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Michael's theory of continuous selections. Development and applications

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Contents

Introduction

A multivalued transformation of a set A into a set B is a map that puts every element $a \in A$ into correspondence with some non-empty subset $F(a)$ of B. In this case a single-valued selection of the multivalued transformation (m-transformation) $F: A \to B$ is a single-valued transformation $f: A \to B$ such that $f(a) \in F(a)$ for every $a \in A$.

In terms of the simplest mathematical category — the category of sets and their maps — the question of finding a single-valued selection from a multivalued transformation is simply the same as the axiom of choice, and so we will not consider it here — one either accepts the axiom of choice or not. Of course, we remain in the usual framework of a positive variant of the axiom of choice.

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In more meaningful categories the question of the existence of single-valued selections for a multivalued map is more subtle and interesting: the first complication consists in defining a multivalued morphism suitably. In practice throughout this survey we will work in the framework of the *category of topological spaces* or in one of its subcategories. The main problem here can be formulated as follows.

What conditions should be imposed on the topological spaces X and Y, on the family of subsets of Y where the multivalued transformation $F : X \to Y$ takes its values, and on the type of continuity of the multivalued map F to guarantee that F has a continuous single-valued selection?

In this survey we will not consider problems on the approximation of multivalued maps by single-valued ones, fixed-point theory, applied aspects of the theory of multivalued maps (differential equations with multivalued right-hand sides, problems of the optimal equation), and many other connections with multivalued maps.

In our view these restrictions are natural, since clearly it is true for multivalued maps that *one cannot envelop the infinite.* An indirect confirmation of this view lies in simply adding up the number of articles in the references given in the surveys [6]–[8] and [113]. There are more than 2500. To touch on them all is not practical.

To be concise, almost without exception we will only look at the theory based on Michael's papers from the mid-50s, [57]–[62]. There are four main selection theorems that have already established their place in the mathematical folklore: Michael's 0-dimensional, convex-valued, compact-valued and finite-dimensional theorems. Almost everything that is known now in the theory of continuous selections for multivalued maps can be traced back directly to one of these theorems. In fact, this paper is a detailed confirmation of this thesis, in particular beginning at the fourth section. We remark that the 0-dimensional and compact-valued selection theorems are more often used in general topology. In geometrical topology the finite-dimensional selection theorem is more important. As for the convex-valued theorem, in our view everyone who is a professional mathematician should be familiar with it.

§1. Basic concepts and formulations of the main theorems

Definition 1.1. A multivalued map F from a topological space X into a topological space Y is said to be:

a) *lower semicontinuous* if for any non-empty open set G in Y the set

$$F^{-1}(G) = \{x \in X : F(x) \cap G \neq \varnothing\}$$

is open in X;

b) *upper semicontinuous* if for any non-empty open set G in Y the set

$$\{x \in X : F(x) \subset G\}$$

is open in X;

c) *continuous* if it is both lower and upper semicontinuous.

Almost everywhere in this paper we will be dealing with lower semicontinuous mappings. The next result shows clearly why.

Lemma ([58], Proposition 2.2). *Suppose that the multivalued map $F: X \to Y$ is such that for any points $x_0 \in X$ and $y_0 \in F(x_0)$ we can find a neighbourhood U of x_0 and a continuous single-valued selection f of $F|_U$ such that $f(x_0) = y_0$. Then F is lower semicontinuous.*

Example 1.1. Suppose that the map F is the inverse of a single-valued surjective map f, so $F = f^{-1}$. Then the lower semicontinuity of F is equivalent to f being open. In this case the continuous single-valued selections of the multivalued map $F = f^{-1}$ are precisely the continuous sections of the surjection f.

Example 1.2. Suppose that we are given a continuous map g of a subset A of a topological space X into a topological space Y, and let the multivalued map $F_g : X \to Y$ be defined by

$$F_g(x) = \begin{cases} \{g(x)\} & \text{if } \ x \in A, \\ Y & \text{if } \ x \notin A. \end{cases}$$

If A is closed in X, then F_g is lower semicontinuous.

In this case, the continuous single-valued selections of F_g are precisely the continuous extensions of the continuous map g from A to the whole space X.

Example 1.3. Let A be a subset of a topological space X and suppose that we are given a continuous single-valued selection g of a map $G|_A$, where G is a lower semicontinuous map from X to a topological space Y. Define the map $F_{g,G} : X \to Y$ by

$$F_{g,G}(x) = \begin{cases} \{g(x)\} & \text{if } \ x \in A, \\ G(x) & \text{if } \ x \notin A. \end{cases}$$

If A is closed in X, then $F_{g,G}$ is lower semicontinuous.

Example 1.4. Suppose that in a topological space X we are given two functions $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$, with $g \leqslant h$ throughout X. If g and h are lower semicontinuous (as real-valued functions), then so is the "interval-valued" map $F(x) = [g(x), h(x)]$.

In this case, a selection f of F is a continuous function $f: X \to \mathbb{R}$ that separates g and h.

We will now state the main selection theorems. They are all concerned with a paracompact domain of definition of a multivalued map. We recall that if X is Hausdorff, then its *paracompactness* is equivalent to the fact that in any open cover of X we can refine a locally finite continuous partition of unity. The class of *paracompact spaces*, that is, the class of all paracompact Hausdorff spaces, is contained in the class of all normal spaces, and contains the classes of all compact and all metrizable spaces. As a rule, selection theorems are used either for compact or for metrizable spaces X. For the dimension of a normal space X we will throughout mean its Lebesgue dimension $\dim X$.

***Theorem* 1.1** (the 0-dimensional selection theorem, [57], Theorem 2). *Any lower semicontinuous map of a 0-dimensional paracompact space X into a complete metrizable space Y that takes non-empty closed values has a continuous single-valued selection.*

***Theorem* 1.2** (the convex-valued selection theorem, [57], Theorem 1). *Any lower semicontinuous map of a paracompact space X into a Banach space Y that takes non-empty closed convex values has a continuous single-valued selection.*

***Theorem* 1.3** (the compact-valued selection theorem, [61], Theorem 1). *Any lower semicontinuous map of a paracompact space X into a complete metrizable space Y that takes non-empty closed values has an upper semicontinuous compact selection, which in turn has a lower semicontinuous compact selection.*

To formulate n-dimensional selection theorems we need additional definitions. A space Y is called *n-connected* if we can retract any continuous image of the k-dimensional sphere ($k \leqslant n$) in Y. We write this $Y \in C^n$. Y is called *locally n-connected* if for any neighbourhood U of any of its points we can find a neighbourhood V of this point such that any continuous image of the k-dimensional sphere ($k \leqslant n$) lying in the neighbourhood V can be retracted in U. We use the notation $Y \in LC^n$. Finally, a family $\mathscr{L}$ of subsets of Y is *equilocally n-connected* if for any neighbourhood U of any point in any member of the family $\mathscr{L}$ we can find a neighbourhood V of this point such that if a member S of the family $\mathscr{L}$ intersects V, then any continuous image of the k-dimensional sphere ($k \leqslant n$) lying in $S \cap V$ can be retracted in $S \cap U$. We denote this by $\mathscr{L} \in ELC^n$.

***Theorem* 1.4** (the n-dimensional selection theorem, [59], Theorem 1.2). *Let A be a closed subset of a space X that is paracompact, $\dim_X(X \setminus A) \leqslant n+1$, and let Y be a complete metric space. Let $\mathscr{L}$ be an equilocally n-connected family of non-empty closed subsets of Y, and F a lower semicontinuous map from X to Y, all of whose values are members of the family $\mathscr{L}$. Then every selection of the map $F|_A$ can be extended to a selection of $F|_U$ for some open set $U \supset A$. If all the members of $\mathscr{L}$ are n-connected, then U can be taken to be the entire space X.*

The most commonly used version of this theorem is when $A = \emptyset$ and all values of $F(x)$ are n-connected. Then the statement of the theorem consists in the existence of at least one continuous single-valued selection.

§2. Applications of the theory of continuous selections

1. The Bartle–Graves theorem.

We begin with some standard results from linear algebra. If an operator L is a linear map from a finite-dimensional vector space X (over $\mathbb{R}$ or $\mathbb{C}$) into a vector space Y, then X is isomorphic to the direct sum of Y with the kernel of L, $\operatorname{Ker}(L)$. An analogous result is true for a finite-dimensional space Y and an arbitrary locally convex (infinite-dimensional) topological vector space X. Here the isomorphism, and also the operator L, are considered in the category of topological vector spaces. However, if *both* the spaces X and Y are infinite-dimensional, there can be no question of this kind of topological isomorphism. In fact, for any separable Banach space Y there exists a continuous linear operator L mapping l_1, the space of summable sequences, into Y. If we assume that l_1 is isomorphic to the direct sum $Y \oplus \operatorname{Ker}(L)$, then we find that Y is isomorphic to a complementary subspace of l_1. But in l_1 an infinite-dimensional subspace is complementary only if it is isomorphic to the whole of l_1, [86]. Thus, all infinite-dimensional separable Banach spaces are isomorphic to l_1, giving a contradiction.

It turns out that in this situation *a homeomorphism between X and $Y \oplus \operatorname{Ker}(L)$ nevertheless exists.*

***Theorem* 2.1** [10]. *If L is a continuous linear operator that maps a Banach space X into a Banach space Y, then the space X is homeomorphic to the direct sum $Y \oplus \operatorname{Ker}(L)$.*

Proof. By Banach's open map theorem L is an open map from X into Y, that is, the multivalued map $F = L^{-1}$ is a lower semicontinuous map from the paracompact space Y into the Banach space X (any metric space is paracompact). But then the values of F are closed and convex in X: they are simply parallel translations of the kernel of L. By Theorem 1.2, F has a selection $g : Y \to X$. Then another selection for F will be the map $f : Y \to X$ that is given by $f(y) = g(y) - g(0)$, $y \in Y$. Here $f(0) = 0$. It remains to define the homeomorphism h mapping X into the direct sum $Y \oplus \operatorname{Ker}(L)$ by $h(x) = (L(x), x - f(L(x)))$. □

Remark 1. This theorem is of course true for any pair of spaces (X, Y) where Banach's open map theorem and the selection theorem 1.2 can be applied. For example, it holds for Fréchet spaces (complete metrizable locally convex vector spaces). In fact, we require complete metrizability for both X and Y, but it is enough to require *only* that the kernel $\operatorname{Ker}(L)$ of L is locally convex (see [62], Corollary 7.3 or [4], Chapter 2, Proposition 7.1). We give a new proof of this generalization of Theorem 2.1 in §6.

Remark 2. Theorem 2.1 is the simplest version of some results of Bartle and Graves and of Michael. We now state a stronger (parametric) version. For given Banach spaces X and Y, we let Z denote the set of continuous linear surjections from X onto Y. We take Z to be non-empty and use the usual sup-norm. We define the function $m : Z \oplus Y \to [0, \infty)$ by $m(L, y) = \inf\{\|x\| : x \in L^{-1}(y)\}$.

***Theorem* 2.2** ([58], Corollary 7.5). *Let $h : T \to Z$ and $g : T \to Y$ be continuous maps of some topological space T. Then for any $\lambda > 1$ there exists a continuous map $f : T \to X$ such that:*

a) $f(x) \in [h(x)]^{-1}(g(x))$ *for all* $x \in X$;

b) $\|f(x)\| < \lambda \cdot m(h(x), g(x))$ *for all* $x \in X$;

c) *if* $h(x_1) = \alpha h(x_2)$ *and* $g(x_1) = \beta g(x_2)$ *for* $x_1, x_2 \in X$ *and scalars* $\alpha \neq 0, \beta$, *then* $f(x_1) = (\beta/\alpha) f(x_2)$.

As a particular case we see that in the proof of Theorem 2.1 the selection f of the map L^{-1} can be chosen so that $f(\alpha x) = \alpha f(x)$: for f to be linear, additivity *alone* is not enough. But in principle it cannot be attained, for the reasons given at the start of this subsection.

2. The problem of whether there is a homeomorphism between a Hilbert space and any separable Banach space.

This problem, due to Banach and Mazur, was resolved positively in 1967 by Kadets [45]. In a sequence of papers he proved a succession of theorems on replacing norms in Banach spaces with bases by equivalent, but "smoother" norms. However, what about spaces that do not have a basis? We recall that in 1967 the question of whether or not such spaces existed was open.

***Theorem* 2.3** [5]. *If every infinite-dimensional Banach space with a basis is homeomorphic to the Hilbert space l_2, then every infinite-dimensional separable Banach space is homeomorphic to l_2.*

Proof. We will use Pelczyński's decomposition method, which is a version of the Cantor–Bernstein theorem in the category of Banach spaces and the homeomorphisms between them. Thus, suppose that X is an infinite-dimensional separable Banach space. By a theorem due to Banach, it contains an infinite-dimensional closed subspace Z with a basis. We let $\approx$ denote the fact that the spaces are homeomorphic and apply the hypotheses of the theorem. Then by Theorem 2.1 we obtain

$$X \approx Z \oplus (X/Z) \approx l_2 \oplus (X/Z) \approx l_2 \oplus l_2 \oplus (X/Z) \approx l_2 \oplus X.$$

On the other hand, X is isomorphic to the factor space of the space l_1 and again using Theorem 2.1 we find that $l_1 \approx X \oplus Y$ for some closed subspace $Y \subset l_1$. Then

$$l_2 \approx c_0[l_1] \approx c_0[X \oplus Y] \approx X \oplus c_0[X \oplus Y] \approx X \oplus c_0[l_1] \approx X \oplus l_2,$$

that is, X is homeomorphic to l_2. Here, for a Banach space B, $c_0[B]$ denotes the space of all sequences of elements of B that converge to zero with the standard max-norm, and we use without proof the easily verified fact that the space $c_0[l_1]$ has a Schauder basis and therefore, by hypothesis, is homeomorphic to l_2. □

3. Retracts of zero-dimensional metric spaces.

One of the remarkable properties of the Cantor set is that any of its compact subsets is one of its retracts. An analogous result holds for zero-dimensional metric spaces and their complete subsets (Mazurkiewicz's theorem). In fact, for a metric space (X, ρ) and any of its complete subsets A, we consider the multivalued map $R : X \to A$ that equals A for points in $X \backslash A$ and is the identity map on A. By Example 1.2 the map R is lower semicontinuous and *if* X is 0-dimensional, then applying Theorem 1.1 to the map R we obtain a continuous selection of this map, which is also a retract of X on A.

Now we give yet another version of the same theme. Suppose that X is paracompact and 0-dimensional and is the image of some complete metrizable space Y under some continuous open map $g : Y \to X$. Then Y contains some homeomorphic image of the paracompact space X. The set we require that is homeomorphic to X will be the image of X under a continuous section f of F, the inverse of g. This section exists by Theorem 1.1.

Again, let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and let $g : X \to \mathbb{C}$ be a continuous map of a zero-dimensional paracompact space X. Then for every x we can choose a solution z of the equation $P(z) = g(x)$, so that the solution depends *continuously* on x; $z = z(x)$. To prove this it is enough to use the fact that any analytic map is open, in order to obtain a continuous selection $z = z(x)$ of the map $F = P^{-1} \circ g$.

4. Strictly regular maps and Hurewicz fiberings.

We will discuss Ferry's result [36] and a sketch of how the finite-dimensional selection theorem is used to prove it. We say that a map $f : X \to Y$ between two metrizable separable spaces is strictly regular if for any $y \in Y$ and any $\varepsilon > 0$, in some neighbourhood U of y the inverse images of all points are pairwise ε-homotopic, that is, in the time of the homotopy the points do not shift by more than ε.

***Theorem* 2.4.** *Let f be a strictly regular proper map on a finite-dimensional complete space. If all the fibres are ANR-compact, then f is a fibering in the sense of Hurewicz.*

Sketch of the proof. For any point $y \in Y$ we can find a compact ANR-neighbourhood M of the inverse image $f^{-1}(y)$ which can be retracted to the inverse image itself. Since the inverse images of compact sets under the map f are compact, there exists a neighbourhood W of y such that $f^{-1}(W) \subset M$. For every point $z \in W$ we define $H(z)$ to be the set of all continuous maps from M into $f^{-1}(W)$ that retract the whole compact set M onto the inverse image $f^{-1}(z)$. The topology in $H(z)$ is induced by its embedding in the metric space $C(M, f^{-1}(W))$ of all continuous maps of M into $f^{-1}(W)$. We may assume that X lies in a Hilbert cube and take the sup-metric in $C(M, f^{-1}(W))$. The main point of the proof consists in showing that we can apply Theorem 1.4 to the map $H : W \to C(M, f^{-1}(W))$, with $\dim W < \infty$. Next we fix a selection s of H that is continuous in z and selects a retract of the compact set M onto the inverse image $f^{-1}(z)$. In the next subsection we look at the case when the convex-valued selection theorem is used in roughly the same way, only to prove the local triviality of a regular map.

5. Topologically regular maps.
Now we introduce some results from [88], see also [92].

Definition 2.1. a) Let (X, ρ) be a metric space and $\exp_M(X)$ the family of all closed subsets of X that are homeomorphic to a fixed space M. *The distance between two elements* of the family $\exp_M(X)$ is the infimum of the $\varepsilon > 0$ for which we can find a homeomorphism between these elements that shifts points by a distance no greater than ε.

b) A map p from the metric space X onto a topological space Y is called *topologically regular* if the inverse map p^{-1} maps Y continuously into $\exp_M(X)$ for some M, where we take the metric given in a) on $\exp_M(X)$; (see also [32]).

***Theorem* 2.5.** *Let p be a continuous map of a locally compact metric space X into a perfectly normal space Y such that all its inverse-images $p^{-1}(y)$, $y \in Y$, are homeomorphic to the interval $I = [0,1]$. Then, if the map p is topologically regular, it is a locally trivial fibering.*

Proof. We fix a point $y_0 \in Y$ and let c_0, d_0 be the end points of the arc $p^{-1}(y_0)$. We take $0 < 2\varepsilon_0 < \rho(c_0, d_0)$. By the definition of topological regularity, we can find a neighbourhood U of y_0 such that for all $y \in U$ the arcs $p^{-1}(y)$ are the results of some homeomorphic ε_0-shifts of the arc $p^{-1}(y_0)$. Then for these y we can distinguish the ends of the arc $p^{-1}(y)$: exactly one end of the arc $p^{-1}(y)$ lies in an ε_0-neighbourhood of c_0 (we will denote it by $c(y)$) and the other end of the arc $p^{-1}(y)$ lies in an ε_0-neighbourhood of d_0 (we will denote it by $d(y)$). Let G be a neighbourhood of the arc $p^{-1}(y_0)$ whose closure $\mathrm{cl}(G)$ is compact. Since the map is topologically regular, it is easy to see that it is open. Let W be a neighbourhood of y_0 such that $\mathrm{cl}(W) \subset U \cap p(G)$. We define a multivalued map F of the perfectly normal space $\mathrm{cl}(W)$ into the Banach space $C(E)$ of all continuous functions on the compact space $E = \mathrm{cl}(G)$ as follows:

$$F(y) = \{f \in C(E) \mid f \text{ maps the arc } p^{-1}(y) \text{ homeomorphically onto } [0,1] \\ \text{and } f(c(y)) = 0,\ f(d(y)) = 1\}.$$

Obviously, $F(y)$ is non-empty and convex. The lower semicontinuity of $F : \mathrm{cl}(W) \to C(E)$ is easily verified. Unfortunately, $F(y)$ is not closed in $C(E)$, and so we cannot apply Theorem 1.2. We can, however, apply the selection theorem for open-valued maps (see §4, subsection 1 below) to obtain a continuous single-valued selection $s : \mathrm{cl}(W) \to C(E)$ of F. Then the trivialization of the map p over the neighbourhood W is given in the standard way:

$$\Psi(x) = (p(x), [s(p(x))](x)) \in W \times [0,1], \quad x \in p^{-1}(W).$$

□

We note that: a) in Theorem 2.5 the dimensional restrictions on the image of the map p are removed; b) an analogous result is proved in [93] in the case when the inverse-images of the points are homeomorphic to a one-dimensional polyhedron, and X and Y are compact metrizable spaces; c) a case of interest is the two-dimensional disk as a layer, when there is no "good" convex structure in the space of homeomorphisms, but nonetheless it is known [56] that the space of homeomorphisms of the disk is an absolute retract.

6. The problem of groups of homeomorphisms.

Is the group of homeomorphisms of an n-dimensional compact manifold an l_2-manifold? This is one of the few fundamental problems in the theory of infinite-dimensional manifolds that was posed when the subject originated at the end of the 60s, and had not made at least some appreciable progress up until 1972. The problem reduces to the following (see [115]). Is the group Auth_n of homeomorphisms of the n-dimensional disk that are the identity on the boundary of the disk an absolute retract? For $n = 1$ the group Auth_1 is well known: it consists of all continuous strictly increasing maps of the interval $[0, 1]$ onto itself. Anderson proved that Auth_1 is homeomorphic to l_2. We will see how the finite-dimensional selection theorem, Theorem 1.4, was used in [56] to prove the same result for Auth_2. It may be that an analogue of Mason's construction will also yield a proof for $n = 3$.

As Auth_n is contractible (Alexander's result), it is enough to prove that Auth_n is an absolute neighbourhood retract. In order to do this, using Torunczyk's theorem [105] we need only find a basis of open sets in Auth_n such that any non-empty intersection of members of this basis has trivial homotopic groups π_k for all $k \in \mathbb{N}$. We now give the construction of such a basis in the case $n = 2$. For the disk in two dimensions we will take the square $S = [0, 1] \times [0, 1]$. Auth_2 is a complete space in the metric $d(f, g) = \sup\{\max(\rho(f(x), g(x)), \rho(f^{-1}(x), g^{-1}(x)))\}$.

Let L_n be the lattice in S of closed intervals dividing the sides of S into 2^n equal subintervals. We take a fixed homeomorphism $f \in \mathrm{Auth}_2$. The image $f(L_n)$ will be a strongly twisted lattice that has the same endpoints on its boundary as the lattice L_n. For every vertical of the lattice $f(L_n)$ we can choose a tubular neighbourhood such that these vertical neighbourhoods are disjoint and polygonal with vertices at points with rational coordinates. We will denote these vertical neighbourhoods by $V_1, V_2, \ldots, V_n$. We can construct horizontal tubes $H_1, H_2, \ldots, H_n$ similarly. We emphasize that a vertical element in the lattice $f(L_n)$ intersects a horizontal element in one point, whereas a vertical tube can intersect a horizontal tube in a very complicated manner. To obtain an invariant definition of the basis of open sets we forget about the homeomorphism f and leave only the system of vertical and horizontal tubes $V_1, \ldots, V_n, H_1, \ldots, H_n$. We define an open set $G = G(V_1, \ldots, V_n, H_1, \ldots, H_n)$ as the set of homeomorphisms $g \in \mathrm{Auth}_2$ that carry vertical and horizontal elements of the lattice L_n into corresponding vertical and horizontal tubes. We call the set of all neighbourhoods $G = G(V_1, \ldots, V_n, H_1, \ldots, H_n)$ for all $n \in \mathbb{N}$ and for all possible tubes $V_1, \ldots, V_n, H_1, \ldots, H_n$ an HVT-basis. It is not hard to show that HVT is in fact a basis in Auth_2, and further, the intersection of two members of HVT lies in HVT (or is empty). In other words, HVT is a base of open sets in Auth_2 and to apply Torunczyk's theorem we have to show that each element $G \in HVT$ is weakly homotopic to zero. Here we use Michael's Theorem 1.4. For the finite-dimensional paracompact space X we take $K \times [0,1]$, where K is a finite-dimensional connected compact set (the image of the sphere) in G, the closed set A coincides with $K \times \{0\}$, the complete metric space Y is G itself, and the multivalued lower semicontinuous maps at different stages of the proof are chosen differently with the aim of retracting K to a point in the interior of G.

7. Soft maps. The functor of probability measures.

The concept of a soft (n-soft) map was introduced by Shchepin and was an adequate carrier in the category of maps for the concept of absolute extensor (extensor in dimension n) in the category of compact sets. A detailed survey of soft maps is given in [100].

A map $f: X \to Y$ is said to be *soft with respect to the pair* $A \subset Z$ if for any map $g: Z \to Y$ and any selection $h: A \to X$ of the multivalued map $f^{-1} \circ g|_A : A \to X$ there exists a continuous extension of this selection to some selection $\widehat{h}: Z \to X$ of the multivalued map $f^{-1} \circ g : Z \to X$.

If a map is soft with respect to all paracompact spaces (paracompact spaces of dimension n) and their closed subspaces, then it is called a *soft* (*n-soft*) map.

We give an example that will be useful to us in §6. If X is compact, we let $P(X)$ denote the set of all regular Borel probability measures on it. We consider $P(X)$ as embedded in the space dual to the Banach space $C(X)$ of continuous functions on X; we take $P(X)$ with the weak-$*$ topology. Then $P(X)$ is a convex compact space, and any continuous map $f: X \to Y$ will induce a map $P(f): P(X) \to P(Y)$. To determine the value of the measure $P(f)(\mu)$ on a subset $Z \subset Y$ we need only calculate the value of $\mu(f^{-1}(Z))$. Thus we have constructed a covariant functor from the category of compact spaces and their continuous maps into the category of convex compact spaces and their linear (affine) maps. Details of the properties of the *functor of probability measures* are given, for instance, in [33] and [34]. Here we remark that a map $P(f)$ is soft if the maps f we take are open maps between metrizable compact spaces X and Y. In fact, in this case $P(X)$ and $P(Y)$ can be considered as convex compact subsets in a Hilbert space and all the values of the map $[P(f)]^{-1} \circ g$ will be convex compact sets. As f is open, it follows that $P(f)$ is open [28], that is, $[P(f)]^{-1}$ is a lower semicontinuous map, and so if A is closed in the paracompact set Z we can apply Theorem 1.2 to the map $[P(f)]^{-1} \circ g$ and the pair (Z, A). Thus $P(f)$ is a soft map.

The finite-dimensional selection theorem (with $n = 1$) is used in [35] to prove that the continual functor $\exp^c$ of the exponential preserves the 1-softness of a map f between Peano continua provided $\exp^c(f)$ is open.

For an example of the use of the (zero-dimensional) selection theorem we refer the reader to [44], where this theorem is used to prove that any Dugundji space is an absolute extensor for zero-dimensional compact sets.

8. Various examples.

a) *The continuity of choice in the definition of continuity.* Let $C(X, Y)$ be the space of continuous maps from one metric space to another. The definition of continuity is

$$(*) \qquad \begin{array}{l} \forall f \in C(X,Y) \quad \forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0: \\ \qquad \forall x' \in X \ \ (\rho(x,x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon). \end{array}$$

Then, to every triple $(f, x, \varepsilon) \in C(X, Y) \times X \times \mathbb{R}_{>0}$ there corresponds some subset (non-empty by definition) $\Delta(f, x, \varepsilon)$ of the set of positive numbers, namely the set of all δ in $(*)$. We obtain a multivalued map $\Delta : C(X, Y) \times X \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with non-empty convex (not closed) values. Our statement is that for X locally compact, this map is lower semicontinuous and has a continuous single-valued selection. In other words, δ can always be considered as a continuous function of f, x, ε.

b) *The continuity of choice in the Stone–Weierstrass theorem.* Let X be a normed space and V a convex everywhere dense subset of it. Then every $x \in X$ can be approximated to within an arbitrary accuracy of $\varepsilon > 0$ by some $v \in V$. Our assertion is that v may be chosen in a single-valued continuous way that depends on x and ε. More formally, there exists a continuous map $v : X \times \mathbb{R}_{>0} \to V$ such that $\|x - v(x, \varepsilon)\| < \varepsilon$ for every $(x, \varepsilon) \in X \times \mathbb{R}_{>0}$. As an illustration, we find that in the Stone–Weierstrass theorem the approximating polynomial (a member of the algebra of functions) can be chosen to depend continuously both on the approximant and the degree of accuracy of the approximation. The proof for fixed $\varepsilon > 0$ proceeds by using the scheme of proof for the convex-valued selection theorem, Theorem 1.2. However, instead of taking open ε-balls with centres at every point of X we take only those balls with centres at points of V. Then ε_n-approximations are simply joined by broken lines; let $\varepsilon_n \to 0$.

c) *More on approximations.* Let V be a closed convex subset of a Banach space X and let $\rho(x, V)$ be the usual distance from a point to the set. For any $\varepsilon > 0$ and any $x \in X$ we define the set $P_V^\varepsilon(x) = \{v \in V \mid \|x - v\| \leqslant \rho(x, V) + \varepsilon\}$. Clearly $P_V^\varepsilon(x)$ is non-empty, closed and convex in X. After checking that $P_V^\varepsilon : X \to X$ is lower semicontinuous we can apply Theorem 1.2 to show that the ε-approximations $v \in V$ of elements x in the space X can be chosen to depend continuously on x; $v = v(x) \in P_V^\varepsilon(x)$. An analogous process was used in [49] to construct a continuous sample from the values of the operator of generalized rational approximation.

d) *Complemented spaces.* In a Hilbert space every closed subspace has a complement. An example is the orthogonal complement. For a long time it was an open question whether there existed closed non-complemented subspaces in any Banach space that was not isomorphic to a Hilbert space. A positive example was given in 1971 by Lindenstrauss who used Dvoretskii's well-known theorem on almost Euclidean sections of spheres in a Banach space. Thus there always exist non-complemented spaces. But we can ask whether in *complemented* spaces it is possible to choose the complement in some single-valued and continuous manner? An affirmative answer is given in [89].

e) *Fixed points of multivalued maps.*

Theorem ([31], Theorem 11.6). *Let C be a convex, but not necessarily closed, subset of a Banach space E, and let F be a lower semicontinuous map of C into itself with convex closed values. Then if the closure of $F(C)$ is compact in C, the map F has a fixed point $x_0 \in C$; $x_0 \in F(x_0)$.*

Proof. We apply the convex-valued Theorem 1.2 to F. Let f be a selection of F. Then $f(x) \in F(x) \subset \mathrm{cl}(F(x)) \subset C$. We can apply Schauder's theorem to the single-valued continuous map f of the convex set C into itself, giving a fixed point $x_0 \in C$ such that $x_0 = f(x_0) \in F(x_0)$.

§3. Proofs of the main theorems. Selection characteristics of paracompactness and of properties of normality type

1. Proof of the zero-dimensional selection theorem, Theorem 1.1.
We take a cover of the metric space (Y, ρ) by open balls $B(y, \varepsilon)$ with radius ε and centred at each point $y \in Y$. The family of sets $\{F^{-1}(B(y,\varepsilon))\}$, $y \in Y$, is then an open cover of X. The fact that X is zero-dimensional and paracompact guarantees that we can find an open cover $\{W_\alpha\}, \alpha \in A$, that is a refinement of $\{F^{-1}(B(y,\varepsilon))\}$, $y \in Y$, consisting of pairwise disjoint sets W_α. For each $\alpha \in A$ we take an arbitrary element $y_\alpha \in Y$ such that $W_\alpha \subset F^{-1}(B(y_\alpha, \varepsilon))$ and we define the map $f_\varepsilon : X \to Y$ by $f_\varepsilon(x) = y_\alpha$ if $x \in W_\alpha$. The fact that f_ε is continuous follows from the fact that $W_\alpha, \alpha \in A$, are disjoint, and we see from the construction of f_ε that $\mathrm{dist}(f_\varepsilon(x), F(x)) < \varepsilon$ for all $x \in X$. In brief, we have constructed a single-valued continuous *ε-selection* f_ε of the multivalued map F. Conveniently, in addition in every set $F(x)$ we can find a point $x(\varepsilon)$ such that $\rho(f_\varepsilon(x), x(\varepsilon)) < \varepsilon$.

Now we set $\varepsilon = 1$ and we take the previous paragraph as the first step in an induction proof. The second step goes as follows. We put every point $x \in X$ in correspondence with the set $F_2(x)$ given by the intersection of $F(x)$ with the open ball $B(f_1(x), 1)$. The set $F_2(x)$ is non-empty as, by construction, f_1 is the 1-selection of F. Thus, F_2 maps the zero-dimensional paracompact space X into the metric space Y. It is not hard to show that since F is lower semicontinuous and f_1 is continuous, F_2 is also lower semicontinuous. We now make the same construction as in the previous paragraph for F_2 with $\varepsilon = 1/2$. We obtain a continuous single-valued map $f_2 : X \to Y$ such that for every $x \in X$ there is $x_2 \in F_2(x)$ with $\rho(f_2(x), x_2) < 2^{-1}$. Moreover, by construction, $\rho(f_1(x), f_2(x)) < 1 + 2^{-1}$, $\rho(x_1, x_2) < 1 + 1$.

Proceeding by induction (with $\varepsilon = 2^{-n}$) we obtain a sequence of lower semicontinuous maps $F_n : X \to Y$ and a sequence of continuous single-valued maps $f_n : X \to Y$, and for every $x \in X$ a sequence of points $x_n \in F_n(x)$ such that

$$F_n(x) \subset F_{n-1}(x) \subset \cdots \subset F(x), \qquad \rho(f_n(x), x_n) < 2^{-n},$$
$$\rho(f_n(x), f_{n+1}(x)) < 2^{-n+1} + 2^{-n}, \quad \rho(x_n, x_{n+1}) < 2^{-n} + 2^{-n}$$

for all natural numbers n.

It follows from the third inequality and the fact that $F(x)$ is dense that for every $x \in X$ the sequence $\{x_n\}$ has a limit, which we will denote by $f(x) \in F(x)$. The first inequality implies that the sequence of functions f_n converges pointwise to f. Finally, the second inequality implies that $\{f_n\}$ converges uniformly to f and so the limit function is continuous. □

Remark. Instead of assuming that the *whole* metric space Y is complete and that the values $F(x)$ of the lower semicontinuous map F are closed, it is sufficient to simply assume that *the values* $F(x)$ *themselves,* $x \in X$, are complete.

2. Proof of the convex-valued selection theorem, Theorem 1.2.

The idea of the proof is the same as for Theorem 1.1. First, for fixed $\varepsilon > 0$ we construct an ε-selection of the lower semicontinuous map $F : X \to Y$ with convex non-empty values. To do this, as in the previous theorem, we look at the cover of the space X by the sets $F^{-1}(B(y, \varepsilon))$, $y \in Y$. In this cover we take some locally finite continuous partition of unity $\{e_\alpha\}, \alpha \in A$, and for every index α we choose an arbitrary element y_α in Y such that $\operatorname{supp}(e_\alpha) \subset F^{-1}(B(y_\alpha, \varepsilon))$. Then we define a map $f_\varepsilon : X \to Y$ by

$$f_\varepsilon(x) = \sum e_\alpha(x) y_\alpha, \qquad \alpha \in A.$$

Since the partition of unity is locally finite, f_ε is continuous. Further, if for some index α, $e_\alpha(x)$ is positive, then by construction the point y_α lies within ε of the set $F(x)$. But then since $f_\varepsilon(x)$ is a convex combination of all such points y_α, it also lies within ε of $F(x)$.

Now the proof follows that of Theorem 1.1. We set $\varepsilon = 1$ and consider the multivalued map $F_2(x) = \operatorname{conv}(F(x) \cap B(f_1(x), 1))$, use the procedure we have described for $\varepsilon = 1/2$, and so on. In doing this we use in addition a purely technical assertion: a multivalued map that is the convex hull of a lower semicontinuous map is itself lower semicontinuous. This proves Theorem 1.2. □

3. Proof of the compact-valued selection theorem, Theorem 1.3.

The proof is very different from that of the previous theorems. The only common idea is that of some induction process on ε_n-approximations, $\varepsilon_n = 2^{-n}$. Our starting point is the following observation. In order that the open sets $\{F^{-1}(B(y, \varepsilon))\}$ (see the proofs of Theorems 1.1 and 1.2) form a cover of the *whole* paracompact set X it is in fact not necessary to consider open balls $B(y,\varepsilon)$ with centres *at all points* y of the metric space Y. We can "economise" and for every $x \in X$ take an open ball $B(y,\varepsilon)$ with its centre at some point y that is chosen in some arbitrary way from the set $F(x)$. In other words, we can start with an arbitrary, and not necessarily continuous selection of the multivalued map (the axiom of choice!) and then using the process of ε_n-approximations attempt to improve this selection.

A rigorous version of this proof will be given below in §4 (the method of covers) and in §6 the compact-valued theorem will be deduced as a corollary of the 0-dimensional selection theorem.

4. Converse selection theorems.

***Theorem* 3.1.** *Suppose that the T_1-space X is such that any lower semicontinuous map from X into any Banach space that has non-empty closed and convex values has a selection. Then X is paracompact.*

Proof [4]. For any open cover $\omega = \{G_\alpha\}, \alpha \in A$, of X we consider the Banach space $Y = l_1(A)$ that consists of all maps $y : A \to \mathbb{R}$ for which $y(\alpha)$ is non-zero only for no more than a countable set of α and for which $||y|| = \sum |y(\alpha| < \infty$. We consider the following multivalued map $F : X \to Y$:

$$F(x) = \{y \in l_1(A) \mid y \geq 0, \|y\| = 1, y(\alpha) = 0 \quad \text{if } x \notin G_\alpha\}.$$

In other words, $F(x)$ is the standard simplex in the closed subspace $l_1(A_x)$ of the space $l_1(A)$; $A_x = \{\alpha \in A : x \in G_\alpha\}$. That F is lower semicontinuous is clear.

From the hypothesis of the theorem, F has a selection $f : X \to l_1(A)$. For every $\alpha \in A$ we define the function $e_\alpha : X \to [0,1]$ by $e_\alpha(x) = [f(x)](\alpha)$. The set of functions $\{e_\alpha\}$ is almost what we require; they are continuous, they sum at every point to unity, and $\operatorname{supp}(e_\alpha) \subset G_\alpha$. Only the local finiteness does not suffice.

Let $e(x) = \sup\{e_\alpha(x) | \alpha \in A\}$, and for any $\alpha \in A$ let $V_\alpha = \{x | e_\alpha(x) > e(x)/2\}$. Then $\{V_\alpha\}$ is the desired cover refined from the cover $\{G_\alpha\}$ of X. Let us prove this. For every $x \in X$ we take $\beta = \beta(x)$ such that $e_\beta(x) > 0$. Then for some *finite* set of indices $\Gamma(x) \subset A$ we have

$$1 - \sum e_\alpha(x) < e_\beta(x)/2, \qquad \alpha \in \Gamma(x).$$

The continuity of a finite number of functions shows that this inequality holds in some neighbourhood $U(x)$ of x. But then in this neighbourhood for every index $\gamma \notin \Gamma(x)$ we have $e_\gamma(x) < e_\beta(x)$. Thus, the function $e(\cdot)$ in the neighbourhood $U(x)$ is the maximum of a finite number of continuous functions, and therefore is itself continuous. As $e(\cdot)$ is continuous, the V_α are open, and every one is contained in G_α. Local finiteness follows from the construction: in fact, if $\gamma \notin \Gamma(x)$, then $V_\gamma \cap U(x) = \emptyset$. Finally, since $e(\cdot)$ is strictly positive, it follows that $\{V_\alpha\}, \alpha \in A$, covers the whole of X. This proves Theorem 3.1. □

We now give a list of other selection theorems that are expressed in the form of an equivalence between some conditions on a T_1-space X. (We have numbered these to coincide with the numbering in [58].) We denote by $C'(Y)$ ($F(Y)$) the family of all non-empty convex subsets of the Banach space Y that are either compact or coincide with the whole of Y (are closed in Y).

Theorem 3.1′. *The following properties of the T-space X are equivalent:*

a) *X is normal;*

b) *any lower semicontinuous map from X into $\mathbb{R}$ with values in $C'(\mathbb{R})$ has a selection;*

c) *if Y is a separable Banach space, then any lower semicontinuous map from X to Y with values in $C'(Y)$ has a selection.*

Theorem 3.2′. *The following properties of the T-space X are equivalent:*

a) *X is collectively normal;*

b) *if Y is a Banach space, then any lower semicontinuous map from X to Y with values in $C'(Y)$ has a selection.*

Theorem 3.1″. *The following properties of the T-space are equivalent:*

a) *X is normal and countably paracompact;*

b) *any lower semicontinuous map from X into $\mathbb{R}$ with values in $F(\mathbb{R})$ has a selection;*

c) *if Y is a separable Banach space, then any lower semicontinuous map from X to Y with values in $F(Y)$ has a selection.*

Theorem 3.2″: *see earlier, Theorems* 1.2 *and* 3.1.

Theorem 3.1‴: *see below,* §4, *subsection* 1.

According to these results, if a space is *countably paracompact*, then this implies that it is possible to refine any cover of the space that consists of a *countable* number of open sets to some locally finite open cover; *collective normality* of a space means that for any disjoint locally finite family of closed subsets of the space we may include every member of the family in some open subset such that the open sets we obtain are pairwise disjoint.

§4. Generalizations of the main selection theorems. The method of covers. Theorems on "the modulus of countable sets". Amalgamated selection theorems

1. In each of Theorems 1.1 – 1.4 we have conditions on:

I) the domain of definition X of the multivalued map F;

II) the space Y where $F(x)$ takes its values;

III) the family of subsets of Y whose members are the values $F(x)$;

IV) the type of continuity of the multivalued map $F(x)$,

that are sufficient for there to exist a continuous single-valued selection of F.

The first and in many respects the most important questions are connected with whether these conditions are also necessary. For definiteness we take the convex-valued selection theorem, Theorem 1.2.

Question 1. To what extent is it essential to make the following hypotheses in order to prove Theorem 1.2?

I) X is paracompact;

II) Y is a Banach space;

III) a) the map F has convex values;

b) these values are closed;

IV) F is lower semicontinuous.

The difficulty with formally giving a single solution to all these questions is connected with the fact that these questions are not independent. For example, if we reduce the class of Banach spaces $\{Y\}$ where the lower semicontinuous map can take its values, then we weaken the conditions on the corresponding class of spaces $\{X\}$, and paracompactness in its pure form becomes an unnecessarily strong hypothesis. A similar situation obtains if we restrict the family of subsets where the values of F lie: formally we can restrict this family to one member, and then the paracompactness of the domain of definition, the type of continuity of the map, and so on, are no longer essential. Thus, we can formulate the question better as follows:

Question 1′. Suppose that in the "four-parameter" set of hypotheses in Theorem 1.2 the ranges of three of these parameters are fixed as given in the theorem. Is the fourth condition *necessary* to obtain a continuous single-valued selection?

To a large extent, an unending set of questions that are currently open can be obtained by fixing any three of the parameters in a manner *different* from the formulation of the theorem. Most often, even a hypothetical answer for the fourth parameter remains unclear. For example, what can we say about the class of T_1-spaces X for which any lower semicontinuous map with convex closed values in *any reflexive* Banach space has a continuous single-valued selection? A very unexpected example due to Nedev [78] shows that besides all the paracompact spaces, this class contains, as a minimum, one that is *not paracompact*: the set of all ordinals less than the first uncountable one with the order topology. As another example, what can we say about the class of topological vector spaces $\{Y\}$ for which any lower semicontinuous map with convex closed values and with any *metrizable* (*completely metrizable, compact, n-dimensional, ...*) domain of definition has a continuous single-valued selection?

But we must return to question 1, or more precisely, question 1′. In a large measure one answer is already known, and has been given (Theorem 3.1).

Answer I. If we fix hypotheses II)−IV) in the convex-valued selection theorem, Theorem 1.2, then hypothesis I), that the domain of definition is paracompact, is necessary.

Answer II. In the proof of Theorem 1.2, the requirement that Y be a Banach space is not formally used to its full extent: we only need Y to be locally convex and completely metrizable. Thus (for fixed I), III) and IV)) we have three questions:

a) Is it necessary that Y be locally convex?

b) Is it necessary that Y be metrizable?

c) If the answers to a) and b) are both positive, then is it necessary that Y be complete?

II a). We do not have any results in this direction (even under the hypothesis of complete metrizability). As a corollary of the universality of the 0-dimensional selection theorem we will prove (see §6 below) a generalization of one of Michael's results to complete but not necessarily locally convex spaces. We remark that this question is closely connected with one of the old problems of infinite-dimensional topology: is there an absolute retract of any infinite-dimensional completely metrizable vector space (see [115])?[(1)]

II b). For this part the situation is as follows. In practice, for any concrete non-metrizable locally convex space from the standard list (inductive limits, Banach spaces in weak topologies, spaces of continuous functions with the topology of pointwise convergence) we can give a corresponding counterexample (see for example [25]). But up to now there is no proof that the condition that Y be measurable is necessary in the general case.

Under the assumption that Y is *normal* the best possible positive answer has been obtained by Nedev and Valov [79], [80].

***Theorem* 4.1.** *For any normal locally convex space Y the following are equivalent*:

a) *Y is a closed separable weakly compact subspace of some Fréchet space*;

b) *for any normal space X, any lower semicontinuous map from X into Y with closed convex values has a lower semicontinuous closed selection.*

II c). Here we have a positive answer in the case of *normed spaces*. From Klee's results it follows that a normed space that is not complete is not complete in the topological sense, and so there is no extensor for the class of paracompact spaces, and moreover there is no positive solution to the selection problem (see the references in [58], p. 364). The following example due to Michael is also relevant here.

Example. *There exists a lower semicontinuous map from the interval $I = [0, 1]$ into a normed space Y that is not complete and in which all the values are closed and convex, but for which there is no continuous selection.*

Proof. For the space Y we take the subspace of the Banach space l_1 that consists of all sequences that have non-zero entries only for a finite set of indices.

(1) Recently R. Canty has constructed a counterexample. Dranishnikov's example on infinite-dimensional compact sets with finite cohomological dimension was essential to this.

Then we may assume that the set of indices for the sequences in l_1 is precisely the set of all rational points in I with some fixed numberings. Then the mapping

$$F(x) = \begin{cases} C & \text{if } x \text{ is irrational,} \\ C \cap \{y \in Y \mid y(r_n) \geq 1/n\} & \text{if } x = r_n, \end{cases}$$

where $C = \{y \in Y \mid y \geqslant 0\}$, is the map we seek.

Answer III a). §5 of this survey will deal with the question of convexity.

Answer III b). Sometimes we can do without assuming that the values of a multivalued map are closed (under the assumption that they are convex).

Theorem ([58], Theorem 3.1‴). *For T_1-spaces X, the following are equivalent*:

a) *X is perfectly normal*;

b) *for any separable Banach space Y and for any lower semicontinuous map from X to Y with values that are convex of type* (D) *there exists a continuous single-valued selection.*

A convex set in a Banach space is of type (D) if it contains all interior points (in the convex sense) of its closure; a point in the closure of a convex set is called interior (in the convex sense) if it does not lie on any supporting hyperplane for the set. The following are of type (D): a) all closed convex sets; b) all convex sets that contain at least one interior point (in the usual metric sense); c) all finite-dimensional convex sets; d) the set S of all continuous functions that are strictly increasing on the interval $[0, 1]$ and keep the endpoints of the interval fixed, $S \subset C[0, 1]$. This last example is important in the proof of Theorem 2.5 (see §2 above, on topologically regular maps). The hypothesis that Y is separable is essential in this theorem, as we see from [58], Example 6.3.

We now give one version where the requirement that the values are closed is dropped, not for each set individually but for all the values $F(x)$ at once.

***Theorem* 4.2.** *For any open subset G of any Banach space Y, and for any lower semicontinuous map of an arbitrary paracompact space X into G with convex closed* (*in G*) *values there exists a continuous single-valued selection.*

Recently Gutev proved an analogous theorem for maps of a countable-dimensional metric space into a G_δ-subset of a Banach space.

Answer IV. The assumption of lower semicontinuity for a multivalued map is of course *not* a necessary condition in order for it to have at least one continuous selection (compare Proposition 1.1, where the question was about the existence of a large number of local continuous selections). Clearly, if from a map H we can manage to *refine* some lower semicontinuous map F, then after this we may restrict the problem to obtaining a selection of F, which will then automatically be a selection of H. This simple observation has been made at different times by various authors. We give a result due to Lindenstrauss [52]; for a generalization see [97].

***Theorem* 4.3.** *Let H be a convex-valued closed-valued map of a metric space M into a separable Banach space Y, such that for any countable compact set $K \subset M$ the restriction $H|_K$ has a continuous single-valued selection. Then the map H has a continuous single-valued selection on the whole of M.*

Proof. For any point $x \in M$ and for any countable compact set K that contains x we can define a non-empty convex subset $F(x, K)$ of the set $H(x)$:

$$F(x, K) = \{h(x) \mid h \text{ is a selection of the map } H|_K\}.$$

Now we define the closed convex subset $F(x)$ of $H(x)$ as the intersection of the closures of all the sets $F(x, K)$ with respect to all countable compact sets K that contain x. It is clear that the set $F(x)$ is non-empty and the map $F: M \to Y$ is lower semicontinuous. □

Remark. In this theorem it is not sufficient to assume that sequences in M converge.

The question of whether a lower semicontinuous map can be refined to a given m-map was also studied in [39] by Gel'man, who restricted consideration to a metrizable domain of definition X and a map F with convex compact values in some convex paracompact subset Y of a Banach space. In this case every map F defines some operator $L(F)$ by

$$L(F)(x_0) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\bigcup_{\delta>0}\left(\bigcap\{F_\varepsilon(x) \mid x \in U_\delta(x_0)\}\right)\right).$$

If we look at the transfinite iterates of L, then we obtain the following theorem [39].

***Theorem* 4.4.** *For a map F of a metric space X into a convex compact subset of a Banach space Y with convex closed values to have a continuous selection, it is necessary and sufficient that the sequence $\{L^\alpha(F)\}$ stabilizes at some step α_0, and further $(L^{\alpha_0}(F))(x) \neq \emptyset$ for all $x \in X$.*

Finally, we give a recent result due to Gutev [43].

***Theorem* 4.5.** *Any closed-valued lower quasi-semicontinuous map of a topological space into a complete metric space has a lower semicontinuous selection.*

Here a multivalued map $H: X \to Y$ is called *lower quasi-semicontinuous* if for every $x \in X$, every neighbourhood $V = V(x)$ and every $\varepsilon > 0$ we can find a point $x' \in V$ such that for any point $y \in H(x')$ there exists a neighbourhood U of x such that $y = \cap\{B_\varepsilon(H(z)) | z \in U\}$, where $B_\varepsilon(S)$ is an open ε-neighbourhood of the subset S in the metric space (Y, ρ).

2. The method of covers.

Both the terminology and methods introduced by Choban have, as a result of careful analysis of the proof of the compact-valued theorem, Theorem 1.3, axiomatized the method used in such a proof, turning it into a good working tool to obtain theorems on compact-valued selections of multivalued maps.

Suppose we are given a topological space X, a metric space (Y,ρ), and a multivalued map $F: X \to Y$. Suppose further that we are given three types of objects:

a) a countable spectrum $p = \{(p_n, A_n)\}$ of discrete (pairwise disjoint) indexed sets A_n and their maps p_n

$$\{*\} = A_0 \underset{p_0}{\longleftarrow} A_1 \underset{p_1}{\longleftarrow} \dots \underset{p_{n-1}}{\longleftarrow} A_n \underset{p_n}{\longleftarrow} A_{n+1} \longleftarrow \dots;$$

we denote the limit of this spectrum by A;

b) a sequence $\gamma = (\gamma_n)$ of covers (not necessarily open) of the topological space X, indexed for each n by the set A_n, that is, $\gamma_n = \{V_{n,\alpha} | \alpha \in A_n\}$;

c) a sequence $\omega = (\omega_n)$ of systems of open subsets (not necessarily covers) of the metric space Y, indexed for each n by the set A_n, that is, $\omega_n = \{W_{n,\alpha} | \alpha \in A_n\}$.

We assume that the triple (p, γ, ω) satisfies the following axioms.

MC1. $\sup\{\text{diam}\, W_{n,\alpha} | \alpha \in A_n\} \leqslant 2^{-n}$;

MC2. $W_{n,\alpha} \supset \bigcup\{\text{cl}(W_{n+1,\beta}) | \beta \in p_n^{-1}(\alpha)\}$;

MC3. $V_{n,\alpha} = \bigcup\{V_{n+1,\beta} | \beta \in p_n^{-1}(\alpha)\}$;

MC4. $\text{cl}(V_{n,\alpha}) \subset F^{-1}(W_{n,\alpha})$;

MC5. if $\alpha^* = (\alpha_n) \in A$, then the intersection $D(\alpha^*) = \cap W_{n,\alpha_n}$ is non-empty.

For a complete metric space (Y,ρ) the axiom MC5 is a consequence of MC1 and MC2.

Now we define two multivalued selections G and H of the map $\text{cl}(F)$ as follows:

$$G(x) = \bigcup\{D(\alpha^*) \mid \alpha^* \in A,\ x \in \cap V_{n,\alpha_n}\},$$

$$H(x) = \bigcup\{D(\alpha^*) \mid \alpha^* \in A,\ x \in \cap\, \text{cl}(V_{n,\alpha_n})\}.$$

It turns out that under the fixed properties MC1–MC5 the triple (p,γ,ω) clearly manages to take those properties from the covers γ_n that guarantee the compactness and appropriate semicontinuity of the maps G and H.

***Theorem* 4.5.** *In the notation introduced earlier*:

a) *if the covers* γ_n, $n \in \mathbb{N}$, *are pointwise finite, then the sets* $G(x)$ *are compact*;

b) *if the covers* γ_n, $n \in \mathbb{N}$, *are open, then* G *is lower semicontinuous*;

c) *if the covers* γ_n, $n \in \mathbb{N}$, *are locally finite, then the sets* $H(x)$ *are compact and* H *is lower semicontinuous.*

We now state some results from a series due to Choban [16], [17].

***Theorem* 4.6.** *Any continuous closed-valued map of a normal space* X *into a completely metrizable space has an upper semicontinuous compact-valued* *selection, which has a lower semicontinuous compact-valued selection. If the space X has zero measure, then there exists a continuous single-valued selection of such a multivalued map.*

***Theorem* 4.7.** *If X is a T_1-space, the following are equivalent:*

a) *X is weakly paracompact;*

b) *any closed-valued lower semicontinuous map of X into a complete metric space has a compact-valued lower semicontinuous selection.*

Using the method of covers, in [112] a proof of Theorem 3.2′ is presented with simultaneous "filtration" through all Banach spaces. This method is applied in [76] and [42] to obtain selection theorems that connect the "collective normal" Theorem 3.2 with the corresponding 0-dimensional and finite-dimensional selection theorems.

3. "Countable" selection theorems.

In 1974 at the International Congress of Mathematicians in Vancouver, Michael announced the following theorem [66].

***Theorem* 4.8.** *Any lower semicontinuous map from a countable regular space into a space with the first axiom of countability has a single-valued continuous selection.*

The proof was published in 1981 in [68]. In addition six theorems appeared, connected with discarding the need for the values of a multivalued map to be closed on a countable subset of the domain of definition, and a theorem that connected the 0-dimensional and convex-valued theorems with Theorem 4.8. By way of a remark, Michael noted that these theorems also hold for σ-discrete sets, although in Theorem 4.8, for this to be true, regularity must be changed to paracompactness. In 1978 in [18] Choban used the method of covers to prove a stronger statement.

***Theorem* 4.9.** *The compact-valued selection theorem remains true if the subset L of all points in the domain of definition where the values of the lower semicontinuous map are not closed is σ-discrete. Moreover, a compact-valued selection of the given map can be constructed so that for $x \in L$ its values are finite.*

***Theorem* 4.10.** *Any lower semicontinuous map F of the 0-dimensional paracompact space X into a complete metric space has a continuous single-valued selection if the subset of all points in X where its values are not closed is σ-discrete.*

***Theorem* 4.11.** *Theorem* 4.7 *(on the characterization of weak paracompactness) holds if we omit the assumption that the values of the lower semicontinuous map are closed on an arbitrary σ-discrete subset of the domain of definition.*

We must not fail to point out that Theorem 4.9 (without the statement that the values of the selections are finite on the σ-discrete set) and Theorem 4.10, together with Theorem 4.8 (directly for a σ-discrete set), were all announced by Choban in 1970 [20].

Kolesnikov proved the following [48].

***Theorem* 4.12.** *Suppose that X is completely regular (collectively normal) and Y is a complete metric space. Let $C \subset X$ be a countable (σ-discrete) subset of X and $F : X \to Y$ a map such that $F(x) = Y$ for $x \notin C$ and $\operatorname{cl} F(x) = Y$ for $x \in C$. Then F has a selection if Y is either locally connected or is an ANR.*

Michael strengthened Kolesnikov's results somewhat in [69].

4. Amalgamated selection theorems.
The start of this area of research was a short paper [74] where the following theorem was proved, which coincides with the convex-valued selection theorem for $Z = \emptyset$ and with the 0-dimensional selection theorem for $Z = X$. The inequality $\dim_X Z \leqslant 0$ implies that for any subset E of X that is closed in X the inclusion $E \subset Z$ implies that $\dim E \leqslant 0$.

***Theorem* 4.13.** *Let X be paracompact, Y a Banach space, and Z a subset of X with $\dim_X Z \leqslant 0$. Then any lower semicontinuous map F from X to Y with convex values $F(x)$ for all $x \notin Z$ has a continuous single-valued selection.*

***Theorem* 4.14** (Theorem 1.2 and Lemma 6.16 in [67]). *Let X be paracompact, Y a Banach space, and Z a subset of X with $\dim_X Z \leqslant n+1$. Then any closed-valued lower semicontinuous map F from X to Y such that the values $F(x)$ are convex for $x \notin Z$ has a continuous single-valued selection if for all $x \in Z$ the values $F(x)$ are n-connected and the family $\{F(x) | x \in Z\}$ is uniformly equilocally n-connected.*

For versions of amalgamations of these theorems with "countable" theorems see [73].

§5. Attempts to waive convexity. Convexity in metric spaces. Paraconvexity. Topological convexity

1. We begin with an example ([58], Example 6.1), to show that in Theorem 1.2 the condition that the values $F(x)$ of the map F be convex is essential.

***Example* 5.1.** *There exists a lower semicontinuous (and even continuous) map of the interval $[0, 1]$ into the Euclidean plane whose values are the graphs of continuous functions on some intervals, and which does not have a continuous single-valued selection.*

Proof. Naturally, we use $\sin(1/x)$. More precisely, we put each $t \in (0, 1]$ in correspondence with the set

$$F(t) = \{(x, y) \mid x \in [t/2, t], y = \sin(1/x)\},$$

and we put $F(0) = \{(0, y) | y \in [-1, 1]\}$. □

In this example the values of $F(t)$ as $t \to 0$ are very non-convex. The natural way to try to waive convexity is not so radical.

Let Y be a normed space and P a non-empty closed subset of Y. We fix $R > 0$ and for any open ball D of radius R we define the number

$$\delta(D, P) = \big(\sup\{\operatorname{dist}(q, P) \mid q \in \operatorname{conv}(D \cap P)\}\big)/R.$$

If D does not intersect P, then $\delta(D, P) = 0$. We define $h_P(R)$ to be the supremum of the set $\{\delta(D, P)\}$ for all open balls D of radius R.

Roughly speaking, we consider all simplexes of radius R with vertices in the set P and we look at how far from P points of the same simplex go: $\operatorname{dist}(q, P) \leqslant h_P(R) \cdot R$, $q \in \operatorname{conv}(D \cap P)$. It is clear that $h_P(R)$ cannot exceed 2, and examples in the space l_∞ show that in general we cannot obtain a smaller upper bound. In a Euclidean space Y for any closed $P \subset Y$ and any $R > 0$ a better estimate holds, $h_P(R) \leqslant 1$. Klee has shown that such an estimate for any P when $\dim E > 2$ *characterizes* the spaces with scalar product among the normed spaces. Thus, every non-empty closed subset P of the normed space Y can be put in correspondence with some function $h_P : (0, \infty) \to [0, 2]$.

Definition 5.1. The function h_P constructed above is called the *function of non-convexity* of the non-empty closed set P.

If the identity $h_P \equiv 0$ holds, then P is convex. The further h_P is from zero, the less convex P is. For instance, in the Euclidean plane, for a semicircle of radius r we have $h_P(r) = 1$, and for a parabola, the function of non-convexity monotonically tends to 1 from below.

Definition 5.2 [63]. Let $\alpha \in [0, 1)$. A non-empty closed subset P of a normed space Y is called *α-paraconvex* if its function of non-convexity h_P nowhere exceeds α.

***Theorem* 5.1.** *Let $h : (0, \infty) \to [0, 1)$ be a fixed monotone non-decreasing function. Then any lower semi-continuous map F of a paracompact space X into a Banach space Y, whose function of non-convexity $h_{F(x)}$ takes values that are strictly less than h, has a continuous single-valued selection.*

Establishing α-paraconvexity and further verifying that the function $h_{F(x)}$ majorizes the non-convex values of $F(x)$ for some fixed function $h : (0, \infty) \to [0, 1)$ is itself rather hard. In the definition of paraconvexity there are six quantifiers $(\exists, \forall, \forall, \forall, \forall, \exists)$. Nonetheless there has been success in obtaining some results for a finite-dimensional Euclidean space and graphical maps.

***Theorem* 5.2** [98]. *For any $n \in \mathbb{N}$ and any $C \geqslant 0$ there exists $\alpha = \alpha(n, C) \in [0, 1)$ such that the graph of any Lipschitz function of n real variables with constant C and with a closed convex domain of definition is α-paraconvex in $(n+1)$-dimensional Euclidean space.*

***Theorem* 5.3** [91]. *For fixed $n \in \mathbb{N}$ and fixed $C > 0$ for the set of graphs of all polynomials of the form*

$$P(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad |a_i| \le C, \quad |a_i/a_n| \le C,$$

with closed convex domain of definition there exists a function $h : (0, \infty) \to [0, 1)$ that strictly majorizes the functions of non-convexity of all such graphs in the Euclidean plane.

Clearly, using Theorem 5.1, in Theorem 1.2 the requirement that the values $F(x)$ of the lower semicontinuous map F be convex can be changed to the requirement that $F(x)$ lies in the set of graphs in Theorems 5.2, 5.3. We note that for this, the system of coordinates in which we consider the value $F(x)$ as a graph can depend arbitrarily on $x \in X$. The condition of uniform boundedness of the coefficients of the polynomials is essential in this theorem. An interesting open question is the case of polynomials of several real variables.

2. In the theory of differential inclusions (Cauchy's problem with a multivalued right-hand side) there is a series of papers on continuous selections of maps with non-convex values. The details of this theory reduce to posing selection problems for maps with values in the particular Banach space $L_1(\mu)$ and with a particular version of non-convexity: convexity by switching (the native term) or the decomposability of the subset in the space $L_1(\mu)$.

Let T be a compact space with a non-negative non-atomic Borel measure μ on it, let Z be a separable Banach space, and $L_1(T, Z)$ the Banach space of Bochner integrable classes of summable maps from T to Z. A subset D of $L_1(T, Z)$ is decomposable if for any $f, g \in D$ and any μ-measurable subset $A \subset T$ there is a map in D that equals f on A and g on $T \setminus A$.

Theorem (Friszkowski [37], 1983). *Any lower semicontinuous map of a compact space into the Banach space $L_1(T, Z)$ with non-empty closed decomposable values has a continuous single-valued selection.*

The first result of this kind was obtained by Antosiewicz and Cellina [2] in 1975 in the case when the compact space to be mapped is a compact space S in the Banach space of continuous maps of the interval $[0, 1]$ into the space $\mathbb{R}^n$, and the multivalued map $F : S \to L_1([0, 1], \mathbb{R}^n)$ is defined by

$$F(s) = \{u \in L_1([0, 1], \mathbb{R}^n) \mid u(t) \in P(t, s(t)) \text{ almost everywhere on } [0, 1]\}.$$

All cases were examined for a fixed compact-valued map $P : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$, relative to which there were solutions of the differential inclusion $x'(t) \in P(t, s(t))$. The results in [2] are that for a bounded map P that is measurable in t (for all x) and continuous in x (for almost all t) the function F defined above has a selection, and the corresponding differential inclusion has a solution. Next, in [9] the condition of continuity in x was weakened to lower semicontinuity. The theorem we gave earlier was proved by analysing the proofs in [2] in abstract terms.

For generalizations and different applications see, for instance, [41], [10], [12] and the references cited there. We also make a simple observation that shows that decomposable subsets of $L_1[0,1]$ are in general not paraconvex, that is, the techniques of the last subsection do not apply to them. If $f, g \in L_1[0,1]$, then the decomposable set D is the family of functions $\{(1-\chi)f+\chi g\}$, where χ is changed by the characteristic functions of measurable subsets from being identically zero to being identically one. But the whole set D is centrally symmetric relative to $(f+g)/2$ and lies on a sphere with centre at this point. Thus, the set D made up of the functions f and g and all "natural" paths between them is decomposable, but not α-paracompact for any $\alpha < 1$.

3. In subsection 1 we looked at the case when the metric structure of the (Banach) space Y combined well with the canonical structure of the convexity of Y as a vector space and we estimated the "degree of non-convexity" of the values of the multivalued map. However, instead of non-convex sets in "standard convexity", we can consider convex sets in some "non-standard convexity".

Definition 5.3 [62]. Let Δ^n be the standard unit simplex with n vertices in n-dimensional Euclidean space. The *convex structure of a metric space* (Y,ρ) is a sequence of pairs $\{(M_n,k_n)\}_{n=1}^{\infty}$, where M_n is a subset of the nth Cartesian power Y^n and k_n are maps $k_n : M_n \times \Delta^n \to Y$ such that the following conditions hold:

a) if $x \in M_1$, then $k_1(x,1) = x$;

b) if $x \in M_n$, then $\partial_i x \in M_{n-1}$ and if further $t_i = 0$ for $t \in \Delta^n$, then $k_n(x,t) = k_{n-1}(\partial_i x, \partial_i t)$ where ∂_i is the operator that omits the ith coordinate;

c) if $x \in M_n$ and $x_i = x_{i+1}$, then for $t \in \Delta^n$

$$k_n(x,t) = k_{n-1}(\partial_i x, t_1, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n);$$

d) for fixed $x \in M_n$, the map $k_n(x,\cdot)$ is continuous;

e) for every $\varepsilon > 0$ there exists a neighbourhood V of the diagonal in $Y \times Y$ such that for all $n \in N$ and all $x, y \in M_n$ if $(x_i, y_i) \in V$, $1 \leqslant i \leqslant n$, then $\rho(k_n(x,t), k_n(y,t)) < \varepsilon$ for all $t \in \Delta^n$.

Conditions a)–c) give the usual compatability conditions for curvilinear simplexes $k_n(x,\cdot)$ with respect to their faces. Condition e) is essential, and in real examples it is hard to verify: the ε-closeness of points of curvilinear simplexes with the same coordinates should ensure some closeness (independent of the dimension!) of the vertices of these simplexes.

Definition 5.4. A subset C of a metric space Y with a convex structure $\{(M_n,k_n)\}$ is said to be *convex* if for any $n \in \mathbb{N}$ and any $x_1 \in C, \ldots, x_n \in C$ the n-tuple $x = (x_1, \ldots, x_n)$ lies in M_n and $k_n(x,t) \in C$ for all $t \in \Delta^n$.

The proof of the corresponding analogue of the Selection Theorem 1.2 for metric spaces with convex structure does not proceed analogously to that of Theorem 1.2. In Theorem 1.2 the desired selection is constructed as a uniform limit of a *sequence* of *continuous maps* f_n with values ε_n-close to the values of the multivalued map, but, in general *lying outside* these values: an exterior approximation. Here the maps f_n in general are *discontinuous* and their values *lie in* the values of the given multivalued map: it is an interior approximation.

***Theorem* 5.4** [62]. *Any lower semicontinuous map F from a paracompact space X into a metric space (Y,ρ) with convex structure $\{(M_n,k_n)\}$, whose values are complete with respect to the metric ρ and convex relative to the structure $\{(M_n,k_n)\}$, has a continuous single-valued selection.*

In essence the proof is the same as the method of covers, where compact-valued selections are approximated by finite-valued (discontinuous) selections. Here at the nth step we have to look at convex combinations (relative to the structure $\{(M_n,k_n)\}$) of elements of these finite-valued selections.

Curvilinear simplexes in the space with the given metric cannot be defined simultaneously in all dimensions, as we did in Definition 5.3, but we can begin with curvilinear segments, and then give an inductive construction on the dimension. Thus we arrive at the definition of a *geodesic structure* on a metric space.

Definition 5.5 [62]. A *geodesic structure of a metric space* (Y,ρ) is a pair (M,k), where M is a subset of $Y\times Y$ and k is a map $k: M\times[0,1]\to Y$ such that the following conditions are satisfied:

a) if $(x,x)\in M$, then $k(x,x,t)=x$;

b) if $(x_1,x_2)\in M$, then $k(x_1,x_2,0)=x_1$, $k(x_1,x_2,1)=x_2$;

c) if $(x_1,x_2,t)\in M\times[0,1]$ and $(k(x_1,x_2,t),x_2)\in M$, then $k(k(x_1,x_2,t),x_2,s)=$ $=k(x_1,x_2,t+s(1-t))$ for all $s\in[0,1]$;

d) for fixed $(x_1,x_2)\in M$ the map $k(x_1,x_2,\cdot)$ is continuous;

e) for each $\varepsilon>0$ there exist neighbourhoods $W\subset V$ of the diagonal in $Y\times Y$ such that if $(x,y)\in V$, then $\rho(x,y)<\varepsilon$, and such that if $(x_1,x_2)\in M$, $(y_1,y_2)\in M$, $(x_1,y_1)\in V$, $(x_2,y_2)\in W$, then $(k(x_1,x_2,t),k(y_1,y_2,t))\in V$ for all $t\in[0,1]$.

We should remark that condition e) in the last definition is strongly analogous to the condition in Definition 5.3 for $n=2$. This is connected with the need to proceed inductively with the construction of the curvilinear simplexes. To construct a curvilinear triangle with ordered vertices x_1,x_2,x_3 we first take the "segment" with endpoints at x_2 and x_3 and then join its points by means of "segments" to the vertex x_1: that this is well defined follows from c). "Simplexes" of arbitrary dimensions are defined analogously. We will say that a subset G of a metric space Y with geodesic structure (M,k) is a *geodesic* set if $x_1,x_2\in G$ implies that the pair (x_1,x_2) lies in M, and all the points on the segment $k(x_1,x_2,t)$ lie in G.

***Theorem* 5.5.** *For any geodesic structure in a metric space there exists a convex structure in this space such that any geodesic set is convex with respect to this convex structure.*

A fairly typical example is a Riemannian manifold Y with Riemannian metric ρ. We take L to be the set of all pairs of points on the manifold Y for which there exists exactly one shortest geodesic connecting the two points, and let $h : L \times [0, 1] \to Y$ be the natural map that translates every point of the triple (x_1, x_2, t) to the corresponding point of the shortest geodesic with endpoints x_1 and x_2.

***Theorem* 5.6** ([62], Proposition 6.1). *On any compact Riemannian manifold there exists a geodesic structure (M, k) such that $M \subset L$, $k = h|_M$ and for every point $y \in Y$ there exists a neighbourhood V with $V \times V \subset M$.*

We mention the paper [26] by Curtis on the contractibility of the hyperspace of subsets of a metric continuum. In this, condition c) from Definition 5.3 is omitted, but condition e) is strengthened to a condition of uniform type ($\forall \varepsilon > 0 \, \exists \delta > 0 \ldots$) and with these changes he gives a proof of the analogue to Theorem 5.4. It is possible to construct such a modified convex structure in the space of maximal arcs in the hyperspace $C(X)$, which consists of subcontinua of the given metric continuum X. Any such maximal arc γ connects some one-point arc $e(\gamma) = \gamma(0)$ continuously in $C(X)$ with the whole space X. One of the main results in [26] is that $C(X)$ is contractible if and only if the map e^{-1} has a lower semicontinuous selection, that is, if e is inductively open. The proof is based on finding a single-valued continuous selection for some lower semicontinuous selection of e^{-1}, that is, finding a section for e.

4. Topological convexity.
A systematic study of topological convex structures was first made, so it seems, in the papers of van de Vel [106], [107] and in his joint papers with van Mill [109], [110] (see also the recent paper [108]).

Definition 5.6. A family of subsets of a given set X is called a *convex structure* if:

CONV 1. The empty set and the whole set X lie in the family.

CONV 2. The family is closed under the intersection of an arbitrary number of its elements.

CONV 3. The family is closed under the union of an arbitrary number of its elements that are linearly ordered by inclusion.

If we remove the condition that the whole set X lies in the family from the above axioms, then the family is called a convex system. Elements of a convex structure (system) are called convex sets.

As in the case of standard convexity, the *convex hull* $\operatorname{conv} A$ of a subset A of a set X with a convex structure (system) is the intersection of all convex sets that contain A; a *polytope* is the convex hull of a finite set; a *half-space* is a convex (non-empty) subset, whose complement is also convex. For a convex structure we can introduce analogues $S_1 - S_4$ to the separability axioms $T_1 - T_4$ for topological spaces. For instance, S_1: all sets consisting of a single point are convex. Similarly, S_4: if two convex sets are disjoint, then they lie in some mutually complementary half-spaces.

Definition 5.7. *A topological convex structure* is a set with a convex structure, determined by the topology, in which all polytopes are closed. If, further, the closures of the convex sets are closed, then the structure is *closure-stable.*

Definition 5.8. A topological convex structure is called *uniformizable* if there exists a uniformity μ generating the given topology such that for a uniform cover $U \in \mu$ we can find a uniform cover $V \in \mu$ for which the convex hull of the star of any convex set with respect to V lies in the star of this convex set with respect to U. If this uniformity μ is induced by some metric d, then the topological convex structure is said to be *metrizable*, and the metric d is said to be *compatible.*

***Theorem* 5.6.** *Suppose that a closure-stable topological structure has properties S_1 and S_4, that the convex sets in it are connected, and the polytopes are compact. Let C be a non-empty convex set, and ω a cover of C by open convex sets. Then the nerve of the cover ω is contractible.*

Finally, we state the analogues of the selection Theorems 3.1′ and 3.2″ for topological convex structures.

***Theorem* 5.7.** *Suppose that under the hypotheses of the last theorem there is a metric d that is compatible with a topological structure Y. Then*:

a) *if X is normal, and Y is separable, then any lower semicontinuous map from X to Y with compact convex values has a continuous selection*;

b) *if X is paracompact, then any lower semicontinuous map from X to Y with d-complete convex values has a continuous selection.*

We stress that there is a difference in the approaches to defining convex structures even on the level of metric spaces. Michael has this version of the "exterior" definition. Van de Vel makes all the constructions in an "interior" way. Michael constructs a selection as a uniform limit of single-valued continuous maps ("exterior approximations") or 2^{-n}-continuous maps ("interior approximations"), whereas van de Vel cuts off open convex sets from the complete convex values of a multivalued map until only a single-point set remains.

§6. Selections and averaging operators. The universality of the 0-dimensional selection theorem

1. The main aim of this subsection is to show that as corollaries of the simple 0-dimensional selection theorem, Theorem 1.1, we can obtain both the compact-valued and convex-valued selection theorems. We will call this property of Theorem 1.1 its *universality.*

This idea of universality for this theorem, it seems, was first introduced in [1], where the 0-dimensional selection theorem was used to derive a compact-valued selection theorem with *transposed order* of lower and upper semicontinuous selections. An analogous result was obtained independently in [99], where such a "transposition" theorem was mistakenly called a compact-valued theorem. The main thrust of [99] is nonetheless to deduce the convex-valued Theorem 1.2 from the 0-dimensional theorem. This is done by using Milyutin maps, that is, continuous surjections that have a regular linear averaging operator. This idea of using Milyutin maps in the case of multivalued operators with compact domains of definition is due to Shchepin and was announced in the Tiraspol' Symposium on Topology in 1985. In [99] the proof is given for a wide class of paracompact spaces (compact, strongly paracompact, p-paracompact ...), and in [94] it is given for all paracompact spaces. It remains to point out that universality can easily be obtained for metrizable spaces by using Choban's results [19].

If X is completely regular, we let $C(X)$ denote the Banach space of all continuous bounded real functions on X with the usual sup-norm; it is useful to identity $C(X)$ with the space $C(\beta X)$ of continuous functions on the Stone–Čech compactification βX of the space X. We let $P(X)$ denote the space of all regular probability Borel measures on X; the topology on $P(X)$ is induced by the weak-$*$ topology from the space dual to $C(\beta X)$.

Definition 6.1 [102]. A continuous surjection $f: X \to Y$ is called a *Milyutin map* if there exists a continuous map $\nu : Y \to P(X)$ such that $\operatorname{supp}\nu(y) \subset f^{-1}(y)$ for all $x \in X$, where $\operatorname{supp}\nu(y)$ is the support of the measure $\nu(y)$.

2. ***Theorem* 6.1.** *Any paracompact space X is the image of some zero-dimensional paracompact space X_0 for some Milyutin perfect inductively open map p.*

***Corollary* 1.** *The compact-valued selection Theorem* 1.3 *is a consequence of the* 0-*dimensional selection Theorem* 1.1.

***Corollary* 2.** *The convex-valued selection Theorem* 1.2 *is a consequence of the* 0-*dimensional selection Theorem* 1.1.

***Corollary* 3.** *Let Y be a locally convex topological vector space, M a metrizable subset of Y, and F a lower semicontinuous map from X to M such that*:

a) *all the values $F(x)$ are complete with respect to some metric compatible with the topology induced from Y*;

b) *the closed convex hull of any compact set that lies in any value $F(x)$ is compact.*

Then F has a continuous single-valued selection f such that $f(x) \in \operatorname{cl}(\operatorname{conv} F(x))$ *for any $x \in X$.*

***Corollary* 4.** *Suppose that X and Y are completely metrizable topological vector spaces, and that L is a linear continuous surjection from Y onto X, with kernel* $Z = \operatorname{Ker} L$ *a locally convex space. Then the map L has a continuous section, and in particular, Y is homeomorphic to* $X \oplus \operatorname{Ker} L$.

Proof of Corollary 1. Let $F: X \to Y$ be a lower semicontinuous map from the paracompact space X into the metric space (Y, ρ) with values $F(x)$ that are complete in Y. Let $p: X_0 \to X$ be the map in Theorem 6.1. Then we may apply Theorem 1.1 to $F \circ p : X_0 \to Y$ and so it has a single-valued selection $g : X_0 \to Y$. The map p is closed, and so p^{-1} is upper semicontinuous. Thus, $g \circ p^{-1}$ is an upper semicontinuous selection of F. Since p is perfect, all the inverse images $p^{-1}(x)$ are compact. But then the values of the selection $g \circ p^{-1}$ are also compact. Further, since p is inductively open, we can find some lower semicontinuous selection of p^{-1}, say $G : X \to X_0$, $G(x) \subset p^{-1}(x)$. Then the composition $g \circ G$ is a lower semicontinuous compact-valued selection of $g \circ p^{-1}$. □

Proof of Corollary 2. Let $F : X \to Y$ be a lower semicontinuous map of a paracompact space X into a Banach space Y that has closed convex values $F(x)$. Let $p : X_0 \to X$ be the map in Theorem 6.1. Then we may apply Theorem 1.2 to $F \circ p : X_0 \to Y$ and so it has a continuous single-valued selection $g : X_0 \to Y$. Let $\nu : X \to P(X_0)$ be the map that is associated with the Milyutin map p.

We define the single-valued map $f : X \to Y$ by the equality

$$f(x) = \int g \, d\nu(x),$$

where the integral is taken over the compact set $p^{-1}(x)$ that contains the support of the measure $\nu(x)$. By the definition of integration with respect to a probability measure, its value lies in the closed convex hull of the values of the integrand. But, by construction, $g(p^{-1}(x)) \subset F(x)$, and since $F(x)$ is closed and convex, $f(x) \in F(x)$. □

Proof of Corollary 3. In the last proof it is enough to use the fact that in order for the integral to exist over a compact set relative to the probability measure in the given case it is sufficient that the closed convex hulls $\mathrm{cl}(\mathrm{conv}(g \circ p^{-1}(x))$ of the compact sets $g \circ p^{-1}(x) \subset F(x)$ are compact for $x \in X$. □

The proof of Corollary 4 is analogous to that of Corollary 3. □

Remark 1. The proof of Corollary 3 gives a new proof of the theorem in [65] and strengthens its statement somewhat: property b) need not be relative to the whole set M, but just the values $F(x)$. Corollary 4, which generalizes the Bartle–Graves theorem, gives a new proof of Corollary 7.3 in [62] (see also Proposition 7.1 in Chapter 2 of [4]).

Remark 2. Corollaries 3 and 4 can be strengthened somewhat if we note the fact that to have an integral we do not need the space containing the values of the integrand to be locally convex. It is sufficient that the closed convex hull of the set of values of the continuous function to be integrated over the compact set is itself compact.

3. *Proof of Theorem* 6.1. We consider the set of all distinguished locally finite open covers of the given paracompact space X, that is, the set of all pairs

(γ, e), where $\gamma = \{G_\alpha\}$ is a locally finite open cover, and $e = \{e_\alpha\}$ is some local partition of unity that is fixed for γ and refined in γ; α runs through some discrete set of indices $A = A(\gamma)$.

First we construct some Milyutin map for every pair (γ, e). For this we define the set $X_{\gamma,e}$ to be a subset of the Cartesian product of X with the discrete set $A(\gamma)$:

$$X_{\gamma,e} = \{(x, \alpha) \mid x \in \operatorname{supp}(e_\alpha)\}$$

and the map $p_\gamma : X_{\gamma,e} \to X$ as the natural projection onto the first component. As e is locally finite and $\operatorname{supp}(e_\alpha)$ is closed, p_γ is closed. All the inverse images of points of this map are finite, that is, compact. Hence p_γ is perfect, and $X_{\gamma,e}$ is paracompact as the inverse image of a paracompact space under a perfect map. The map p_γ is a Milyutin map. In fact, let

$$p_\gamma^{-1}(x) = \{(x, \alpha_1), \dots, (x, \alpha_n)\}.$$

Then we take the value of the measure $\nu(x)$ at the point (x, α_i) to equal $e_{\alpha_i}(x)$. It is clear that $\nu(x)$ is a probability measure, and its support lies in $p_\gamma^{-1}(x)$. The fact that $\nu(x)$ depends continuously on x follows from the fact that e is locally finite and the functions $e_\alpha, \alpha \in A(\gamma)$, are continuous.

Now we consider a partial product (pull-back) of the maps $p_{\gamma,e}$ over all pairs (γ, e). Stated less formally, we embed X diagonally in its Cartesian degree of cardinality, equal to the cardinality of the set of all γ, and we define the set X_0 to be the following subset of the Cartesian product of the paracompact space X and the Cartesian product of the discrete sets $A(\gamma)$ for all γ; $\gamma \in \Gamma$:

$$X_0 = \{(x, \{\alpha(\gamma)\}_{\gamma \in \Gamma}) \mid x \in \operatorname{supp}(e_{\alpha(\gamma)}), \gamma \in \Gamma\}$$

and the map $p : X_0 \to X$ as the natural projection onto the first component. This completes the construction. The proof is in [94].

§7. Miscellaneous results

1. The characterization of metrizability in the compact case.

Suppose that a compact set K is such that for any zero-dimensional compact set S and any closed lower semicontinuous map $F: S \to K$ there exists a continuous single-valued selection. A simple observation, due to Magerl [54], is that in this case K is metrizable. In fact, by Alexandrov's theorem, any compact subset $T \subset K$ is the continuous image $g(A)$ of some closed subset A of some power D^τ of the two-point set D. But, once we have solved the selection problem for zero-dimensional compact sets with values in K we can continue the map $g : A \to T$ to a continuous map of the whole of the power D^τ of D. Thus, the compact set T is dyadic, and consequently so is K. Then, using a theorem of Efimov, K is metrizable. Here is a more fundamental result from [54].

***Theorem* 7.1.** *Let $\alpha(\Gamma)$ be the one-point bicompactification of an uncountable discrete set Γ that lies in some locally convex vector space Y. Let $X = \exp_3(\alpha(\Gamma))$ be a compact set that consists of no more than three-point subsets of $\alpha(\Gamma)$. Then the map F that associates with every $x \in X$ the convex hull of elements that are re-entrant in x is lower semicontinuous and does not have a continuous single-valued selection.*

Corollary. *For any convex compact set K, the condition that it is metrizable is equivalent to a condition on the existence of selections for arbitrary lower semicontinuous maps with a compact domain of definition, taking values in the convex compact subsets of K.*

Proof. We will deduce that K is metrizable from the solubility of the selection problem. The fact that K is dyadic was proved at the start of this subsection. If we assume that K is not metrizable, then using another of Efimov's theorems, K contains a copy of $\alpha\Gamma$ for some uncountable discrete set Γ. It remains to apply Theorem 7.1.

The next question now seems natural. Is a convex compact set with the property that all its convex compact subsets are dyadic metrizable [54]? Valov has found a positive solution to this question [80] and thus also an alternative proof of the last corollary.

2. Measurable selections.

If we compare the proofs of Michael's selection theorems with the proof of one of the main theorems on obtaining *measurable selections*—theorems due to Kuratowski and Ryll-Nardzewski [51], the overlap is clear: in both cases the desired selection is constructed as a limit of 2^{-n}-selections. Thus it is natural to look for a single approach to the proofs of these theorems,. This has also been done by Lindenstrauss in [52]. In fact, he considered the family Ω of subsets of an arbitrary set X, with X and the empty set as members, that is closed under finite intersections and countable unions. A multivalued map F from such a space (X, Ω) into a topological space Y is called Ω-measurable if the inverse image $F^{-1}(G)$ of any open set G lies in the family Ω. The space (X, Ω) is called (k, n)-paracompact if for any cover $\Omega_1 \subset \Omega$, with $\operatorname{card} \Omega_1 < k$, there exists a cover $\Omega_2 \subset \Omega$ that is a refinement of Ω_1 such that:

a) the dimension of the nerve $N(\Omega_2) \leqslant n$;

b) there exists an Ω-measurable map $F: X \to N(\Omega_2)$ such that $F^{-1}(St(e_B)) \subset B$ for all $B \in \Omega_2$ and the open star of the vertex e_B corresponding to B in this nerve.

Now for the abstract version of the operator of taking the convex hull. This is a map H that puts every subset A of Y in correspondence with some subset $H(A) \subset Y$ such that $H(\{y\}) = \{y\}$, and from $A \subset B$ it follows that $H(A) \subset H(B)$ and $H(A) = H(H(A))$. If H leaves all balls relative to some metric d fixed, then H is called compatible with d. Further, an operator H in a topological space Y is called n-convex if for any simplicial complex Δ of dimension $\leqslant n$, and any map g of its vertices into Y, there exists a continuous extension f of g to Δ such that for any simplex $S \in \Delta$ the image $f(S)$ lies in the H-hull of the image of its vertices under g.

Theorem 7.2. *Let (X, Ω) be (n,k)-paracompact, Y a k-bounded complete metric space, and H the operator of taking the convex hull that is compatible with the metric and n-convex. Then any Ω-measurable map from X to Y whose values coincide with their convex hulls has an Ω-measurable single-valued selection.*

Here k-bounded means there exists a ε-net of cardinality strictly less than k for any $\varepsilon > 0$.

3. Is it possible to manage without convexity in the infinite-dimensional case? In the case when the lower semicontinuous maps have n-dimensional paracompact domains of definition, Theorem 1.4 gives sufficient (and almost necessary) conditions for the existence of selections. Is it possible to obtain a topological analogue of the selection Theorem 1.2 in infinite dimensions, that does not use "additional" structure, namely the structure of convexity?

The following example due to Pixley [87] shows that a positive answer to this question, if it exists at all, must be highly non-trivial. In this example all the values of the lower semicontinuous map are contractible, locally contractible, and locally contractible "with a single speed", and nonetheless there are no continuous single-valued selections.

Theorem 7.3. *There exists a lower semicontinuous mapping F of the Hilbert cube into itself such that:*

a) *all the values of $F(x)$ not consisting of a single point are homeomorphic either to a finite-dimensional cube or to the entire Hilbert cube*;

b) *the family of values $\{F(x)\}$ is a uniform and equilocal absolute extensor* (UE-LAE);

c) *F has no selection, and moreover for some point $x \in Q$ there is no selection of the restriction of F to any neighbourhood of this point.*

We restrict ourselves to a construction that is a modification of Borsuk's well-known example in "the theory of retracts" of a locally contractible metric compact set that is not an ANR. For every $n \geqslant 2$ let

$$E_n = \{x \in Q \mid (n+1)^{-1} \le x_1 \le n^{-1} \text{ and } x_i = 0 \text{ for all } i > n+1\},$$
$$X_n = \{x \in E_n \mid \text{ one of the following conditions is satisfied: } x_1 = (n+1)^{-1}, x_1 = n^{-1},\ x_2 = 0,\ x_i \in \{0,1\} \text{ for some } 2 < i \le n+1\}.$$

We define the map $F: Q \to Q$ as follows. If $x \in \bigcup\{X_n \mid n \geqslant 2\}$ or $x_1 = 0$, then $F(x) = \{x\}$. If $x \in E_n \setminus X_n$, then $F(x) = X_n$. In all remaining cases $F(x) = Q$.

Finally, the point x in condition c) can be chosen as follows: the first coordinate is equal to zero, and the rest are equal to 1/2.

4. Continuous multivalued selections.

***Theorem* 7.4** [82]. *Let F be a lower semicontinuous map from a paracompact space X into a complete metric space Y with closed connected values, and let the family $\{F(x)\}, x \in X$, of these values be equilocally connected. Then any continuous multivalued selection of F can be extended from any closed subset of X to a continuous multivalued selection of F on the whole of X. Moreover, an analogous result holds for continuum-valued selections.*

Here, an equilocal connection of the family $\{S\}$ of subsets of a metric space Y is defined by analogy with the property ELC^n as follows. For any $S' \in \{S\}$, any $y \in S'$ and any neighbourhood $W(y)$ there exists a neighbourhood $V(y) \subset W(y)$ such that for any element $S'' \in \{S\}$ and for any points $a, b \in V(y) \cap S''$ in the set $W(y) \cap S'$ there exists a connected subset that contains both a and b.

Gutev [42] proved a collectively normal version of Theorem 7.4.

Two further papers of Michael's.

a) If $F : X \to Y$ is a multivalued map, we let F^* denote the map from X to the graph Γ_F of F:

$$F^*(x) = \{x\} \times F(x) \subset X \times Y.$$

If the family of values of F^* is equilocally n-connected, then so is F. Thus, since F is an ELC^n-map it follows that the family $\{F(x)\}$ is an ELC^n-family. The converse is false: here is a simple counterexample. Let $X = \mathbb{N}$, $y = \mathbb{R}$ and $F(k) = \{0, 1/k\}$. Then F is an ELC^n-map for any n, but the family $\{F(n)\}$ is not a ELC^0-family.

***Theorem* 7.5** [71]. *The finite-value Theorem* 1.4 *can be strengthened in two directions*:

a) *the hypothesis that the family of values $\{F(x)\}$ is equilocally n-connected can be changed to the hypothesis that the map F is equilocally n-connected*;

b) *the hypothesis that the values $F(x)$ are closed in Y can be changed to the hypothesis that the values of $F^*(x)$ are closed in some G_δ-subset of the Cartesian product $X \times Y$.*

b) In [90] Saint-Raymond proved the following fact. If F is a lower semicontinuous map of an n-dimensional compact metric space X into a Banach space Y with convex closed values $F(x)$ that contain $0 \in Y$, then from the condition $\dim F(x) > n$ for all $x \in X$ it follows that there exists a selection f of F such that $f(x) \neq 0$ for all $x \in X$. Roughly speaking, in the values $F(x)$ there is enough room to avoid the origin. Michael developed this theme in [60].

On the one hand, the condition of finite-dimensionality is essential even if we replace lower semicontinuity by continuity. In fact, consider a map of the Hilbert cube Q into the Hilbert space that for each $x \in Q$ shifts Q onto $-x$. If this continuous map had a selection g that avoids the origin, then the single-valued map $g(x)+x$ would carry the cube Q continuously into itself without fixed points. On the other hand, there are positive results.

Theorem 7.6. *Let X be a topological space, Y a Banach space, and F a continuous map from X to Y with closed convex infinite-dimensional values. Suppose further that if $y \in F(x)$, $y \neq 0$, then $(y/\|y\|) \in F(x)$. Then F has a selection that avoids the origin.*

Theorem 7.7. *Let F be a lower semicontinuous map of a paracompact space X into a Banach space Y with convex closed values, and let $E \subset Y$ be closed, $Z = F^{-1}(Y) = \{x \in X \mid F(x) \cap E \neq \emptyset\}$, and suppose that $\dim X < \dim F(x) - \dim(\operatorname{conv}(F(x) \cap E))$ for all $x \in Z$. Then F has a continuous selection that avoids E.*

6. Selections and regularizability.

One version of the selection problem is the question of selections in the family $\exp X$ of all closed subsets of a topological space X. In $\exp X$ we take the Vietoris topology, which coincides with the topology given by the Hausdorff metric in the metric case. More precisely, the question goes as follows. Does there exists a continuous map $f\colon \exp X \to X$ such that $f(A) \in A$ for all $A \in \exp X$? It is clear that for the order compact set X we can take $f(A) = \min A$. In 1981 in [111] it was shown that (in the compact case) this is the only possibility.

Theorem 7.8. *If X is a compact space, then the following are equivalent:*

a) *X can be ordered;*

b) *X has a selection from $\exp X$ into X;*

c) *X has a selection from $\exp_2 X$ into X.*

Here $\exp_2 X$ is the subspace of $\exp X$ that consists of all subsets of X with no more than two points. For a continuum X this was proved by Michael in 1951, and for zero-dimensional metric compact spaces by Yang in 1971. In [50], published in 1970, it was shown that for a locally compact separable metric space condition c) of Theorem 7.8 is equivalent to a condition that this space can be embedded in the real line.

7. Selections with non-metrizable images.

As we have already remarked, in [25] there is a series of examples that show that for limits of metrizable topological vector spaces, the convex selection theorem does not, as a rule, hold. In some cases it is nonetheless possible to obtain some positive results connected with the Lindelöf property. By developing methods from [52] the following theorem is proved in [25].

Theorem 7.9. *Any lower semicontinuous map from a paracompact space X into the space $Y = C_0(\Gamma)$ with convex compact values has a continuous selection if Γ is a discrete set and every point of X has a neighbourhood that is the continuous image of some separable semi-metrizable space.*

Here $C_0(\Gamma)$ is the space of real functions on Γ that degenerate at infinity, that is, $f \in C_0(\Gamma)$ if for any neighbourhood U of the origin there exists a finite subset of Γ whose dual is mapped by f into U.

In the non-convex case, we mention some results of Kolesnikov [47]. We state one theorem.

***Theorem* 7.10.** *Any continuous map of a zero-dimensional paracompact space into a regular pointwise perfect rarified space with arbitrary values has a continuous single-valued selection.*

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