright n} = \bigcup_{i < n} N(A_{z \upharpoonright i, z(i)})$ for all $n > 0$. It follows from above that $X = \bigcup_{n \in \omega} N(A_{z \upharpoonright n, z(n)})$. In addition, $A_{z \upharpoonright n, z(n)} \subset F_{z \upharpoonright n} = X \setminus \bigcup_{i < n} N(A_{z \upharpoonright i, z(i)})$

for all $n > 0$, which implies that $A := \bigcup_{n \in \omega} A_{z \upharpoonright n, z(n)}$ is a closed discrete subset of X . It suffices to note that $N(A) = X$. \square

We recall from [5] that a topological space X is called *subparacompact* if every open cover of X has a σ -locally finite closed refinement.

Lemma 2.3. *Suppose that X is a subparacompact topological space which can be covered by ω_1 -many of its Lindelöf subspaces. Then X has the property D_ω .²*

In particular, every subparacompact space of size ω_1 has the property D_ω .

Proof. Let $\mathcal{L} = \{L_\xi : \xi < \omega_1\}$ be an increasing cover of X by Lindelöf subspaces, let τ be the topology of X , and let $N : X \rightarrow \tau$ be a neighbourhood assignment. Construct by induction a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ of (possibly empty) countable subsets of X such that

- (i) $L_0 \subset N(C_0)$;
- (ii) $C_\alpha \cap N(\bigcup_{\xi < \alpha} C_\xi) = \emptyset$ for all $\alpha < \omega_1$; and
- (iii) $L_\alpha \setminus N(\bigcup_{\xi < \alpha} C_\xi) \subset N(C_\alpha)$ for all $\alpha < \omega_1$.

Set $C = \bigcup_{\alpha < \omega_1} C_\alpha$. The subparacompactness of X yields a closed cover $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ of X which refines $\mathcal{U} = \{N(x) : x \in C\}$ and such that each \mathcal{F}_n is locally finite. Since every element of \mathcal{U} contains at most countably many elements of C , so do elements of \mathcal{F} . Therefore for every $F \in \mathcal{F}_n$ such that $C \cap F \neq \emptyset$ we can write this intersection in the form $\{x_{n,F,m} : m \in \omega\}$. Now it is easy to see that $A_{n,m} := \{x_{n,F,m} : F \in \mathcal{F}_n, C \cap F \neq \emptyset\}$ is a closed discrete subset of X and $\bigcup_{n,m \in \omega} A_{n,m} = C$. \square

Remark 2.4. What we have actually used in the proof of Lemma 2.3 is the following weakening of subparacompactness: every open cover \mathcal{U} which is closed under unions of its countable subsets admits a σ -locally finite closed refinement. We do not know whether this property is strictly weaker than subparacompactness.

Corollary 2.5. *Let X be a countably tight paracompact topological space of density ω_1 . Then X has the property D_ω .*

Proof. Let $\{x_\alpha : \alpha < \omega_1\}$ be a dense subspace of X . Since X has countable tightness, $X = \bigcup_{\alpha < \omega_1} \overline{\{x_\xi : \xi < \alpha\}}$. It suffices to note that the closure of any countable subspace of a paracompact space is Lindelöf. \square

It is well known [9, Theorem 4.4] (and it easily follows from corresponding definitions) that any Lindelöf space of size $< \mathfrak{d}$ has the Menger property. The same argument shows that every topological space of size $< \mathfrak{d}$ has the property E_ω^* . Combining this with Theorem 2.1 and Lemma 2.3, we get the following corollary, which implies the first of the results mentioned in our abstract.

Corollary 2.6. *Suppose that X is a subparacompact topological space of size $|X| < \mathfrak{d}$ which can be covered by ω_1 -many of its Lindelöf subspaces. Then X is a D -space.*

3. CONCERNING THE EXISTENCE OF A MICHAEL SPACE

A topological space X is said to be *productively Lindelöf* if $X \times Y$ is Lindelöf for all Lindelöf spaces Y . It was asked in [19] whether productively Lindelöf spaces are D -spaces. The positive answer to the above question has been proved consistent,

²By the methods of [15] the submetalindelöfness is sufficient here.

and in a stream of recent papers (see the list of references in [19]) several sufficient set-theoretical conditions were established. The following statement gives a uniform proof for some of these results. In particular, it implies [16, Theorems 5 and 7] and [1, Corollary 4.5] and answers [17, Question 15] in the affirmative.

A Lindelöf space Y is called a *Michael space* if $\omega^\omega \times Y$ is not Lindelöf.

Proposition 3.1. *If there exists a Michael space, then every productively Lindelöf space has the Menger property.*

We refer the reader to [11], where the existence of a Michael space was reformulated in a combinatorial language and a number of set-theoretical conditions guaranteeing the existence of Michael spaces were established.

In the proof of Proposition 3.1 we shall use set-valued maps; see [13]. By a *set-valued map* Φ from a set X into a set Y we understand a map from X into $\mathcal{P}(Y)$ and write $\Phi : X \rightrightarrows Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of Y). For a subset A of X we set $\Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y$. A set-valued map Φ from a topological spaces X to a topological space Y is said to be

- *compact-valued* if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous* if for every open subset V of Y the set $\Phi_C^{-1}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in X .

The proof of the following claim is straightforward.

- Claim 3.2.* (1) Suppose that X, Y are topological spaces, X is Lindelöf, and $\Phi : X \rightrightarrows Y$ is a compact-valued upper semicontinuous map such that $Y = \Phi(X)$. Then Y is Lindelöf.
- (2) If $\Phi_0 : X_0 \rightrightarrows Y_0$ and $\Phi_1 : X_1 \rightrightarrows Y_1$ are compact-valued upper semicontinuous, then so is the map $\Phi_0 \times \Phi_1 : X_0 \times X_1 \rightrightarrows Y_0 \times Y_1$ assigning to each $(x_0, x_1) \in X_0 \times X_1$ the product $\Phi_0(x_0) \times \Phi_1(x_1)$.

Proof of Proposition 3.1. Suppose, contrary to our claim, that X is a productively Lindelöf space which does not have the Menger property and Y is a Michael space. It suffices to show that $X \times Y$ is not Lindelöf.

Indeed, by [23, Theorem 8] there exists a compact-valued upper semicontinuous map $\Phi : X \rightarrow \omega^\omega$ such that $\Phi(X) = \omega^\omega$. By Claim 3.2(2) the product $\omega^\omega \times Y$ is the image of $X \times Y$ under a compact-valued upper semicontinuous map. By the definition of a Michael space, $\omega^\omega \times Y$ is not Lindelöf. By applying Claim 3.2(1) we can conclude that $X \times Y$ is not Lindelöf either. \square

By a result of Tall [16] the existence of a Michael space implies that all productively Lindelöf analytic metrizable spaces are σ -compact. Combining recent results obtained in [1] and [12], we can consistently extend this result to all Σ_2^1 definable subsets of 2^ω .

Theorem 3.3. *Suppose that $\text{cov}(\mathcal{M}) > \omega_1$ and that there exists a Michael space. Then every productively Lindelöf Σ_2^1 definable subset of 2^ω is σ -compact.*

Proof. Let X be a productively Lindelöf Σ_2^1 definable subset of 2^ω .

If X cannot be written as a union of ω_1 -many of its compact subspaces, then it contains a closed copy of ω^ω [12], and hence the existence of the Michael space implies that X is not productively Lindelöf, a contradiction.

Thus X can be written as a union of ω_1 -many of its compact subspaces, and therefore it is σ -compact by [1, Corollary 4.15]. \square

We do not know whether the assumption $\text{cov}(\mathcal{M}) > \omega_1$ can be dropped from Theorem 3.3.

Question 3.4. Suppose that there exists a Michael space. Is every coanalytic productively Lindelöf space σ -compact?

By [18, Proposition 31] the affirmative answer to the question above follows from the Axiom of Projective Determinacy.

4. LOCALLY FINITE UNIONS

Theorem 4.1. *Suppose that X is a locally D -space which admits a σ -locally finite cover by Lindelöf spaces. Then X is a D -space.*

Proof. Let $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$ be a cover of X by Lindelöf subspaces such that \mathcal{F}_n is locally finite. Fix $F \in \mathcal{F}_n$. For every $x \in F$ there exists an open neighbourhood U_x of x such that \bar{U}_x is a D -space. Let C_F be a countable subset of F such that $F \subset \bigcup_{x \in C_F} U_x$. Then $\mathcal{Z}_F = \{\overline{F \cap U_x} : x \in C_F\}$ is a countable cover of F consisting of closed D -subspaces of X such that $F \cap Z$ is dense in Z for all $Z \in \mathcal{Z}_F$. It follows from the above that X admits a σ -locally finite cover consisting of closed D -subspaces. Since a union of a locally finite family of closed D -subspaces is easily seen to be a closed D -subspace, X is a union of an increasing sequence of its closed D -subspaces. Therefore it is a D -space by results of [3]. \square

Corollary 4.2. *If a topological space X admits a σ -locally finite locally countable cover by topological spaces with the Menger property, then it is a D -space. In particular, a locally Lindelöf space admitting a σ -locally finite cover by topological spaces with the Menger property is a D -space.*

Proof. The second part is a direct consequence of the first one since every σ -locally countable family of subspaces of a locally Lindelöf space is locally countable.

To prove the first assertion, note that by local countability every point $x \in X$ has a closed neighbourhood which is a countable union of its subspaces with the Menger property, and hence it has the Menger property itself. Therefore X is a locally D -space. It now suffices to apply Theorem 4.1. \square

It is known that every Lindelöf \mathcal{C} -scattered space is \mathcal{C} -like and that \mathcal{C} -like spaces have the Menger property; see [20, p. 247] and references therein. Thus Corollary 4.2 implies Theorems 2.2 and 3.1 from [10].

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