



Uniform convexity and the splitting problem for selections

Maxim V. Balashov^a, Dušan Repovš^{b,*}

^a Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutski str. 9, Dolgoprudny, Moscow region, Russia 141700

^b Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia 1000

ARTICLE INFO

Article history:

Received 26 December 2008

Available online 18 June 2009

Submitted by M. Quincampoix

Keywords:

Splitting problem

Set-valued mapping

Uniformly continuous selection

Uniform convexity

Modulus of convexity

Reflexive Banach space

ABSTRACT

We continue to investigate cases when the Repovš–Semenov splitting problem for selections has an affirmative solution for continuous set-valued mappings. We consider the situation in infinite-dimensional uniformly convex Banach spaces. We use the notion of Polyak of uniform convexity and modulus of uniform convexity for arbitrary convex sets (not necessary balls). We study general geometric properties of uniformly convex sets. We also obtain an affirmative solution of the splitting problem for selections of certain set-valued mappings with uniformly convex images.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

The questions concerning continuity of set-valued mappings and existence of continuous, uniformly continuous and Lipschitz continuous selections of set-valued mappings have for a long time been the central questions of nonsmooth analysis [2,3]. The classical Michael theorem [16] guarantees the existence of continuous selections for lower semicontinuous set-valued mappings with convex closed images. However, the condition of lower semicontinuity for a set-valued mapping is not typical for (many) problems in which the set-valued mappings are represented as the intersection of two set-valued mappings. This occurs e.g. in approximation theory [5,14].

It is well known [2,3] that even the intersection of Lipschitz continuous set-valued mappings with convex compact images, defined on \mathbb{R}^n , is only upper semicontinuous. In certain minimization problems [19] and problems of stability of functionals [5] it is necessary to obtain uniformly continuous selections and explicit estimates for their moduli of continuity. This explains the necessity for additional constraints on the type of convexity of the set-valued mappings under consideration.

Let E be a Banach space. The *diameter* of the subset $A \subset E$ is defined as $\text{diam } A = \sup_{x_1, x_2 \in A} \|x_1 - x_2\|$. Let ∂A be the boundary of the set A , $\text{int } A$ the interior of A , and $\text{cl } A$ the closure of A . Let $\langle p, x \rangle$ be the value of the functional $p \in E^*$ at the point $x \in E$. We define the *closed ball* with center $a \in E$ and radius r as follows: $B_r(a) = \{x \in E \mid \|x - a\| \leq r\}$. Following [19], we define uniformly convex set as follows:

Definition 1.1. (See [19].) Let E be a Banach space and $A \subset E$ a closed convex set. The *modulus of convexity* $\delta_A : (0, \text{diam } A) \rightarrow [0, +\infty)$ is the function defined by

* Corresponding author.

E-mail addresses: balashov@mail.mipt.ru (M.V. Balashov), dusan.repovs@guest.arnes.si (D. Repovš).

$$\delta_A(\varepsilon) = \sup \left\{ \delta \geq 0 \mid B_\delta \left(\frac{x_1 + x_2}{2} \right) \subset A, \forall x_1, x_2 \in A: \|x_1 - x_2\| = \varepsilon \right\}.$$

Definition 1.2. (See [19].) Let E be a Banach space and $A \subset E$ a closed convex set. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0, \text{diam } A)$, then we call the set A *uniformly convex (with modulus $\delta_A(\cdot)$)*.

Definition 1.1 is very similar to the well-known definition of the modulus of convexity for uniformly convex function [23, Chapter 4, §7]. If the set A is bounded and has the center of symmetry then $\delta_A(\cdot)$ is the modulus of convexity for space E with the ball A [8,12]. Note that, as in the case of the bodies with center of symmetry (under assumption $A \neq E$, see [12, Part e]), it suffices to choose points $x_1, x_2 \in \partial A$, i.e.

$$\delta_A(\varepsilon) = \sup \left\{ \delta \geq 0 \mid B_\delta \left(\frac{x_1 + x_2}{2} \right) \subset A, \forall x_1, x_2 \in \partial A: \|x_1 - x_2\| = \varepsilon \right\}.$$

The properties of uniformly convex sets were used in [11,19] for the proof of convergence of minimizing sequences in certain extremal problems. Similar constructions appeared in approximation theory (see for example [5], [6, p. 12]). We plan to consider the entire class of uniformly convex sets and apply their properties for the solution of the splitting problem for selections.

The splitting problem for selections was formulated in [20]. Let $F_i: X \rightarrow 2^{Y_i}$, $i = 1, 2$, be any (lower semi)continuous mappings with closed convex images and let $L: Y_1 \oplus Y_2 \rightarrow Y$ be any linear surjection. The splitting problem is the problem of representing any continuous selection $f \in L(F_1, F_2)$ in the form $f = L(f_1, f_2)$, where $f_i \in F_i$ are some continuous selections, $i = 1, 2$. Some special cases of this problem in finite-dimensional spaces were considered in [17,21].

In [4] we obtained new results for finite-dimensional spaces and proved that there exist approximate solutions of the splitting problem for Lipschitz selections in the Hilbert space. We also wish to mention [13] and [15], where related questions were considered.

2. Uniformly convex sets and their properties

Note that if a set is uniformly convex then it is also strictly convex, i.e. its boundary contains no nondegenerate segments.

Lemma 2.1. Let $A \subset E$ be a closed and uniformly convex set with modulus $\delta_A(\cdot)$ and suppose that $A \neq E$. Then for any $\lambda \in (0, 1)$, $\varepsilon \in (0, \text{diam } A)$ the following inequality holds

$$\delta_A(\lambda\varepsilon) \leq \lambda\delta_A(\varepsilon).$$

Note that for any uniformly convex unit ball A , the inequality $\delta_A(\lambda\varepsilon) \leq \lambda\delta_A(\varepsilon)$, for all $\lambda, \varepsilon \in (0, 1)$, follows from [12, Lemma 1.e.8].

Proof. Let's fix $\varepsilon \in (0, \text{diam } A)$, $\alpha > 0$ and $\lambda \in (0, 1)$. Choose points $x_1, x_2 \in \partial A$, such that $\|x_1 - x_2\| = \varepsilon$ and $\delta_A(\varepsilon) + \alpha > \delta$, where $\delta = \sup\{r \geq 0 \mid B_r(z) \subset A\}$ and $z = \frac{1}{2}(x_1 + x_2)$.

For any k we define a point $a_k \in \partial A$ with $\|a_k - z\| \leq \delta + \frac{1}{k}$. Let y_i^k be the homothetic image of the point x_i under the homothety with center a_k and coefficient λ , $i = 1, 2$; let z_k be the homothetic image of the point z under the homothety with center a_k and coefficient λ .

We have $\|y_1^k - y_2^k\| = \lambda\varepsilon$ and $\|z_k - a_k\| \leq \lambda\delta + \lambda\frac{1}{k}$. It follows from the inclusions $y_i^k \in A$, $i = 1, 2$, that

$$\delta_A(\lambda\varepsilon) \leq \|z_k - a_k\| \leq \lambda\delta + \lambda\frac{1}{k} \leq \lambda\delta_A(\varepsilon) + \lambda\alpha + \lambda\frac{1}{k}.$$

By taking limits $\alpha \rightarrow +0$, $k \rightarrow \infty$ we get the following inequality:

$$\delta_A(\lambda\varepsilon) \leq \lambda\delta_A(\varepsilon). \quad \square$$

The following corollary follows from Lemma 2.1.

Corollary 2.1. The modulus of convexity is a strictly monotone function and moreover, the function $\varepsilon \rightarrow \frac{\delta_A(\varepsilon)}{\varepsilon}$ is also monotone.

Lemma 2.2. Let $A \subset E$ be a closed and uniformly convex set with modulus $\delta_A(\cdot)$. Let $\varepsilon \in (0, \text{diam } A)$, $p_1, p_2 \in \partial B_1^*(0)$, $x_i = \arg \max_{x \in A} \langle p_i, x \rangle$, $i = 1, 2$. If $\|p_1 - p_2\| < \frac{4\delta_A(\varepsilon)}{\varepsilon}$ then $\|x_1 - x_2\| < \varepsilon$.

Proof. Suppose that $\|x_1 - x_2\| \geq \varepsilon$. Define $\delta = \delta_A(\|x_1 - x_2\|)$. We have $B_\delta(\frac{x_1+x_2}{2}) \subset A$. By hypotheses of the lemma,

$$\langle p_1, x_1 \rangle = \max_{x \in A} \langle p_1, x \rangle \geq \max_{x \in B_\delta(\frac{x_1+x_2}{2})} \langle p_1, x \rangle = \frac{1}{2} \langle p_1, x_1 + x_2 \rangle + \delta$$

and in the same way $\langle p_2, x_2 \rangle \geq \frac{1}{2} \langle p_2, x_1 + x_2 \rangle + \delta$. Hence

$$\langle p_1, x_1 \rangle - \langle p_1, x_2 \rangle \geq 2\delta, \quad \langle p_2, x_2 \rangle - \langle p_2, x_1 \rangle \geq 2\delta.$$

Adding the last two inequalities

$$\langle p_1 - p_2, x_1 - x_2 \rangle \geq 4\delta$$

we obtain $\|p_1 - p_2\| \cdot \|x_1 - x_2\| \geq 4\delta$ and

$$\|p_1 - p_2\| \geq \frac{4\delta_A(\|x_1 - x_2\|)}{\|x_1 - x_2\|} \geq \frac{4\delta_A(\varepsilon)}{\varepsilon},$$

where the last inequality follows by Corollary 2.1. \square

Let us denote $\varphi(\varepsilon) = \frac{4\delta_A(\varepsilon)}{\varepsilon}$. We obtain the following corollary:

Corollary 2.2. Let $A \subset E$ be a closed and uniformly convex set with modulus $\delta_A(\cdot)$. Let $p_1, p_2 \in \partial B_1^*(0)$, $x_i = \arg \max_{x \in A} \langle p_i, x \rangle$, $i = 1, 2$. Then

$$\varphi(\|x_1 - x_2\|) \leq \|p_1 - p_2\|.$$

Proof. Let $\|x_1 - x_2\| = \varepsilon$. By Lemma 2.2 we then obtain that $\varphi(\varepsilon) = \frac{4\delta_A(\varepsilon)}{\varepsilon} \leq \|p_1 - p_2\|$. \square

Remark 2.1. Suppose that the convex closed bounded subset A of a Banach space E has uniformly continuous supporting elements, i.e. that there exists a continuous function $\varphi: [0, \text{diam } A] \rightarrow [0, +\infty)$, $\varphi(0) = 0$, such that for any unit vectors $p_1, p_2 \in E^*$ and $x_i = \arg \max_{x \in A} \langle p_i, x \rangle$, $i = 1, 2$:

$$\varphi(\|x_1 - x_2\|) \leq \|p_1 - p_2\|.$$

Then there exists $C > 0$ such that

$$\delta_A(\varepsilon) \geq C \cdot \int_0^{\frac{\varepsilon}{2}} \varphi(t) dt, \quad \forall \varepsilon \in (0, \text{diam } A).$$

The proof of this fact has not been published yet, however, it is too long to be included in this paper.

The supporting function of the set $A \subset E$ is defined by $s(p, A) = \sup_{x \in A} \langle p, x \rangle$, $p \in E^*$. This is a positively uniform convex closed function (see [2,18]). For the set A we define the barrier cone by $b(A) = \{p \in E^* \mid s(p, A) < +\infty\}$, i.e. $b(A)$ is the domain of the supporting function.

The fact that every uniformly convex set which does not coincide with the entire space is bounded was stated in [11]. We shall prove a more precise result.

Theorem 2.1. Let E be a Banach space and let $A \subset E$ a closed and uniformly convex subset with modulus $\delta_A(\cdot)$. Then for any $\varepsilon \in (0, \text{diam } A)$

$$\text{diam } A \leq \left(\left\lceil \frac{\varepsilon}{\delta_A(\varepsilon)} \right\rceil + 1 \right) \cdot \varepsilon,$$

where $\lceil x \rceil$ is the largest integer $\leq x$.

Proof. For any unit vector $p \in b(A)$ and any $t > 0$ we define a convex closed set:

$$A_p(t) = A \cap \{x \in E \mid \langle p, x \rangle \geq s(p, A) - t\}.$$

We obtain from the definition of the supporting function that $A_p(t) \neq \emptyset$ for any $t > 0$, $p \in b(A)$, $\|p\| = 1$, and if $0 < t_1 < t_2$ then $A_p(t_1) \subset A_p(t_2)$.

We shall show that for any unit $p \in b(A)$ the following holds

$$\lim_{t \rightarrow +0} \text{diam } A_p(t) = 0.$$

Suppose that for some unit $p \in b(A)$ there exist $d > 0$ and $t_k \rightarrow +0$ with $\text{diam } A_p(t_k) \geq d$. The latter means that there exist points x_k^1, x_k^2 from $A_p(t_k)$ with $\|x_k^1 - x_k^2\| > d/2$, for all k . It follows from uniform convexity of the set A that

$$B_{\delta_A(\frac{d}{2})} \left(\frac{x_k^1 + x_k^2}{2} \right) \subset A.$$

However, by taking the supporting functions of the sets from this inclusion we obtain the following:

$$s\left(p, B_{\delta_A(\frac{d}{2})}\left(\frac{x_k^1 + x_k^2}{2}\right)\right) = \frac{1}{2}(\langle p, x_k^1 \rangle + \langle p, x_k^2 \rangle) + \delta_A\left(\frac{d}{2}\right) \geq s(p, A) - t_k + \delta_A\left(\frac{d}{2}\right) > s(p, A).$$

The last inequality holds for sufficiently large k (when $\delta_A(\frac{d}{2}) > t_k$). This contradiction shows that $\text{diam } A_p(t) \rightarrow 0, t \rightarrow +0$.
By the completeness of A we conclude that

$$\bigcap_{t>0} A_p(t) = \{a(p)\}.$$

We have thus proved that for any unit vector $p \in b(A)$ there exists $a(p) = \arg \max_{x \in A} \langle p, x \rangle$.

Let's fix arbitrary points $x, y \in \partial A$. By the separation theorem there exist unit vectors $q_1, q_2 \in E^*$ such that $\langle q_1, x \rangle = s(q_1, A), \langle q_2, y \rangle = s(q_2, A)$. If $q_1 \neq -q_2$ then let $D = \partial B_1^*(0) \cap \text{cone}\{q_1, q_2\}$. If $q_1 = -q_2$ then choose any $q_3 \in \partial B_1^*(0)$ with $s(q_3, A) < +\infty$ and define $D = \partial B_1^*(0) \cap \text{cone}\{q_1, q_2, q_3\}$. Note that for any $q \in D, s(q, A) < +\infty$.

By [10, Theorem 11.9] for any 2-dimensional subspace $\mathcal{L} \subset E^*$ the length of the curve $\mathcal{L} \cap \partial B_1^*(0)$ is less than 8 (in the $\|\cdot\|_*$ -norm). Thus the length of D is less than 4. Choose $N = \lceil \frac{\varepsilon}{\delta_A(\varepsilon)} \rceil + 1$ and points $\{p_i\}_{i=0}^N \in D$ which decompose the length of D into N equal parts; $p_0 = q_1, p_N = q_2$ and $\|p_{i-1} - p_i\| < \frac{4}{N} \leq \varphi(\varepsilon), i = 1, \dots, N$.

By the previous considerations we obtain that for all $1 \leq i \leq N - 1$ there exists $x_i \in \partial A$ with $\langle p_i, x_i \rangle = s(p_i, A)$. By Lemma 2.2 we have $\|x_{i-1} - x_i\| < \varepsilon$ and

$$\|x - y\| \leq \sum_{i=1}^N \|x_{i-1} - x_i\| \leq \varepsilon \cdot N.$$

The points x, y are arbitrary boundary points of A , hence $\text{diam } A \leq \varepsilon \cdot N$. \square

Corollary 2.3. *By Theorem 2.1 we have*

$$\delta_A(\varepsilon) \leq \frac{\varepsilon^2}{\text{diam } A - \varepsilon}, \quad \forall \varepsilon \in (0, \text{diam } A).$$

This means that $\delta_A(\varepsilon) \leq C \cdot \varepsilon^2$ for any convex closed bounded set A .

For balls this statement follows from the well-known Day–Nordlendar theorem [8] which asserts that if E is a Banach space then the modulus of convexity for E , i.e. the modulus of convexity for the unit ball, satisfies the estimate $\delta_E(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}, \forall \varepsilon \in (0, 2)$.

Next we shall prove a result which is very close to the Day–Nordlendar theorem.

Theorem 2.2. *Let E be a Banach space and $A \subset E$ a closed and uniformly convex set with modulus $\delta_A(\cdot), \text{diam } A = 1$. Let $r_0 > 0$ and $a \in E$ be such that $B_{r_0}(a) \subset A$. Then for all $\varepsilon \in (0, 1)$:*

$$\delta_A(2r_0\varepsilon) \leq \frac{1}{2}(1 - \sqrt{1 - \varepsilon^2}). \tag{2.1}$$

In (2.1) the equality takes place when A is the Euclidean ball of diameter 1 in the Euclidean space (with $r_0 = \frac{1}{2}$).

Proof. Without loss of generality we can assume that $a = 0$. Let $B = A \cap (-A)$. Note that the set B is bounded, has a nonempty interior ($B_{r_0}(0) \subset B$) and its center of symmetry in zero. Hence we can consider the set B as the ball of radius $\frac{1}{2}$ and we have:

$$B_{r_0}(0) \subset B \subset B_{\frac{1}{2}}(0). \tag{2.2}$$

Let's say a few words about the second inclusion in (2.2). If $x \in B$, then $-x \in B$, and $2\|x\| = \|x - (-x)\| \leq \text{diam } B = 1$. Therefore $B \subset B_{\frac{1}{2}}(0)$. By $\|\cdot\|_B$ we denote the new norm with the unit ball $2B$.

For any convex closed bounded set $C \subset E$ we shall consider the modulus of convexity:

$$\delta_C^B(\varepsilon) = \sup \left\{ \delta \geq 0 \mid \delta \cdot 2B + \frac{x_1 + x_2}{2} \subset C, \forall x_1, x_2 \in \partial C: \|x_1 - x_2\|_B = \varepsilon \right\}.$$

Let $x_1, x_2 \in B$ and $\|x_1 - x_2\|_B = \varepsilon \in (0, 1)$. From $\delta_A^B = \delta_{-A}^B$ we have

$$\frac{x_1 + x_2}{2} + 2B\delta_A^B(\varepsilon) \subset A, \quad \frac{x_1 + x_2}{2} + 2B\delta_{-A}^B(\varepsilon) = \frac{x_1 + x_2}{2} + 2B\delta_A^B(\varepsilon) \subset -A.$$

By definition, $B = A \cap (-A)$, so we obtain that $\frac{x_1+x_2}{2} + \delta_A^B(\varepsilon) \cdot 2B \subset B$ and thus $\delta_A^B(\varepsilon) \leq \delta_B^B(\varepsilon)$ for all $\varepsilon \in (0, 1)$. From the equality $\delta_B^B(\varepsilon) = \frac{1}{2}\delta_{2B}^B(2\varepsilon)$, using Day–Nordlander theorem [8, Theorem 3.3.1] for the unit ball $2B$, we obtain for all $\varepsilon \in (0, 1)$

$$\delta_B^B(\varepsilon) = \frac{1}{2}\delta_{2B}^B(2\varepsilon) \leq \frac{1}{2}\left(1 - \sqrt{1 - \frac{(2\varepsilon)^2}{4}}\right),$$

and $\delta_A^B(\varepsilon) \leq \frac{1}{2}(1 - \sqrt{1 - \varepsilon^2})$ for all $\varepsilon \in (0, 1)$.

We conclude from inclusions (2.2), that for any $x_1, x_2 \in E$ the inequalities $2r_0\|x_1 - x_2\|_B \leq \|x_1 - x_2\| \leq \|x_1 - x_2\|_B$ hold. If $x_1, x_2 \in \partial A$, $\|x_1 - x_2\|_B = \varepsilon$ and $\varepsilon \in (0, 1)$, then $\delta_A(2r_0\varepsilon) \leq \delta_A(\|x_1 - x_2\|)$. Since for any $\delta \geq 0$ the condition $\frac{x_1+x_2}{2} + \delta B_1(0) \subset A$ implies the condition $\frac{x_1+x_2}{2} + \delta \cdot 2B \subset A$, it follows that $\delta_A(\|x_1 - x_2\|) \leq \delta_A^B(\varepsilon)$. Therefore we get the formula (2.1).

An easy calculation shows that in the case when the set A is a Euclidean ball of diameter 1 in the Euclidean space with $r_0 = \frac{1}{2}$ we get the equality in the formula (2.1). \square

Theorem 2.3. *In every Banach space E there exists a closed uniformly convex set A if and only if the space E admits an equivalent uniformly convex norm.*

Proof. Due to Theorem 2.1 we must consider only bounded sets. If the space E admits an equivalent uniformly convex norm then the unit ball of this norm is a uniformly convex set. Let us prove the converse statement.

Let $A \subset E$ be closed and uniformly convex set with modulus δ_A . Suppose that $0 \in \text{int } A$. As we can see from the proof of Theorem 2.2, the set $B = A \cap (-A)$ is a uniformly convex ball of equivalent norm. \square

Note that a Banach space which is equivalent to a uniformly convex space, is reflexive [8]. Thus we can further use reflexivity without loss of generality. The reflexivity of the Banach space with bounded nonsingleton uniformly convex set was mentioned in [19]. We also note that nonreflexive spaces (e.g., the spaces $C([0, 1])$, $L_1([0, 1])$, $L_\infty([0, 1])$, l_1 , l_∞) do not contain uniformly convex sets.

Recall that in any finite-dimensional Banach space the class of strictly convex compacta coincides with the class of uniformly convex sets. This fact easily follows from compactness of sets from two classes. It is well known [8] that in infinite-dimensional spaces there exist strictly but nonuniformly convex balls.

We wish to mention an important class of uniformly convex sets. Let E be a uniformly convex Banach space. The set $A \subset E$ is strongly convex with radius $R > 0$ [18, Chapters 3, 4] (or R -convex [9]) if $A = \bigcap_{x \in X} B_R(x) \neq \emptyset$, where $X \subset E$ an arbitrary subset. It is easy to see that the modulus of convexity for A is $\delta_A(\varepsilon) \geq R\delta_E(\frac{\varepsilon}{R})$ for all $\varepsilon \in (0, \text{diam } A)$. Here δ_E is the modulus of convexity for the space E .

3. Applications to the set-valued analysis and the splitting problem for selections

Let $\{F(t)\}_{t \in T}$ be any collection of convex closed sets and let $\text{diam } F(t) \geq r_0 > 0$ for all t . Suppose that each set $F(t)$ is uniformly convex with modulus $\delta_t(\varepsilon)$. Then under the assumption that $\delta(\varepsilon) = \inf_{t \in T} \delta_t(\varepsilon) > 0$ for all $\varepsilon \in (0, r_0)$, the set $F = \bigcap_{t \in T} F(t)$ is uniformly convex with modulus $\delta_F(\varepsilon) \geq \delta(\varepsilon)$ for all $\varepsilon \in (0, r_0)$ (this set can also be empty or a singleton). Note that Lemmata 2.1, 2.2 and Theorem 2.1 are valid for the function $\delta(\varepsilon)$ and the set F .

Consider as an example the set A , which can be represented as the intersection of closed balls of radius 1 in Hilbert space \mathcal{H} . The modulus of convexity for the unit ball from \mathcal{H} is $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \geq \frac{\varepsilon^2}{8}$ for all $\varepsilon \in (0, 2)$ and $\delta_A(\varepsilon) \geq \delta_{\mathcal{H}}(\varepsilon)$. By Corollary 2.2 we have that $\varphi(\varepsilon) \geq \varepsilon/2$ and $\|x_1 - x_2\| \leq 2\|p_1 - p_2\|$. So we conclude that the gradient $\nabla s(p, A) = \arg \max_{x \in A} \langle p, x \rangle$ of supporting function for the set A is a Lipschitz function with respect to p . This result was proved in [18] by different methods.

Next we shall consider set-valued mappings $F : T \rightarrow 2^E \setminus \emptyset$ from a metric space (T, ρ) to a Banach space E . Suppose that there exists $r_0 > 0$ such that for any $t \in T$ we can find a point $a(t) \in E$ with $B_{r_0}(a(t)) \subset F(t)$. Suppose that any set $F(t)$ is closed and uniformly convex with modulus $\delta_t(\varepsilon)$, $\varepsilon \in (0, \text{diam } F(t))$. If $\delta(\varepsilon) = \inf_{t \in T} \delta_t(\varepsilon) > 0$ for all $\varepsilon \in (0, 2r_0]$ then we say that the images $F(t)$, $t \in T$, are uniformly convex with modulus $\delta(\varepsilon)$, $\varepsilon \in (0, 2r_0]$. It's easy to see that Lemmata 2.1, 2.2 and Theorem 2.1 are valid for any set $F(t)$ when instead of the modulus $\delta_{F(t)}$ we take the modulus δ .

For an increasing function $\delta : [0, d] \rightarrow [0, \Delta]$ we define the inverse function δ^{-1} as follows: for $x_0 \in [0, \Delta]$ let $\delta^{-1}(x_0) = y_0 \in [0, d]$. Here $\delta(y_0 - 0) \leq x_0 \leq \delta(y_0 + 0)$; $\delta(y_0 \pm 0) = \lim_{y \rightarrow y_0 \pm 0} \delta(y)$. Note that the function δ^{-1} is continuous on the segment $[0, \Delta]$.

We shall use $\text{conv } A$ to denote the convex hull of the set A . The Hausdorff distance $h(A, B)$ between sets A and B in a Banach space E is defined as follows:

$$h(A, B) = \inf\{r > 0 \mid A \subset B + B_r(0), B \subset A + B_r(0)\}.$$

Theorem 3.1. Let (T, ρ) be a metric space and E a reflexive Banach space. Suppose that the set-valued mappings $F_i : T \rightarrow 2^E \setminus \emptyset$, $i = 1, 2$, have convex closed images. Let F_i , $i = 1, 2$, be uniformly continuous in the Hausdorff metric, i.e. there exist nonnegative infinitely small at zero numerical functions ω_i , such that for all $t_1, t_2 \in T$ we have the following:

$$h(F_i(t_1), F_i(t_2)) \leq \omega_i(\rho(t_1, t_2)).$$

Let the images $F_1(t)$ be uniformly convex with modulus $\delta(\varepsilon)$, $\varepsilon \in (0, 2r_0]$. Let $\Delta_0 = \delta(2r_0)$ and $H(t) = F_1(t) \cap F_2(t) \neq \emptyset$ for all $t \in T$.

Then

$$h(H(t_1), H(t_2)) \leq \omega_1(\rho(t_1, t_2)) + 2\omega_2(\rho(t_1, t_2)) + f(\omega_1(\rho(t_1, t_2)) + \omega_2(\rho(t_1, t_2))), \tag{3.3}$$

where

$$f(x) = \begin{cases} \delta^{-1}(\frac{x}{2}), & x < 2\Delta_0, \\ \frac{Mx}{2\Delta_0}, & x \geq 2\Delta_0, \end{cases} \tag{3.4}$$

and $M = \sup_{t \in T} \text{diam } F_1(t) \leq r_0(\lceil \frac{r_0}{\delta(r_0)} \rceil + 1)$.

Proof. We define $\omega_1 = \omega_1(\rho(t_1, t_2))$, $\omega_2 = \omega_2(\rho(t_1, t_2))$. Let $b_1 \in H(t_1)$. Let's fix $k > 1$. We shall prove that there exists point $a(t_2) \in H(t_2)$ such that

$$\|a(t_2) - b_1\| \leq f(\omega_1 + k\omega_2) + \omega_1 + 2k\omega_2. \tag{3.5}$$

We obtain from formula (3.5) the following:

$$h(H(t_1), H(t_2)) \leq f(\omega_1 + k\omega_2) + \omega_1 + 2k\omega_2,$$

and keeping in mind that the function f is continuous from the right (see (3.4)), we take the limit $k \rightarrow 1 + 0$ and obtain formula (3.3).

Let $b(t_2) \in F_2(t_2)$: $\|b(t_2) - b_1\| \leq kh(F_2(t_2), F_2(t_1)) \leq k\omega_2$. If $b(t_2) \in F_1(t_2)$ then we can take $a(t_2) = b(t_2)$ and we conclude that formula (3.5) is valid. Further we shall assume that $b(t_2) \notin F_1(t_2)$.

Let $c(t_2) \in H(t_2) \subset F_1(t_2)$. Let $b_\pi(t_2)$ be the metric projection of the point $b(t_2)$ onto $F_1(t_2)$. The point $b_\pi(t_2)$ exists because the space E is reflexive. Consider the point $a(t_2)$ which is the nearest to the point $b(t_2)$ of the set $F_1(t_2) \cap \text{conv}\{b(t_2), c(t_2)\}$. By definition, $a(t_2) \in F_1(t_2)$ and $a(t_2) \in \text{conv}\{b(t_2), c(t_2)\} \subset F_2(t_2)$. This implies that $a(t_2) \in H(t_2)$.

Let $z(t_2) = \frac{a(t_2)+b(t_2)}{2}$, $\tilde{z}(t_2) = \frac{a(t_2)+b_\pi(t_2)}{2}$. Since

$$\|z(t_2) - \tilde{z}(t_2)\| = \frac{1}{2}\|b(t_2) - b_\pi(t_2)\|, \quad B_{\delta(\|a(t_2)-b_\pi(t_2)\|)}(\tilde{z}(t_2)) \subset F_1(t_2),$$

it follows from the condition $z(t_2) \notin F_1(t_2)$ that

$$\delta(\|a(t_2) - b_\pi(t_2)\|) \leq \|z(t_2) - \tilde{z}(t_2)\| = \frac{1}{2}\|b(t_2) - b_\pi(t_2)\|. \tag{3.6}$$

So we have following estimate:

$$\begin{aligned} \|b(t_2) - b_\pi(t_2)\| &= \rho(b(t_2), F_1(t_2)) \leq \rho(b_1, F_1(t_2)) + \|b(t_2) - b_1\| \\ &\leq h(F_1(t_1), F_1(t_2)) + k\omega_2 \leq \omega_1 + k\omega_2. \end{aligned}$$

By the last formula and by (3.6) we have that $\delta(\|a(t_2) - b_\pi(t_2)\|) \leq \frac{1}{2}(\omega_1 + k\omega_2)$.

If $\omega_1 + k\omega_2 < 2\Delta_0$ then

$$\|a(t_2) - b_\pi(t_2)\| \leq \delta^{-1}\left(\frac{1}{2}(\omega_1 + k\omega_2)\right).$$

If $\omega_1 + k\omega_2 \geq 2\Delta_0$ then

$$\|a(t_2) - b_\pi(t_2)\| \leq \frac{\omega_1 + k\omega_2}{2\Delta_0} M.$$

Thus in both cases we have $\|a(t_2) - b_\pi(t_2)\| \leq f(\omega_1 + k\omega_2)$. Finally,

$$\|a(t_2) - b_1\| \leq \|a(t_2) - b_\pi(t_2)\| + \|b_\pi(t_2) - b(t_2)\| + \|b(t_2) - b_1\| \leq f(\omega_1 + k\omega_2) + \omega_1 + 2k\omega_2. \quad \square$$

Theorem 3.1 has important consequences. It follows from Corollary 2.3 that the modulus of convexity $\delta(\varepsilon)$ of sets $F_1(t)$ in Theorem 3.1 does not exceed $C \cdot \varepsilon^2$. Hence the Hölder condition with the power no greater than $\frac{1}{2}$ with respect to the Hausdorff metric is typical for the product of intersections of two Lipschitz set-valued mappings. We need to invoke good mutual geometric properties of F_1 and F_2 if we want to obtain power greater than $\frac{1}{2}$ (see for example [18, Theorem 2.2.1]). Under the conditions of Theorem 3.1 the result is the best possible.

Example 3.1. In the Euclidean plane \mathbb{R}^2 with the standard basis $x_1 O x_2$ we consider (for $t \geq 0$)

$$F_1(t) = F_1 = \{(x_1, x_2) \mid x_2 \geq |x_1|^p\} \cap B_1(0), \quad p \geq 2, \quad F_2(t) = \{(x_1, t) \mid x_1 \in \mathbb{R}\}.$$

It is easy to see that if $\varepsilon > 0$ is sufficiently small then the modulus of convexity F_1 equals $\delta_1(\varepsilon) = \frac{\varepsilon^p}{2^p}$ (and it realized on the segment $[(-\frac{\varepsilon}{2}, \frac{\varepsilon^p}{2^p}), (\frac{\varepsilon}{2}, \frac{\varepsilon^p}{2^p})]$). The intersection of $F_1(t)$ and $F_2(t)$ is $H(t) = [-t^{1/p}, t^{1/p}] \times \{t\}$. Let $t_1 > 0, t_2 = 2t_1$. Then

$$h(H(t_1), H(t_2)) \geq (2t_1)^{1/p} - t_1^{1/p} = (2^{1/p} - 1) \cdot |t_2 - t_1|^{1/p} = \frac{2^{1/p} - 1}{2} \delta_1^{-1}(|t_2 - t_1|).$$

Example 3.2. Consider the following extremal problem

$$\min_{x \in A} g(x). \tag{3.7}$$

Suppose that the function g has closed and uniformly convex level sets $\mathcal{L}_g(\beta) = \{x \in E \mid g(x) \leq \beta\}$. The function g itself cannot be convex. We shall consider two problems (3.7) with the same function and convex closed sets $A_i, i = 1, 2$. Suppose that the point u_i is the solution of the problem (3.7) with the set $A = A_i$, i.e. $\{u_i\} = A_i \cap \mathcal{L}_g(\min_{x \in A_i} g(x))$. We shall estimate the value $\|u_1 - u_2\|$ through the distance $h = h(A_1, A_2)$.

Note that for convex functions and sets such problems were considered e.g., in [5,14]. Let $g(u_1) \leq g(u_2)$. Then

$$u_1 = A_1 \cap \mathcal{L}_g(g(u_1)) \subset A_1 \cap \mathcal{L}_g(g(u_2)).$$

Let the set $\mathcal{L}_g(g(u_2))$ be uniformly convex with modulus δ . Let $F_1(A) = \mathcal{L}_g(g(u_2))$ be a constant mapping with the modulus of continuity $\omega_1 = 0$, and let $F_2(A) = A$ be a mapping with the modulus of continuity $\omega_2(t) = t$. By Theorem 3.1 we have

$$h(F_1(A_1) \cap F_2(A_1), F_1(A_2) \cap F_2(A_2)) \leq 2h + f(h),$$

where function f is defined in (3.4) and $M = \text{diam } \mathcal{L}_g(g(u_2)), \Delta_0 = \lim_{\varepsilon \rightarrow \text{diam } \mathcal{L}_g(g(u_2)) - 0} \delta(\varepsilon)$. Therefore for all $t > 1$

$$A_1 \cap \mathcal{L}_g(g(u_2)) \subset A_2 \cap \mathcal{L}_g(g(u_2)) + t(2h + f(h))B_1(0) = u_2 + t(2h + f(h))B_1(0),$$

i.e.

$$\|u_1 - u_2\| \leq 2h(A_1, A_2) + f(h(A_1, A_2)). \tag{3.8}$$

We now consider applications of the above results to the splitting problem for selections.

Example 3.3. (See [20, Question 4.6].) Do there exist for every closed convex sets A, B and $C = A + B$ continuous functions $a : C \rightarrow A$ and $b : C \rightarrow B$ with the property that $a(c) + b(c) = c$ for all $c \in C$? Similar questions were also considered in previous papers, see [13] for details.

Lemma 3.1. Let the space E be uniformly convex with modulus δ_E . Let $A \subset E$ be a closed and uniformly convex set with modulus δ_A , and $B \subset E$ a convex and closed set. Then there exist uniformly continuous functions $a : C \rightarrow A$ and $b : C \rightarrow B$ such that $a(c) + b(c) = c$, for all points $c \in C$.

Proof. Suppose that $0 \notin A$. For any $c \in C$ we define sets $F_1(c) = A, F_2(c) = c - B$. Then (in terms of Theorem 3.1) $\omega_1 = 0, \omega_2(t) = t, F_1$ has uniformly convex images with modulus δ_A . Note that $H(c) = (c - B) \cap A$ is nonempty for all $c \in C$.

Let's define $M = \text{diam } A, \Delta_0 = \lim_{\varepsilon \rightarrow \text{diam } A - 0} \delta_A(\varepsilon)$. By Theorem 3.1

$$h(H(c_1), H(c_2)) \leq 2\|c_1 - c_2\| + f(\|c_1 - c_2\|), \tag{3.9}$$

where f is from (3.4).

Let $r = \inf_{a \in A} \|a\| > 0, R = \sup_{a \in A} \|a\|$. All balls $B_t(0), t \in [r, R]$, are uniformly convex with modulus $\delta(\varepsilon) = R\delta_E(\frac{\varepsilon}{R}), \varepsilon \in (0, 2r]$. Let $a(c) = \arg \min_{x \in H(c)} \|x\|$. Let's define $\Delta_E = \delta(2r)$,

$$f_E(t) = \begin{cases} \delta^{-1}(\frac{t}{2}), & t < 2\Delta_E, \\ \frac{Rt}{\Delta_E}, & t \geq 2\Delta_E. \end{cases}$$

Using (3.8) from Example 3.2 and (3.9) we have

$$\begin{aligned} \|a(c_1) - a(c_2)\| &\leq 2h(H(c_1), H(c_2)) + f_E(h(H(c_1), H(c_2))) \\ &\leq 4\|c_1 - c_2\| + 2f(\|c_1 - c_2\|) + f_E(2\|c_1 - c_2\| + f(\|c_1 - c_2\|)). \end{aligned}$$

So we have built uniformly continuous selections $a(c) \in H(c) \subset A$ and $b(c) = c - a(c) \in B$. \square

Remark 3.1. Note that in the case $E = \mathbb{R}^n$ we can define $a(c)$ as $a(c) = s(H(c))$, where $s(H(c))$ is the Steiner point of the set $H(c)$. The Steiner point is a Lipschitz selection of convex compacta from \mathbb{R}^n with the Lipschitz constant $L_n = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})}$ [3,18]. From this and by formula (3.9) we get

$$\|a(c_1) - a(c_2)\| \leq L_n \cdot (2\|c_1 - c_2\| + f(\|c_1 - c_2\|)).$$

Remark 3.2. Let A and B be closed convex subsets of the reflexive Banach space E and let the set A be strictly convex and bound. Let $0 \in \text{int } A$.

Let $c \in A + B$, $\varrho_A(c, B) = \inf\{t > 0 \mid c \in B + tA\}$, and

$$b(c) = (c - \varrho_A(c, B)A) \cap B.$$

The set $b(c)$ is a point. This follows from the reflexivity of the space E (the set $B + tA$ is closed for all $t \geq 0$) and strictly convexity of the set A . The point $b(c)$ is projection of the point c in the sense of the set A on the set B . Note that in above situation $\varrho_A(c, B) \in [0, 1]$.

If this projection $b(c)$ uniformly continuously depends on c , then $b(c) \in B$ is a uniformly continuous selection of B and $a(c) = c - b(c) \in \varrho_A(c, B)A \subset A$ is a uniformly continuous selection of A .

In particular, if the spaces E and E^* have moduli of convexity of the second order and $A = B_1(0)$ then by the results from [1] we obtain that the projection $b(c)$ satisfies the Lipschitz condition with respect to c . In particular, this takes place in the Hilbert space. It would be very interesting to describe all spaces and pairs of sets (A and B) for which the projection $b(c)$ of the point c in the sense of the set A on the set B satisfies the Lipschitz condition.

Example 3.4. Hereafter, the sum of Banach spaces $E_1 \oplus E_2$ will be defined as follows: $w = (u, v) \in E_1 \oplus E_2$, $\|w\| = \max\{\|u\|_{E_1}, \|v\|_{E_2}\}$.

Lemma 3.2. Let T be a metric space, E_i a reflexive Banach spaces, and $F_i : T \rightarrow 2^{E_i}$ uniformly continuous set-valued mappings with modulus of continuity ω , i.e.

$$h((F_1(t_1), F_2(t_2)), (F_1(t_1), F_2(t_2))) \leq \omega(\rho(t_1, t_2)), \quad \forall t_1, t_2 \in T, \quad i = 1, 2.$$

Suppose that the images $F_i(t)$ are uniformly convex sets with modulus $\delta(\varepsilon)$, $i = 1, 2$ and $\varepsilon \in (0, 2r_0]$; $\Delta_0 = \delta(2r_0)$. Let $\mathcal{L} \subset E_1 \oplus E_2$ be a closed subspace and suppose that there exists $C > 0$ such that for any $w_1 = (u_1, v_1) \in \mathcal{L}$, $w_2 = (u_2, v_2) \in \mathcal{L}$ we have $\|u_1 - u_2\|_{E_1} \geq C\|w_1 - w_2\|$ and $\|v_1 - v_2\|_{E_2} \geq C\|w_1 - w_2\|$ (i.e. \mathcal{L} is not “parallel” to E_1 and E_2).

Let $M = \sup_{t \in T} \text{diam}(F_1(t), F_2(t)) < +\infty$. Define the set-valued map $H(t) = (F_1(t), F_2(t)) \cap \mathcal{L} \neq \emptyset$ for all $t \in T$. Then

$$h(H(t_1), H(t_2)) \leq \omega(\rho(t_1, t_2)) + \frac{1}{C} f(\omega(\rho(t_1, t_2))), \quad \forall t_1, t_2 \in T, \tag{3.10}$$

where the function f is from formula (3.4).

Proof. Let $w_0 \in H(t_0)$. Let's fix $k > 1$. We shall prove that there exists a point a $w_1 \in H(t)$ with the following property:

$$\|w_0 - w_1\| \leq k\omega(\rho(t_0, t)) + \frac{1}{C} f(k\omega(\rho(t_0, t))).$$

Thus

$$h(H(t_1), H(t_2)) \leq k\omega(\rho(t_0, t)) + \frac{1}{C} f(k\omega(\rho(t_0, t)))$$

and we obtain (3.10) by taking the limit $k \rightarrow 1 + 0$.

Let $w \in (F_1(t), F_2(t))$ be a point such that $\|w_0 - w\| \leq k\omega(\rho(t_0, t))$. Define $w_1 \in H(t)$ to be the point from the set $H(t)$ which is the nearest to the point w_0 (w_1 exists by the reflexivity of E_i , $i = 1, 2$).

Let $w_2 = \frac{1}{2}(w + w_1)$. If $z \in \mathcal{L}$ is the middle point of the segment $[w_1, w_0]$ then

$$\|w_2 - z\| = \frac{1}{2}\|w - w_0\| \leq k\frac{1}{2}\omega(\rho(t_0, t)).$$

Thus we must require $\delta(C\|w - w_1\|) \leq k\frac{1}{2}\omega(\rho(t_0, t))$. Otherwise we would have, since \mathcal{L} is “parallel” neither to E_1 nor to E_2 , the following contradiction:

$$z \in B_{\delta(C\|w - w_1\|)}^{E_1 \oplus E_2}(w_2) \cap \mathcal{L} \subset (F_1(t), F_2(t)) \cap \mathcal{L} = H(t),$$

with the inequality $\|w_1 - w_0\| \leq \|z - w_0\|$.

If $k\omega(\rho(t_0, t)) < 2\Delta_0$ then

$$\|w - w_1\| \leq \frac{1}{C}\delta^{-1}\left(k\frac{1}{2}\omega(\rho(t_0, t))\right).$$

If $k\omega(\rho(t_0, t)) \geq 2\Delta_0$ then

$$\|w - w_1\| \leq \frac{1}{C} \frac{k\omega(\rho(t_0, t))}{2\Delta_0} M.$$

In both cases $\|w - w_1\| \leq \frac{1}{C} f(k\omega(\rho(t_0, t)))$. Finally,

$$\|w_0 - w_1\| \leq \|w_0 - w\| + \|w - w_1\| \leq k\omega(\rho(t_0, t)) + \frac{1}{C} f(k\omega(\rho(t_0, t))). \quad \square$$

Remark 3.3. The result (3.10) of Lemma 3.2 is exact. Let T be the space of convex closed bounded subsets of the Hilbert space \mathcal{H} with the Hausdorff distance, $E = \mathcal{H}$. Define set-valued mappings $F_i : T \rightarrow 2^{\mathcal{H}}$, $i = 1, 2$, as follows:

$$\forall A \in T \quad F_1(A) = A, \quad F_2(A) = B_{\varrho(0,A)}(0),$$

where $\varrho(0, A) = \inf_{a \in A} \|a\|$. Note that $\delta_{F_2}(\varepsilon) = C \cdot \varepsilon^2$ (the modulus of convexity for the Hilbert space). Obviously, $T \ni A \rightarrow F_i(A)$, $i = 1, 2$, are Lipschitz functions in the Hausdorff metric.

Let $L : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$, $L(y_1, y_2) = y_1 - y_2$, $\mathcal{L} = \ker L = \{(y_1, y_2) \in \mathcal{H} \oplus \mathcal{H} \mid y_1 - y_2 = 0\}$. Then

$$(F_1(A), F_2(A)) \cap \mathcal{L} = \{p(A), p(A)\},$$

where $p(A)$ is the metric projection of the zero on the set A . It follows by well-known results of Daniel [7], that $T \ni A \rightarrow p(A)$ is a Hölder function with power $\frac{1}{2}$ in the Hausdorff metric.

Theorem 3.2. Let T be a metric space, E_i uniformly convex Banach spaces, and $F_i : T \rightarrow 2^{E_i}$ uniformly continuous set-valued mappings with modulus of continuity ω , i.e.

$$h((F_1(t_1), F_2(t_2)), (F_1(t_1), F_2(t_2))) \leq \omega(\rho(t_1, t_2)), \quad \forall t_1, t_2 \in T, \quad i = 1, 2.$$

Suppose that images $F_i(t)$ are uniformly convex sets with modulus $\delta(\varepsilon)$, $i = 1, 2$ and $\varepsilon \in (0, 2r_0]$; $\Delta_0 = \delta(2r_0)$. Let $L : E_1 \oplus E_2 \rightarrow E$ be a continuous linear surjection and let $\ker L = \mathcal{L}$.

Suppose that there exists $C > 0$ such that for any $w_1 = (u_1, v_1) \in \mathcal{L}$, $w_2 = (u_2, v_2) \in \mathcal{L}$ we have $\|u_1 - u_2\|_{E_1} \geq C\|w_1 - w_2\|$ and $\|v_1 - v_2\|_{E_2} \geq C\|w_1 - w_2\|$.

Let $f(t) \in L(F_1(t), F_2(t))$ be a uniformly continuous selection. Then there exist uniformly continuous selections $f_i(t) \in F_i(t)$, $i = 1, 2$, with $f(t) = L(f_1(t), f_2(t))$.

Proof. The space $E_1 \oplus E_2$ is uniformly convex with the norm [22]:

$$\|\cdot\|_{uc} = \sqrt{\|\cdot\|_{E_1}^2 + \|\cdot\|_{E_2}^2}.$$

By the inequalities

$$\max\{\|u\|_{E_1}, \|v\|_{E_2}\} \leq \|(u, v)\|_{uc} \leq \sqrt{2} \max\{\|u\|_{E_1}, \|v\|_{E_2}\}, \quad \forall u \in E_1, \forall v \in E_2$$

the norms $\|\cdot\|_{uc}$ and $\max\{\|u\|_{E_1}, \|v\|_{E_2}\}$ are equivalent. Let $w(t)$ be the metric projection of zero onto $L^{-1}(f(t))$ in the space $E_1 \oplus E_2$ with the norm $\|\cdot\|_{uc}$.

By [2, Corollary 3.3.6] the set-valued mapping $t \rightarrow L^{-1}(f(t))$ is uniformly continuous with respect to the Hausdorff distance. By Example 3.2, $w(t) = (u(t), v(t))$ is uniformly continuous and $L^{-1}(f(t)) = w(t) + \mathcal{L}$. Now,

$$H(t) = w(t) + (F_1(t) - u(t), F_2(t) - v(t)) \cap \mathcal{L}$$

is uniformly continuous by Lemma 3.2.

We define $(f_1(t), f_2(t))$ as the metric projection of the zero onto $H(t)$ in the sense of the norm $\|\cdot\|_{uc}$. This projection is uniformly continuous by Example 3.2. \square

Remark 3.4. Note that in the case $E = \mathbb{R}^n$ we can define $(f_1(t), f_2(t))$ as $(f_1(t), f_2(t)) = s(H(t))$, where $s(H(t))$ is the Steiner point of the set $H(t)$.

Consider set-valued mappings F_i , $i = 1, 2$, and the surjection L from Remark 3.3, assuming that $\mathcal{H} = \mathbb{R}^n$. Let $f(A) = 0 \in L(F_1(A), F_2(A))$. The only solution of this splitting problem is the point:

$$f_1(A) = f_2(A) = p(A) = F_1(A) \cap F_2(A),$$

which is the metric projection of zero on the set A in the space \mathbb{R}^n . It follows by Remark 3.3 that in \mathbb{R}^n the order of modulus of continuity for $f_1(A) = p(A)$ and $f_2(A) = p(A)$ is exact.

Acknowledgments

This research was supported by SRA grants P1-0292-0101, J1-9643-0101, J1-2057-0101, and BI-RU/08-09/001. The first author was supported by RFBR grant 07-01-00156, ADAP project “Development of scientific potential of higher school” 2.1.1/500 and REA Program 1.2.1. We thank the referees for comments.

References

- [1] Ya.I. Alber, A.I. Notik, On some estimates for projection operator in Banach space, arXiv:func-an/9311003v1, 25 November 1993.
- [2] J.-P. Aubin, I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, Inc., New York, 1984.
- [3] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Basel, 1990.
- [4] M.V. Balashov, D. Repovš, On the splitting problem for selections, *J. Math. Anal. Appl.* 355 (1) (2009) 277–287.
- [5] V.I. Berdyshev, Continuity of a multivalued mapping connected with the problem of minimizing a functional, *Math. USSR-Izv.* 16 (3) (1981) 431–456.
- [6] V.I. Berdyshev, Stability of minimization of functionals and uniform continuity of metric projection, Doctoral thesis, Moscow, 1987.
- [7] J.W. Daniel, The continuity of metric projection as function of data, *J. Approx. Theory* 12 (3) (1974) 234–240.
- [8] J. Diestel, *Geometry of Banach Spaces – Selected Topics*, Springer-Verlag, Berlin, 1975.
- [9] H. Frankowska, Ch. Olech, R-convexity of the integral of the set-valued functions, in: *Contributions to Analysis and Geometry*, John Hopkins Univ. Press, Baltimore, MD, 1981, pp. 117–129.
- [10] K. Leichtweiß, *Konvexe Mengen*, Springer-Verlag, Berlin, 1980.
- [11] E.S. Levitin, B.T. Polyak, Constrained minimization methods, *Zh. Vychisl. Mat. Mat. Fiz.* 6 (1966) 787–823 (in Russian).
- [12] J. Lindenstrauss, L. Tzafriri, *Geometry of Banach Spaces – II. Functional Spaces*, Springer-Verlag, Berlin, 1979.
- [13] D.T. Luc, M.-L. Martinez-Legaz, A. Seeger, Least deviation decomposition with respect to a pair of convex sets, *J. Convex Anal.* 6 (1) (1999) 115–140, .
- [14] A.V. Marinov, The Lipschitz constants of the metric ε -projection operator in spaces with given modules of convexity and smoothness, *Izv. Mat.* 62 (2) (1998) 313–318 (in Russian); English transl. in: *Izv. Ross. Akad. Nauk Ser. Mat.* 62 (2) (1998) 103–130.
- [15] M.-L. Martinez-Legaz, A. Seeger, A general cone decomposition theory based on efficiency, *Math. Program.* 65 (1994) 1–20.
- [16] E. Michael, Continuous selections, II, *Ann. of Math.* (2) 64 (1956) 562–580.
- [17] L.F. Midolo, G. De Marco, Right inverses of linear maps on convex sets, *Topology Appl.* 156 (7) (2009) 1186–1191.
- [18] E.S. Polovinkin, M.V. Balashov, *Elements of Convex and Strongly Convex Analysis*, Fizmatlit, Moscow, 2007 (in Russian).
- [19] B.T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Soviet Math.* 7 (1966) 72–75.
- [20] D. Repovš, P.V. Semenov, Sections of convex bodies and splitting problem for selections, *J. Math. Anal. Appl.* 334 (2007) 646–655.
- [21] D. Repovš, P.V. Semenov, Hereditary invertible linear surjection and splitting problem for selection, *Topology Appl.* 156 (7) (2009) 1192–1198.
- [22] K.-S. Saito, M. Kato, Uniform convexity of ψ -direct sum of Banach spaces, *J. Math. Anal. Appl.* 277 (2003) 1–11.
- [23] F.P. Vasil'ev, *Numerical Methods for Solution of Extremal Problems*, Nauka, Moscow, 1980 (in Russian).