PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 4, April 2010, Pages 1525–1531
S 0002-9939(09)10217-4
Article electronically published on December 8, 2009

ON NONCONTRACTIBLE COMPACTA WITH TRIVIAL HOMOLOGY AND HOMOTOPY GROUPS

UMED H. KARIMOV AND DUŠAN REPOVŠ

(Communicated by Alexander N. Dranishnikov)

Dedicated to the memory of Professor Evgenij Grigor'evich Sklyarenko (1935-2009)

ABSTRACT. We construct an example of a Peano continuum X such that: (i) X is a one-point compactification of a polyhedron; (ii) X is weakly homotopy equivalent to a point (i.e. $\pi_n(X)$ is trivial for all $n \geq 0$); (iii) X is noncontractible; and (iv) X is homologically and cohomologically locally connected (i.e. X is an HLC and *clc* space). We also prove that all classical homology groups (singular, Čech, and Borel-Moore), all classical cohomology groups (singular and Čech), and all finite-dimensional Hawaiian groups of X are trivial.

1. INTRODUCTION

It is a fundamental fact of homotopy theory that the existence of a weak homotopy equivalence $f : K \to L$ between two CW-complexes K and L implies that f is actually a homotopy equivalence ($K \simeq_w L \Longrightarrow K \simeq L$). Therefore if a CW-complex K has all homotopy groups trivial, then K is necessarily contractible [15].

However, this is no longer true outside the class of CW-complexes, e.g. the Warsaw circle W is an example of a planar noncontractible non-Peano continuum all of whose homotopy groups are trivial (cf. e.g. [11]). The failure of local connectivity of W is crucial, since it is well-known that every planar simply connected Peano continuum must be contractible (cf. e.g. [11]).

In our earlier paper [8] we constructed an example of a noncontractible Peano continuum with trivial homotopy groups. In the present paper we shall construct in some sense a sharper example, namely a noncontractible Peano continuum X which is a one-point compactification of a polyhedron P, which is homologically locally connected (HLC space) and is weakly homotopy equivalent to a point $X \simeq_w *$.

Received by the editors November 25, 2008, and, in revised form, August 7, 2009, and September 18, 2009.

2010 *Mathematics Subject Classification.* Primary 54F15, 55N15; Secondary 54G20, 57M05.

Key words and phrases. Noncontractible compactum, weak homotopy equivalence, reduced complex $\widetilde{K}_{\mathbb{C}}$-theory, admissible spectrum, Peano continuum, infinite-dimensional Hawaiian earrings, Hawaiian group.

We shall also prove that all classical homology groups (singular, Čech, and Borel-Moore), all classical cohomology groups (singular and Čech) and all finite-dimensional Hawaiian earring groups of this space X are trivial. This answers our problem formulated in [8]. We shall also state some new open problems.

2. Preliminaries

We start by fixing some terminology and notation which will be used in the proof. All undefined terms can be found in [4, 6, 8, 9, 15].

For any topological space Z with a base point $z_0 \in Z$ the *reduced* suspension $S(Z, z_0)$ is defined by

$$S(Z, z_0) = (Z \times I)/((Z \times \{0\}) \cup (Z \times \{1\}) \cup (z_0 \times I)),$$

where I is the unit interval $I = [0, 1] \subset \mathbb{R}$, and the *unreduced* suspension $S'(Z)$ of Z is defined by

$$S'(Z) = (Z \times I)/((Z \times \{0\}) \cup (Z \times \{1\})).$$

The *reduced* cone $C(Z, z_0)$ over Z is defined by

$$C(Z, z_0) = (Z \times I)/((Z \times \{1\}) \cup (z_0 \times I)),$$

and the *unreduced* cone $C'(Z)$ over Z is defined by

$$C'(Z) = (Z \times I)/(Z \times \{1\}).$$

The n-dimensional Hawaiian earring ($n = 0, 1, 2, \dots$) is the following subspace of the Euclidean $(n+1)$-space R^{n+1}:

$$\mathcal{H}^n = \{\bar{x} = (x_0, x_1, \dots x_n) \in \mathbb{R}^{n+1} |\ (x_0 - 1/k)^2 + \Sigma_{i=1}^n x_i^2 = (1/k)^2,\ k \in \mathbb{N}\}.$$

In other words, $\mathcal{H}^n$ is a compact bouquet of a countable number of n-dimensional spheres S_k^n of radius $1/k$. The point $\theta = (0, 0, \dots)$ is the base point of $\mathcal{H}^n$. Obviously, $S(\mathcal{H}^n, \theta) \cong \mathcal{H}^{n+1}$ and $\pi_{n+1}(S(\mathcal{H}^n, \theta))$ is an uncountable group, whereas $\pi_{n+1}(S'(\mathcal{H}^n))$ is a countable group.

A *reduced* complex $\widetilde{K}^*_{\mathcal{C}}$-theory is an extraordinary cohomology theory defined on the category of pointed compacta and homotopic mappings with respect to base points. For any compact pair of spaces (X, A) with the base point $x_0 \in A$, (X, A, x_0), there exists the following long exact sequence (cf. e.g. [6, p. 55]):

$$\cdots \to \widetilde{K}^n_{\mathcal{C}}(X/A) \to \widetilde{K}^n_{\mathcal{C}}(X) \to \widetilde{K}^n_{\mathcal{C}}(A) \to \widetilde{K}^{n+1}_{\mathcal{C}}(X/A) \to \cdots . \tag{1}$$

We denote the homotopy classes of mappings with respect to the base point by $[\ ,\ ]$. On the category of connected spaces, there exists a natural isomorphism of cofunctors, for some CW-complex BU (cf. e.g. [9, Theorem 1.32]):

$$\widetilde{K}^0_{\mathcal{C}}(X) \cong [X, BU].$$

Every CW-complex is an absolute neighborhood retract (ANR). Therefore the functor $\widetilde{K}^0_{\mathcal{C}}$ is *continuous*; i.e. if X_i are compact spaces and $X = \varprojlim X_i$, then

$$\widetilde{K}^0_{\mathcal{C}}(X) \cong \varinjlim \widetilde{K}^0_{\mathcal{C}}(X_i).$$

3. The construction of examples and the proofs of the main results

Let $\mathcal{P}$ be an inverse sequence of finite CW-complexes P_i:

$$P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_2} \cdots .$$

Suppose that $P_0 = \{p_0\}$ is a singleton and that all P_i are *regular* finite CW-complexes, i.e. that they admit a finite polyhedral structure. Let $C(f_0, f_1, f_2, \dots)$ be the infinite mapping cylinder of $\mathcal{P}$ (cf. e.g. [10, 12]) and let $\widetilde{\mathcal{P}}$ be its natural compactification by the inverse limit $\varprojlim \mathcal{P}$.

Then the space $\widetilde{\mathcal{P}}$ is an absolute retract (AR) (cf. [10]). Let P^* be the quotient space of $\widetilde{\mathcal{P}}$ by $\varprojlim \mathcal{P}$. Obviously, the space P^* is homeomorphic to the one-point compactification of the countable polyhedron $C(f_0, f_1, f_2, \dots)$.

Let $C(f_n, f_{n+1}, \dots, f_m)$ be the finite cylinder of mappings:

$$P_n \xleftarrow{f_n} P_{n+1} \xleftarrow{f_{n+1}} \cdots P_m \xleftarrow{f_m} P_{m+1}.$$

Clearly, we may assume that

$$P_n \cup P_{m+1} \subset C(f_n, f_{n+1}, \dots, f_m) \subset C(f_0, f_1, f_2, \dots).$$

We shall say that an inverse spectrum $\mathcal{P}$ is *admissible* if the following conditions are satisfied:

(1) every P_i contains only one 0-dimensional cell p_i which is a base point and $f_i(p_{i+1}) = p_i$;

(2) no CW-complex P_i contains any cells of positive dimension less than i; and

(3) for $i \geq 0$, every homomorphism $\widetilde{K}^0_{\mathcal{C}}(f_i)$ is a nontrivial isomorphism.

Such admissible spectra do exist. For example, the inverse spectra constructed by Taylor [13] or the suspension of the spectra constructed by Kahn [7] are admissible spectra (in our sense).

Theorem 3.1. *Let $\mathcal{P}$ be an admissible spectrum. Then the one-point compactification P^* of the countable polyhedron $C(f_0, f_1, f_2, \dots)$ has the following properties:*

(i) X is weakly homotopy equivalent to a point, $P^* \simeq_w *$ (i.e. $\pi_n(X)$ is trivial for all $n \geq 0$);

(ii) P^* is acyclic with respect to all classical homology groups (singular, Borel-Moore, and Čech);

(iii) P^* is acyclic with respect to all classical cohomology groups (singular and Čech);

(iv) P^* is homologically locally connected (HLC);

(v) P^* is cohomologically locally connected (clc); and

(vi) P^* is a noncontractible Peano continuum.

Proof. By the sequence (1), we have the following natural exact sequence:

$$\cdots \to \widetilde{K}^0_{\mathcal{C}}(\widetilde{\mathcal{P}}) \to \widetilde{K}^0_{\mathcal{C}}(\varprojlim \mathcal{P}) \to \widetilde{K}^1_{\mathcal{C}}(P^*) \to \widetilde{K}^1_{\mathcal{C}}(\widetilde{\mathcal{P}}) \to \cdots .$$

As was mentioned before, $\widetilde{\mathcal{P}}$ is an absolute retract; therefore $\widetilde{K}^n_{\mathcal{C}}(\widetilde{\mathcal{P}}) = 0$. On the other hand, the group $\widetilde{K}^0_{\mathcal{C}}(\varprojlim \mathcal{P})$ is nontrivial since the cofunctor $\widetilde{K}^0_{\mathcal{C}}$ is continuous and all homomorphisms $\widetilde{K}^0_{\mathcal{C}}(f_i)$ are nontrivial isomorphisms. It follows by exactness of the sequence (1) that the group $\widetilde{K}^1_{\mathcal{C}}(P^*)$ is also nontrivial and hence the space P^* must be noncontractible, as asserted.

Let us prove that all homotopy groups $\pi_{n\geq 1}(P^*)$ are trivial. Fix a number $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ be any number such that $m > n$. Consider a relative CW-complex $(P^*, C(f_m, f_{m+1}, \dots)^*)$ with the compactification point $*$ as the base point. Let f be any mapping of the sphere S^n with some base point pt to P^* i.e. we have a mapping f of the relative CW-complex (S^n, pt) to $(P^*, C(f_m, f_{m+1}, \dots)^*)$.

By the Cellular Approximation Theorem (cf. e.g. [15]) the mapping f is homotopic relative pt to a cellular map

$$g_m : (S^n, pt) \to (P^*, C(f_m, f_{m+1}, \dots)^*).$$

Note that the space P^* can be represented as the union of the following three spaces:

$$P^* = C(f_0, f_1, \dots, f_{m-1}) \cup C(f_m) \cup C(f_{m+1}, \dots)^*.$$

Observe that CW-complex $C(f_m)$ consists of two 0-dimensional cells, one 1-dimensional cell and some cells of dimension larger than n, since $m > n$ by our choice of the number m. Since the mapping g_m is a cellular mapping of the pairs it follows that the image of g_m lies in the union of the n-dimensional skeleta of P^* and $C(f_{m+1}, \dots)^*$.

Since $C(f_m)$ contains only one 1-dimensional cell e^1 and cells of dimension larger than n, we have

$$\mathrm{Im}(g_m) \subset C(f_0, f_1, \dots, f_{m-1}) \cup e^1 \cup C(f_{m+1}, \dots)^*.$$

The space $\mathrm{Im}(g_m) \subset C(f_0, f_1, \dots, f_{m-1}) \cup e^1$ is contractible with respect to the point $e^1 \cap C(f_{m+1}, \dots)^*$; therefore we may assume that the mapping f is homotopic, with respect to the subspace $C(f_{m+1}, \dots)^*$, to the mapping g_m, the image of which lies in $C(f_{m+1}, \dots)^*$.

Since the relative CW-complexes $(C(f_k, f_{k+1}, \dots)^*, C(f_{k+1}, f_{k+2}, \dots)^*)$ for $k > m$ contain only one 1-dimensional cell plus only cells of the dimensions larger than n, it follows that g_m (and therefore f) is homotopic relative to pt, to the constant mapping to the point $*$. Therefore $\pi_n(P^*, *) = 0$.

The property of homological local connectedness with respect to singular homology HLC at all points, except at the base point, follows by the fact that the space $C(f_0, f_1, f_2, \dots)$, being a CW-complex, is always locally contractible. As we have seen, it is easy to show that $\pi_n(C(f_m, f_{m+1}, f_{m+2}, \dots)^*, *) = 0$. It now follows by the Hurewicz Theorem that singular homology groups are trivial:

$$H_n(C(f_m, f_{m+1}, f_{m+2}, \dots)^*, *) = 0, \quad \text{for } \ n < m.$$

Hence the space P^* is also an HLC space at the base point. The clc property follows from HLC (cf. [4]).

Finally, let us verify the acyclicity of P^*. Since $\pi_n(P^*, *) = 0$ for all $n \geq 0$, it follows by the Hurewicz Theorem that the singular homology groups of P^* are trivial for all n, $H_n(P^*, *) = 0$. It is well-known that all classical homologies are naturally isomorphic on the category of compact metrizable HLC spaces (cf. e.g. [4]). Therefore the space P^* is acyclic in Čech, Borel-Moore, and Vietoris homology theories. By invoking the Universal Coefficients Theorem, we can conclude that over $\mathbb{Z}$, all singular, Čech, Alexander-Spanier, and sheaf cohomology groups of the space P^* are trivial, too. □

4. The infinite-dimensional Hawaiian earrings and the infinite-dimensional Hawaiian group

The n-dimensional *Hawaiian set* of a space X, $n \in \{0, 1, 2, ...\}$, with the base point $x_0 \in X$ is defined as the set of all homotopy classes $[f]$ of the mappings

$$f : (\mathcal{H}^n, \theta) \to (X, x_0)$$

of the n-dimensional Hawaiian earrings $\mathcal{H}^n$ into X. For $n \geq 1$ there is a natural multiplication with respect to which this set is a group. We denote it by $\mathcal{H}_n(X, x_0)$ and call it the *Hawaiian group* in dimension n (cf. [8]).

The Hawaiian groups $\mathcal{H}_n(X, x_0)$ (the set $\mathcal{H}_0(X, x_0)$) are homotopy invariant in the category of all topological spaces with base points and continuous mappings. Note that for the cone over the 1-dimensional Hawaiian earrings the group $\mathcal{H}_1(C(\mathcal{H}^1), pt)$ is nontrivial, for some points pt (cf. [8]).

The space P^* is locally contractible at all points except the base point, and for every natural number n there exists a neighborhood U_* such that $\pi_n(U_*)$ is trivial. Therefore (since $\pi_n(P^*) = 0$) it follows that $\mathcal{H}_n(P^*) = 0$.

Consider a compact bouquet of Hawaiian earrings of all dimensions $\mathcal{H}^\infty = \bigvee_{n=1}^\infty \mathcal{H}^n$ with respect to their base points. Call the space $\mathcal{H}$ the *infinite-dimensional Hawaiian earrings*. There is a natural base point pt. We shall call the set of all homotopy classes of maps $[(\mathcal{H}^\infty, pt), (X, x_0)]$ with the natural multiplication the infinite-dimensional Hawaiian group $\mathcal{H}_\infty(X, x_0)$, where x_0 is the base point of the space X.

Theorem 4.1. *The infinite-dimensional Hawaiian group of the spaces P^* constructed by the admissible spectra of Taylor is nontrivial, $\mathcal{H}_\infty(P^*, *) \neq 0$.*

Proof. The inverse spectrum of Taylor can be described as follows. Let M be the Moore space $= S^{2q-1} \cup_p e^{2q}, p \geq 3$. Toda bracket gives the mapping $f : S^{2(p-1)}(M) \to M$ of the $2(p-1)$-fold suspension of M to M. Let the space P_1 be the singleton $\{p_1\}$, $P_2 = M$, $P_{i+2} = S^{2(p-1)i}(M)$ and $f_2 = f$, $f_{i+1} = S^{2(p-1)}(f_i)$. Then we get the desired inverse spectrum.

According to Adams and Toda we have mappings ϕ and ψ_i such that the composition $\phi f_2 f_3 \cdots f_i \psi_i$ is a nontrivial Toda's element α_i (cf. [1, pp. 12-13], [14]):

$$(2) \qquad \begin{array}{ccc} P_1 \overset{f_1}{\leftarrow} P_2 \overset{f_2}{\leftarrow} P_3 \overset{f_3}{\leftarrow} \cdots \overset{f_{i+1}}{\leftarrow} & P_{i+2} \leftarrow \cdots \\ \varphi \downarrow & \uparrow \psi_i \\ S^{2q} & S^{2q-1+2(p-1)i} \end{array}$$

Define the mapping $f : \mathcal{H}^\infty \to P^*$ as follows. Consider the compact bouquet of spheres $\bigvee_{i=1}^\infty S^{2q-1+2(p-1)i}$. On every sphere $S^{2q-1+2(p-1)i}$ we have a mapping ψ_i to P_{i+2}. The set of all mappings $\{\psi_i\}$ naturally generates the mapping of $\bigvee_{i=1}^\infty S^{2q-1+2(p-1)i}$ to P^*. The space $\bigvee_{i=1}^\infty S^{2q-1+2(p-1)i}$ can be considered as a subspace of $\mathcal{H}^\infty$.

Now let f be the extension of this mapping to the entire space $\mathcal{H}^\infty$ which maps the complement to $*$. The mapping f is an essential mapping.

Indeed, suppose that f were inessential. Then we would have a homotopy $H : \mathcal{H}^\infty \times [0, 1] \to P^*$ such that $H(\theta, 0) = *$. The restrictions of H on every sphere $S^{2q-1+2(p-1)i}$ would be inessential in the space $P^* \setminus \{p_1\}$ for large i.

For simplicity we shall again denote these restrictions by H. So we have for a large i the homotopy

$$H : S^{2q-1+2(p-1)i} \times [0,1] \to P^*,$$

connecting the mapping ψ_i and the constant mapping in $P^* \setminus \{p_1\}$. The CW-complex $S^{2q-1+2(p-1)i} \times [0,1]$ is a $(2q+2(p-1)i)$-dimensional complex.

Choose an integer $m > 2q + 2(p-1)i$ and consider the relative CW-complex $(P^*, C(f_{m+1}, f_{m+2}, \cdots)^*)$. By the Cellular Approximation Theorem the mapping

$$H : S^{2q-1+2(p-1)i} \times [0,1] \to P^* \setminus \{p_1\}$$

is homotopy equivalent to a cellular map with respect to the set $S^{2q-1+2(p-1)i} \times \{1\}$.

Since the complex $C(f_{m+1})$ contains only two 0-cells, one 1-cell, plus cells of dimension larger than $2q+2(p-1)i$, we may assume that the image of the homotopy

$$H : S^{2q-1+2(p-1)i} \times [0,1] \to P^* \setminus \{p_1\}$$

lies in the space

$$C(f_1, \dots, f_{m-1}) \cup e^1 \cup C(f_{m+1}, \dots)^* \setminus \{p_1\}.$$

Now, the space P_2 is a retract of this space. So we have a mapping of the sphere $S^{2q-1+2(p-1)i}$ to the sphere S^{2q}, which should be inessential. But this contradicts the nontriviality of the Toda element mentioned above. Therefore $\mathcal{H}_\infty(P^*, *) \neq 0$. $\square$

5. Epilogue

Since our example in [8] is infinite-dimensional, it is natural to ask the following question [5]:

Problem 5.1. Does there exist a finite-dimensional noncontractible Peano continuum all homotopy groups of which are trivial?

Our Theorem 3.1 gives an answer to our problem from [8] but the cases of finite-dimensional spaces and infinite-dimensional Hawaiian groups remain open:

Problem 5.2. Let P (resp. P^*) be the one-point compactification of any finite-dimensional countable polyhedron by the point $\theta \in P$ (resp. $\theta^* \in P^*$). Suppose that $f : (P, \theta) \to (P^*, \theta^*)$ is any continuous mapping such that

$$\mathcal{H}_n(f) : \mathcal{H}_n(P, \theta) \to \mathcal{H}_n(P^*, \theta^*)$$

is an isomorphism for every $n \in \mathbb{N}$. Is f then a homotopy equivalence?

Problem 5.3. Let P (resp. P^*) be the one-point compactification of a connected polyhedron by the point $\theta \in P$ (resp. $\theta^* \in P^*$). Suppose that $f : (P, \theta) \to (P^*, \theta^*)$ is a continuous mapping such that

$$\mathcal{H}_\infty(f) : \mathcal{H}_\infty(P, \theta) \to \mathcal{H}_\infty(P^*, \theta^*)$$

is an isomorphism. Is f then a homotopy equivalence?

Acknowledgements

This research was supported by the Slovenian Research Agency grants P1-0292-0101, J1-9643-0101 and J1-2057-0101. We thank the referee for comments and suggestions.

References

[1] J. F. Adams, *Talk on Toda's work*, in: Homotopy Theory and Related Topics, Lect. Notes Math. **1418**, Springer-Verlag, Berlin, 1990, pp. 7–14. MR1048172 (91c:55022)

[2] J. F. Adams, *On the groups J(X). IV*, Topology **5** (1966), 21–71. MR0198470 (33:6628)

[3] K. Borsuk, *Theory of Retracts*, Monogr. Mat. **44**, PWN, Warsaw, 1967. MR0216473 (35:7306)

[4] G. E. Bredon, *Sheaf Theory*, 2nd ed., Graduate Texts Math., 170, Springer-Verlag, Berlin, 1997. MR1481706 (98g:55005)

[5] K. Eda, U. H. Karimov, D. Repovš, *A nonaspherical cell-like* 2*-dimensional simply connected continuum and related constructions*, Topology Appl. **156** (2009), 515–521. MR2492298

[6] A. Hatcher, *Vector Bundles and K-Theory*, lecture notes, Cornell University, Ithaca, NY, http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html.

[7] D. S. Kahn, *An example in Čech cohomology*, Proc. Amer. Math. Soc. **16** (1965), 584. MR0179785 (31:4027)

[8] U. H. Karimov, D. Repovš, *Hawaiian groups of topological spaces*, Uspehi. Math. Nauk. **61**:5 (2007), 185–186. (in Russian); English transl. in Russian Math. Surv. **61**:5 (2006), 987–989. MR2328264 (2008d:55009)

[9] M. Karoubi, *K-Theory*, Springer-Verlag, Berlin, 1978. MR0488029 (58:7605)

[10] J. Krasinkiewicz, *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. **92** (1976), 95–112. MR0420546 (54:8560)

[11] S. B. Nadler, Jr., *Continuum Theory. An Introduction*, Monogr. and Textbooks in Pure and Appl. Math., 158, Marcel Dekker, Inc., New York, 1992. MR1192552 (93m:54002)

[12] L. C. Siebenmann, *Chapman's classification of shapes: a proof using collapsing*, Manuscr. Math. **16** (1975), 373–384. MR0431183 (55:4185)

[13] J. L. Taylor, *A counterexample in shape theory*, Bull. Amer. Math. Soc. **81** (1975), 629–632. MR0375328 (51:11523)

[14] H. Toda, *On unstable homotopy of spheres and classical groups*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1102–1105. MR0123329 (23:A657)

[15] G. W. Whitehead, *Elements of Homotopy Theory*, Springer-Verlag, Berlin, 1978. MR516508 (80b:55001)

Institute of Mathematics, Academy of Sciences of Tajikistan, Ul. Ainy 299A, Dushanbe 734063, Tajikistan

E-mail address: umedkarimov@gmail.com

Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P.O. Box 2964, Ljubljana 1001, Slovenia

E-mail address: dusan.repovs@guest.arnes.si