

Continuous Selections of Multivalued Mappings

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Dedicated to the memory of Ernest A. Michael (1925–2013) the founder of selection theory.

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1 Preliminaries

A *selection* of a given multivalued mapping $F : X \rightarrow Y$ with nonempty values $F(x) \neq \emptyset$, for every $x \in X$, is a mapping $\Phi : X \rightarrow Y$ (in general, also multivalued) which for every $x \in X$, selects a nonempty subset $\Phi(x) \subset F(x)$. When all $\Phi(x)$ are

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singletons, a selection is called *singlevalued* and is identified with the usual single-valued mapping $f : X \rightarrow Y$, $\{f(x)\} = \Phi(x)$. As a rule, we shall use small letters $f, g, h, \phi, \psi, \dots$ for singlevalued mappings and capital letters $F, G, H, \Phi, \Psi, \dots$ for multivalued mappings.

There exist a great number of theorems on existence of selections in the category of topological spaces and their continuous (in various senses) mappings. However, the citation index of one of them is by an order of magnitude higher than for any other one: this is the Michael selection theorem for convex-valued mappings [67, Theorem 3.2''], (a) \Rightarrow (b)]:

Theorem 1.1 *A multivalued mapping $F : X \rightarrow B$ admits a continuous singlevalued selection, provided that the following conditions are satisfied:*

- (1) X is a paracompact space;
- (2) B is a Banach space;
- (3) F is a lower semicontinuous (LSC) mapping;
- (4) For every $x \in X$, $F(x)$ is a nonempty convex subset of B ; and
- (5) For every $x \in X$, $F(x)$ is a closed subset of B .

Moreover, the reverse implication (b) \Rightarrow (a) in [67, Theorem 3.2''] states that a T_1 -space X is paracompact whenever each multivalued mapping $F : X \rightarrow B$ with properties (2)–(5) above, admits a continuous singlevalued selection.

If one identifies a multivalued mapping $F : X \rightarrow Y$ with its graph $\Gamma_F \subset X \times Y$ then the *lower semicontinuity* (LSC) of F means exactly the openness of the restriction $\pi_1|_{\Gamma_F} : \Gamma_F \rightarrow X$ of the projection $\pi_1 : X \times Y \rightarrow X$ onto the first factor. In more direct terms, lower semicontinuity of a multivalued mapping $F : X \rightarrow Y$ between topological spaces X and Y means that the (large) preimage $F^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$ of any open set $U \subset Y$ is an open subset of the domain X . Applying the Axiom of Choice, we obtain:

Lemma 1.2 *The following statements are equivalent:*

- (1) $F : X \rightarrow Y$ is a lower semicontinuous mapping;
- (2) For each $(x; y) \in \Gamma_F$ and each open neighborhood $U(y)$ there exists a local singlevalued (not necessarily continuous) selection of F , say $s : x' \mapsto s(x') \in F(x') \cap U(y)$, defined on some open neighborhood $V(x)$.

Therefore, the notion of lower semicontinuity is by definition related to the notion of selection. Symmetrically, if the (large) preimage $F^{-1}(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$ of any closed set $A \subset Y$ is a closed subset of the domain X , then the mapping $F : X \rightarrow Y$ is said to be *upper semicontinuous* (USC). Note that a more useful definition of upper semicontinuity of F is that the (small) preimage $F_{-1}(U) = \{x \in X : F(x) \subset U\}$ of any open $U \subset Y$ is an open subset of the domain X .

Let us now reformulate the other three principal Michael's theorems on selections.

Theorem 1.3 [68] *A multivalued mapping $F : X \rightarrow Y$ admits a continuous singlevalued selection, provided that the following conditions are satisfied:*

- (1) X is a zero-dimensional (in dim-sense) paracompact space;
- (2) Y is a completely metrizable space;
- (3) F is a LSC mapping; and
- (4) For every $x \in X$, $F(x)$ is a closed subset of Y .

Theorem 1.4 [72] A multivalued mapping $F : X \rightarrow Y$ admits a compact-valued USC selection $H : X \rightarrow Y$, which in turn, admits a compact-valued LSC selection $G : X \rightarrow Y$ (i.e. $G(x) \subset H(x) \subset F(x)$, $x \in X$), provided that the following conditions are satisfied:

- (1) X is a paracompact space;
- (2) Y is a completely metrizable space;
- (3) F is a LSC mapping; and
- (4) For every $x \in X$, $F(x)$ is a closed subset of Y .

Theorem 1.5 [68] Let $n \in \mathbb{N}$. A multivalued mapping $F : X \rightarrow Y$ admits a continuous singlevalued selection provided that the following conditions are satisfied:

- (1) X is a paracompact space with $\dim X \leq n + 1$;
- (2) Y is a completely metrizable space;
- (3) F is a LSC mapping;
- (4) For every $x \in X$, $F(x)$ is an n -connected subset of Y ; and
- (5) The family of values $\{F(x)\}_{x \in X}$ is equi-locally n -connected.

A resulting selection of a given multivalued mapping F is practically always constructed as a uniform limit of some sequence of approximate selections. A typical difficult situation arises with the limit point (or the limit subset). Such a limit point (or a subset) can easily end up in the boundary of the set $F(x)$, rather than in the set $F(x)$, if one does not pay attention to a more careful construction of the uniform Cauchy sequence of approximate selections.

In general, for an arbitrary Banach space B , there exists a LSC mapping $F:[0; 1] \rightarrow B$ with convex (nonclosed) values and without any continuous singlevalued selections (cf. [67, Example 6.2] or [85, Theorem 6.1]). On the other hand, every convex-valued LSC mapping of a metrizable domain into a separable Banach space admits a singlevalued selection, provided that all values are finite-dimensional [67, Theorem 3.1''']. Another kind of omission of closedness was suggested in [16, 73]. It turns out that such omission can be made over a σ -discrete subset of the domain.

An alternative to pointwise omission of closedness is to consider some uniform versions of such omission. Namely, one can consider closedness in a fixed subset $Y \subset B$ instead of closedness in the entire Banach space B . Due to a deep result of van Mill, Pelant and Pol [79], existence of selections under such assumption implies that Y must be completely metrizable, or in other words, a G_δ -subset of B .

Due to the Aleksandrov Theorem, each of Theorems 1.3, 1.4 and 1.5 remains valid under a replacement of the entire completely metrizable range space by any of its G_δ -subsets. However, what happens with Theorem 1.1 under such a substitution?

What can one (informally) say concerning the links between the metric structure and the convex structure induced on a G_δ -subset from the entire Banach space?

Thus, during the last two decades one of the most intriguing questions in the selection theory was the following problem:

Problem 1.6 [74] Let Y be a convex G_δ -subset of a Banach space B . Does then every LSC mapping $F : X \rightarrow Y$ of a paracompact space X with nonempty convex closed values into Y have a continuous singlevalued selection?

In the next section we shall present some (partial) affirmative answers, as well as the counterexample of Filippov [30, 31].

2 Solution of the G_δ -Problem

Summarizing the results below, the answer to the G_δ -problem is affirmative for domains which are “almost” finite-dimensional, whereas the answer is negative for domains which are essentially infinite-dimensional, for example, for domains which contain a copy of the Hilbert cube.

For finite-dimensional domains X , the G_δ -problem has an affirmative solution simply because the family of convex closed subsets of a Banach space is ELC^n and every convex set is C^n for every $n \in \mathbb{N}$. Hence Theorem 1.5 can be applied. For a finite-dimensional range B and moreover, for all finite-dimensional values closed in $Y \subset B$, the problem is also trivial, because one can use the compact-valued selection Theorem 1.4 and the fact that the closed convex hull of a finite-dimensional compact space coincides with its convex hull.

As for ways of uniform omission of closedness in the range space let us first consider the simplest case when $Y = G$ is a unique open subset of a Banach space B . Separately we extract the following well-known folklore result (it probably first appeared in an implicit form in Corson and Lindenstrauss [20]).

Lemma 2.1 (Localization Principle) *Suppose that a convex-valued mapping $F : X \rightarrow Y$ of a paracompact domain X into a topological vector space Y admits a singlevalued continuous selection over each member of some open covering ω of the domain. Then F admits a global singlevalued continuous selection.*

Taking for any $x \in X$ and $y \in F(x) \subset G$, an arbitrary open ball D , centered at y , such that the closure $Clos(D)$ is a subset of G , and invoking the Localization Principle we obtain:

Lemma 2.2 *Given any paracompact space X and any open subset G of a Banach space B , every LSC mapping $F : X \rightarrow G$ with nonempty convex values admits a singlevalued continuous selection, whenever all values $F(x)$ are closed in G .*

Somewhat different approach can be obtained using the following:

Lemma 2.3 *For any compact subset K of a convex closed (in G) subset C of an open subset G of a Banach space B , the closed (in B) convex hull $\text{Clos}(\text{conv}(K))$ also lies in C .*

Thus, as it was pointed out in [74, 88], one can affirmatively resolve the G_δ -problem for an arbitrary intersection of countably many open convex subsets of B .

Lemma 2.4 *Let $\{G_n\}, n \in \mathbb{N}$, be a sequence of open convex subsets of a Banach space and $F : X \rightarrow Y = \bigcap_n G_n$ a LSC mapping of a paracompact space X with nonempty convex values. Then F admits a singlevalued continuous selection, whenever all values $F(x)$ are closed in Y .*

In fact, it suffices to pick a compact-valued LSC selection $H : X \rightarrow Y$ of the mapping F (cf. Theorem 1.4). Then the multivalued mapping $\text{Clos}(\text{conv}(H)) : x \mapsto \text{Clos}(\text{conv}(H(x)))$ is a selection of the given mapping F and it remains to apply Theorem 1.1 to the LSC mapping $\text{Clos}(\text{conv}(H))$.

Michael and Namioka [77] characterized those convex G_δ -subsets $Y \subset B$ which are stable with respect to taking closed convex hulls of compact subsets. Note that they essentially used the construction of Filippov’s counterexample [30, 31].

Theorem 2.5 [77] *Let $Y \subset B$ be a convex G_δ -subset of a Banach space B . Then the following statements are equivalent:*

- (1) *If $K \subset Y$ is compact then so is the closed (in Y) convex hull of K ;*
- (2) *For any paracompact space X , each LSC mapping $F : X \rightarrow Y$ with convex closed (in Y) values admits a continuous singlevalued selection;*
- (3) *Same as (2) but with X assumed to be compact and metrizable.*

Moreover, they observed that Theorem 2.5 remains valid for non-convex Y , provided that (1) is modified by also requiring that $K \subset C$ for some closed (in Y) convex subset $C \subset Y$. Hence, the equivalence (2) \Leftrightarrow (3) of Theorem 2.5 holds for any G_δ -subset Y of a Banach space.

Returning to the restrictions for domains, recall that Gutev [35] affirmatively resolved the G_δ -problem for domains X which are either a countably dimensional metric space or a strongly countably dimensional paracompact space. In fact, he proved that in both cases under the hypotheses of the problem, the existence of a singlevalued continuous selection is equivalent to the existence of a compact-valued USC selection. The latter statement is true, because each domain of such type can be represented as the image of some zero-dimensional paracompact space under some closed surjection with all preimages of points being finite.

In 2002, Gutev and Valov [43] obtained a positive answer for domains with the so-called C -property. They introduced a certain enlargement of the original mapping F . Roughly speaking, they defined $W_n(x)$ as the set of all $y \in Y = \bigcap_{n \in \mathbb{N}} G_n$ which are closer to $F(x)$ than to $B \setminus G_n$. It turns out that each of the mappings W_n has an open graph and all of its values are contractible.

Applying the selection theorem of Uspenskii [110] for C -domains, one can first find selections for each W_n and then for the pointwise intersections $\bigcap W_n(x)$ (for

details cf. [43]). Their technique properly works even for arbitrary (non-convex) G_δ -subsets $Y \subset B$. Unfortunately, such a method does not work outside the class of C -domains, because Uspenskii's theorem gives a *characterization* of the C -property.

This was the reason why we stated the following problem in our previous survey:

Problem 2.6 [88] Are the following statements equivalent:

- (1) X is a C -space.
- (2) Each LSC mapping $F : X \rightarrow Y$ to a G_δ -subset of Y of a Banach space B with convex closed (in Y) values admits a continuous singlevalued selection.

Karashev has resolved this problem for weakly infinite-dimensional compact domains.

Theorem 2.7 [53] *Let X be a compact Hausdorff space and suppose that property (2) above holds. Then X is weakly infinite-dimensional.*

Observe that the (non)coincidence of the classes of C -spaces and weakly infinite-dimensional spaces is one of the oldest and still unsolved problems in dimension theory. The advantage of compact spaces is that in this case there is a set of various criteria for weak infinite-dimensionality. In particular, Karashev [53] used the fact that a compact space X is weakly infinite-dimensional if and only if for any mapping $f : X \rightarrow Q$ to the Hilbert cube there exists a mapping $g : X \rightarrow Q$ such that $f(x) \neq g(x)$, for all $x \in X$.

Ending with the affirmative answers, let us recall that in our previous survey we directly suggested (cf. p. 427 in [88]) the area for finding a counterexample. In fact, having Lemma 2.4, one needs to find a convex G_δ -subset Y of a Banach space B , such that Y is not an intersection of countably many open convex sets. Such a situation in fact, appeared in measure theory: for example in the compactum $P[0, 1]$ of probability measures on the segment $[0, 1]$ such is the convex complement of any absolutely continuous measure. To extract the main idea of Filippov's construction we introduce a temporary notion.

Definition 2.8 A convex compact subset K of a Fréchet space B has the *Weizsäcker-Filippov property* (WF-property) if there exists:

- (1) A proper convex G_δ -subset $Y \subset K$ which contains the set $\text{extr}(K)$ of all extreme points of K ; and
- (2) A LSC convex-valued mapping $R : K \rightarrow K$ such that $R(x) \cap \text{extr}(K) \neq \emptyset$, $x \in K$ and $R(\text{conv}A) = \text{conv}A$, for any finite subset $A \subset \text{extr}(K)$.

Lemma 2.9 *If $K \subset B$ has the WF-property then the mapping $F : K \rightarrow Y$ defined by $F(x) = \text{Clos}_Y(R(x) \cap Y)$, $x \in K$, is a counterexample to the G_δ -problem.*

Proof All values $F(x)$ are nonempty because $R(x) \cap \text{extr}(K) \neq \emptyset$ and $\text{extr}(K) \subset Y$. Clearly, $F(x)$ are convex closed (in Y) sets. The mapping $x \mapsto R(x) \cap Y$ is LSC because R is LSC and Y is dense in K . Hence $F : K \rightarrow Y$ is LSC because pointwise-closure operator preserves lower semicontinuity.

Suppose to the contrary, that $f : K \rightarrow Y$ is a singlevalued continuous selection of F . Then $f(x) = x$, provided that x is an extreme point. Moreover, if $x, y \in \text{extr}(K)$ then $f([x, y]) \subset F([x, y]) = [x, y]$ and $f([x, y]) \supset [x, y]$, since $f(x) = x, f(y) = y$, and because of the continuity of f . Similarly, $f(\text{conv}\{x, y, z\}) = \text{conv}\{x, y, z\}$ for each extreme points x, y, z , and so on. Hence, $f(\text{conv}(\text{extr}(K))) = \text{conv}(\text{extr}(K)) \subset K$ is a dense subset of K and $f(K)$ is also dense in K . However, $f(K)$ is compact since it is the image of a compact set K under the continuous mapping f . Therefore $f(K) = K$ which contradicts with $f(K) \subset Y$ and $Y \neq K$. \square

- Theorem 2.10** (1) *The space $P[0; 1]$ of all probability measures on $[0; 1]$ has the WF-property.*
 (2) *In any Banach space there exists a convex compact subset K with the WF-property.*

Proof By the Keller theorem, the convex compact space $P[0; 1]$ can be affinely embedded into the Hilbert space, hence into every Banach space with a Schauder basis (hence into every Banach space). Hence (1) implies (2).

In order to check (1), pick an arbitrary absolutely continuous measure $\mu \in K = P[0; 1]$, for example the Lebesgue measure. For every $m \in K \setminus \{\mu\}$, denote by $l_{m,\mu}$ the infinite ray from the point m through the point μ . Define the convex complement of μ by setting

$$Y = \{m \in K \setminus \{\mu\} : l_{m,\mu} \cap K = [m; \mu]\}.$$

Clearly Y is convex. For every point $x \in [0; 1]$, the Dirac measure δ_x belongs to Y because $(1 - t)\delta_x + t\mu, t > 1$ is not a probability measure. Hence $\text{extr}(K) \subset Y$. Next,

$$Y = \bigcap_{n=1}^{\infty} \{m \in K : (1 - n^{-1}) \cdot \delta_x + n^{-1} \cdot \mu \notin K\}$$

and this is why Y is a proper convex G_δ -subset of K .

Finally, define $R : K \rightarrow K$ by setting $R(m) = \{m' \in K : \text{supp}(m') \subset \text{supp}(m)\}$, where supp denotes the support of the probability measure, i.e. the set of all points $x \in [0; 1]$ with the property that the value of the measure is positive over each neighborhood of the point. It is a straightforward verification that $R : K \rightarrow K$ is a LSC convex-valued mapping and that the equality

$$R\left(\sum \lambda_i \cdot \delta_{x_i}\right) = \text{conv}\{\delta_{x_1}, \dots, \delta_{x_n}\}, \quad x_i \in [0; 1], \quad \lambda_i \geq 0, \quad \sum \lambda_i = 1,$$

is true. \square

A version of the construction was proposed in [91] which (formally) avoids any probability measures and works directly in the Hilbert cube $Q = [0; 1]^{\mathbb{N}}$. Here is a sketch:

- $X = \{x \in Q : x_1 = 1, x_n = x_{2n} + x_{2n+1}, n \in \mathbb{N}\}$;

- $Y = \{x \in X : \sup\{x_n z_n^{-1} : n \in \mathbb{N}\} = \infty\}$, where $z \in X$ is arbitrarily chosen so that $\lim_{n \rightarrow \infty} z(n) = 0$ and $z(n) > 0$ for all n ; and
- $F(x) = \Phi(x) \cap Y$, where $\Phi : X \rightarrow X$ is defined by

$$\Phi(x) = \{y \in X : y_n = 0 \text{ whenever } x_n = 0\}.$$

Let us temporarily say that natural numbers $2n$ and $2n+1$ are *sons* of the number n , which in turn, we shall call the *father* of such *twins*. Thus each natural number has exactly 2 sons, 4 grandsons, etc. and the natural partial order, say $<$, immediately arises on the set \mathbb{N} . With respect to $<$, the set \mathbb{N} can be represented as a binary tree T and every $x \in X$ is a mapping $x : T \rightarrow [0; 1]$ with $x_1 = 1$, $x_n = x_{2n} + x_{2n+1}$, $n \in \mathbb{N}$. In other words, each $x \in X$ defines some probability distribution on each n th level of the binary tree T .

Hence even though all proofs in this construction can be performed directly in the Hilbert cube, the set X is in fact, a “visualization” of the set of all probability measures of the Cantor set and details of the proof look similar to those above.

In conclusion, we mention the paper [32] which demonstrated the essentiality of the G_δ -assumption for Y in Theorem 2.5 of Michael and Namioka. Briefly, it was proved that for every countable $A \subset [0; 1]$, the set $Y = P_A = \{\mu \in P[0; 1] : \text{supp}(\mu) \subset A\}$ has the property (1) from Theorem 2.5. Then by using sets of probability measures with various countable supports, the authors constructed a convex subset $Y \subset \mathbb{R}^2 \times l_2$ with property (1) and without property (2) from Theorem 2.5. Note that, as it was proved by V. Kadets, the property (1) from Theorem 2.5 is equivalent to the closedness of $Y \subset B$ being not only a convex set, but also a *linear subspace* (cf. [75, Proposition 5.1]).

3 Selections and Extensions

There are intimate relations between selections and extensions and typically they appear together: if $A \subset X$ and $f : A \rightarrow Y$ then $\widehat{f} : X \rightarrow Y$ is an extension of f if and only if \widehat{f} is a selection of multivalued mapping $F_A : X \rightarrow Y$ defined by setting $F_A(x) = \{f(x)\}$, $x \in A$, and $F_A(x) = Y$ otherwise. Thus as a rule, each fact concerning existence of singlevalued selections implies some result on extensions. In the other direction, many basic theorems (or some of their special cases) about extensions are special cases of some appropriate selection theorems.

However, extension theory is certainly not simply a subtheory of selection theory: specific questions and problems need specific ideas and methods. For example, selection Theorem 1.1 implies that every continuous map $f : A \rightarrow Y$ from a closed subset A of a paracompact domain X into a Banach space B has a continuous extension $\widehat{f} : X \rightarrow B$ with $f(X) \subset \text{Clos}(\text{conv}(f(A)))$. However, the Dugundji extension theorem states somewhat different: every continuous map $f : A \rightarrow Y$ from a closed subset A of a metric (or stratifiable) domain X into a locally convex topological vector space B has a continuous extension $\widehat{f} : X \rightarrow B$ with $f(X) \subset \text{conv}(f(A))$. Besides

the differences in assumptions and conclusions, these two “similar” theorems are proved by almost disjoint techniques: a sequential procedure of some approximations in the Michael selection theorem and a straightforward answer by a formula in Dugundji extension theorem. It seems that the only common point are continuous partitions of unity.

To emphasize the difference on a more nontrivial level, let us recall that for a wide class of non-locally convex, completely metrizable, topological vector spaces it was proved in [26] that all such spaces are *absolute retracts* (with respect to all metrizable spaces), abbreviated as *AR*'s. At the same time, at present there is no known example of a non-locally convex, completely metrizable, topological vector space E which can be successfully substituted instead of Banach (or Fréchet) spaces B into the assumption of the Michael selection Theorem 1.1. In particular, Dobrowolski stated (private communication) the following:

Problem 3.1 Is the space $l_p, 0 < p < 1$, of all p -summable sequences of reals an absolute selector, i.e. is it true that for every paracompact space (metric space, compact space) and every LSC mapping $F : X \rightarrow l_p$ with convex closed values, there exists a continuous singlevalued selection of F ?

During the last decade one of the most interesting facts concerning relations between selections and extensions was obtained by Dobrowolski and van Mill [27]. To explain their main results recall that $g : X \rightarrow Y$ is said to be an ε -selection of a multivalued mapping $F : X \rightarrow Y$ into a metric space $(Y; d)$ if $dist(g(x), F(x)) < \varepsilon$. Dobrowolski and van Mill used the term ε -near selection for the case when the strong inequality $dist(g(x), F(x)) < \varepsilon$ is replaced by $dist(g(x), F(x)) \leq \varepsilon$. Clearly, for closed-valued mappings 0-near selections are exactly selections.

Definition 3.2 A convex subset Y of a vector metric space $(E; d)$ has the *finite-dimensional selection property* (resp. *finite-dimensional near selection property*) if for every metrizable domain X and every LSC mapping $F : X \rightarrow Y$ with all compact convex and finite-dimensional values $F(x) \subset Y, x \in X$, there exists a continuous singlevalued selection of F (resp., for every $\varepsilon > 0$ there exists a continuous singlevalued ε -near selection of F).

Combining 3.3, 4.1, 5.4 and 6.1 from [27] we formulate the following:

Theorem 3.3 [27] *For any convex subset Y of a vector metric (not necessarily, locally convex) space $(E; d)$ the following statements are equivalent:*

- (1) Y is an *AR*;
- (2) Y has the *finite-dimensional near selection property*.

As for a specific selection theorem we cite:

Theorem 3.4 [27] *Let Y be a convex subset of a vector metric (not necessarily locally convex) space $(E; d)$. Then for every metrizable domain X and every compact-valued and convex-valued LSC mapping $F : X \rightarrow Y$ with $\max\{dim F(x) : x \in X\} < \infty$, there exists a continuous singlevalued selection f of F .*

Continuous singlevalued selections f of a given multivalued mapping F are usually constructed as uniform limits of sequences of certain approximations $\{f_\varepsilon\}$, $\varepsilon \rightarrow 0$, of F . Practically all known selection results are obtained by using one of the following two approaches for a construction of $\{f_n = f_{\varepsilon_n}\}$, $\varepsilon_n \rightarrow 0$. In the first (and the most popular) one, the method of outside approximations, mappings f_n are continuous ε_n -selections of F , i.e. $f_n(x)$ all lie near the set $F(x)$ and all mappings f_n are *continuous*. In the second one, the method of inside approximations, f_n are δ_n -continuous selections of F , i.e. $f_n(x)$ all lie in the set $F(x)$, however f_n are *discontinuous*.

We emphasize that Theorem 3.4 is proved by using the method of inside approximation. This is a rare situation: all previously known to us examples are [10, 70, 78, 85]. However, for non-locally convex range spaces it is an adequate approach since for such spaces Y the intersections of convex subsets with balls are in general, non-convex. Also, compactness is not preserved under the convex closed hull operation.

It is very natural to try to substitute $\dim F(x) < \infty$, $x \in X$, in Theorem 3.4 instead of $\max\{\dim F(x) : x \in X\} < \infty$. It turns out that this is a futile attempt. Namely, 5.6 in [27] implies that Theorem 3.4 becomes false with such a change of the dimensional restriction.

Theorem 3.5 [27] *There exist a linear metric vector space E and a LSC mapping $F : Q \rightarrow E$ from the Hilbert cube Q such that E contains the tower $\{E_n\}$ of closed subsets with the following properties:*

- (1) $Q = \bigcup Q_n$, where $Q_n = F^{-1}(E_n)$;
- (2) The restrictions $F|_{Q_n}$ satisfy all assumptions of Theorem 3.4; and
- (3) For arbitrary choices of continuous selections $f_n : Q_n \rightarrow E$ of $F|_{Q_n}$ their pointwise limit $f = \lim_n f_n$ is not a continuous mapping, whenever such a pointwise limit exists.

Note that due to the Localization principle (Lemma 1.2) the global assumption $\max\{\dim F(x) : x \in X\} < \infty$ in Theorem 3.4 can be replaced by its local version $\max\{\dim F(x) : x \in U(x)\} < \infty$ for some neighborhood $U(x)$ of x . Two slight generalizations of Theorem 3.4 were presented in [91]: in the first one Y was replaced by a G_δ -subset and in the second the closedness restriction for values $F(x)$ was omitted.

Theorem 3.6 [91]

- (1) Let $F : X \rightarrow Y$ be a LSC convex-valued mapping of a paracompact domain X into a G_δ -subset Y of a completely metrizable linear space E . Then F admits a singlevalued continuous selection provided that the values $F(x)$ are closed in Y and that for every $x \in X$ there exists a neighborhood $U(x)$ such that $\max\{\dim F(x') : x' \in U(x)\} < \infty$.
- (2) Let $F : X \rightarrow E$ be a LSC convex-valued mapping of a metrizable domain X into a completely metrizable linear space E . Then F admits a singlevalued continuous selection provided that for every $x \in X$ there exists a neighborhood $U(x)$ such that $\max\{\dim F(x') : x' \in U(x)\} < \infty$.

It is interesting to note that the metrizable of the domain in Theorem 3.6 (2) (in comparison with the paracompactness in (1)) is an essential restriction because the proof is based on the density selection theorem of Michael [69] which works exactly for metrizable spaces.

Under dimensional restrictions for the domain, not for the values of the mapping, van Mill [78, Corollary 5.2] obtained the following:

Theorem 3.7 [78] *Let X be a locally finite-dimensional paracompact space and Y a convex subset of a vector metric space. Then each LSC mapping $F : X \rightarrow Y$ with complete convex values admits a singlevalued continuous selection.*

On the other hand, Example 5.3 [27] shows that in general, Theorem 3.7 does not hold for domains which are unions of countably many finite-dimensional compacta. However, if in the assumptions of Theorem 3.7 one passes to C -domains (which look as approximately finite-dimensional spaces) then exact selections can be replaced by ε -selections [27, Theorem 6.3].

Theorem 3.8 [27] *Let X be a C -space and Y a convex subset of a vector metric space. Then each LSC mapping $F : X \rightarrow Y$ with convex values admits a singlevalued continuous ε -selections for any $\varepsilon > 0$.*

In a voluminous paper, Gutev, Ohta and Yamazaki [42] systematically used selections and extensions for obtaining the criteria for various kinds of displacement of a subset in the entire space. Recall that $A \subset X$ is C -embedded in X (resp., C^* -embedded in X) if every continuous (resp., every bounded continuous) function $f : A \rightarrow \mathbb{R}$ has a continuous extension to entire X . Below are some of their typical results, Theorems 4.3, 4.6, and 6.1.

Theorem 3.9 [42] *For a subset A of X the following statements are equivalent:*

- (1) *A is C^* -embedded in X ;*
- (2) *For every Banach space B , every continuous mapping $F : X \rightarrow B$ with compact convex values $F(x)$ and every continuous selection $g : A \rightarrow B$ of the restriction $F|_A$, there is a continuous extension $f : X \rightarrow B$ of g which is also a selection of F ;*
- (3) *The same as (2), but without convexity of $F(x)$ and without f being a selection of F ;*
- (4) *For every cardinal λ , every continuous maps $g, h : X \rightarrow c_0(\lambda)$ with $g(a) \leq h(a), a \in A$, and every $f : A \rightarrow c_0(\lambda)$ which separates $g|_A$ and $h|_A$, there exists a continuous extension $\widehat{f} : X \rightarrow c_0(\lambda)$ of f .*

Here $c_0(\lambda)$ denotes the Banach space of all mappings x from λ to the reals such that the sets $\{\tau < \lambda : |x(\tau)| \geq \varepsilon\}$ are finite for all $\varepsilon > 0$. Note that $c_0(1) = \mathbb{R}$ and $c_0(\aleph_0) = c_0$.

Theorem 3.10 [42] *For a subset A of X the following statements are equivalent:*

- (1) *A is C -embedded in X ;*

- (2) For every Banach space B , every lower σ -continuous mapping $F : X \rightarrow B$ with closed convex values $F(x)$ and every continuous selection $g : A \rightarrow B$ of the restriction $F|_A$, there is a continuous extension $f : X \rightarrow B$ of g which is also the selection of F ;
- (3) The same as (2), but without convexity of $F(x)$ and without f being a selection of F ;
- (4) For every cardinal λ , every continuous $g_n, h_n : X \rightarrow c_0(\lambda)$, $n \in \mathbb{N}$, with $\liminf g_n(a) \leq \limsup h_n(a)$, $a \in A$, and every $f : A \rightarrow c_0(\lambda)$ which separates $\liminf g_n(a)$ and $\limsup h_n(a)$, there exists a continuous extension $\widehat{f} : X \rightarrow c_0(\lambda)$ of f .

Here lower σ -continuity of a multivalued map means that it is the pointwise closure of a union of countably many continuous compact-valued mappings. Yet another characterization of C -embeddability can be formulated via mappings into open convex subsets of a Banach spaces.

Theorem 3.11 [42] For a subset A of X the following statements are equivalent:

- (1) A is C -embedded in X ;
- (2) For every Banach space B , every open convex $Y \subset B$, every lower σ -continuous mapping $F : X \rightarrow \text{Clos}(Y)$ with closed convex values $F(x)$ and every continuous selection $g : A \rightarrow B$ of $F|_A$ with $g^{-1}(Y) = A \cap F^{-1}(Y)$, there is a continuous extension $f : X \rightarrow B$ of g which is also the selection of F and $f^{-1}(Y) = F^{-1}(Y)$;
- (3) The same as (2), but with continuous compact-valued mapping F .

One more recent paper in which selections and extensions are simultaneously studied is due to Michael [76]. Below we unify 3.1 and 4.1 from [76].

Theorem 3.12 [76] For a metrizable space Y the following statements are equivalent:

- (1) Y is completely metrizable;
- (2) For every paracompact domain X and every LSC mapping $F : X \rightarrow Y$ with closed values, there exists a LSC selection $G : X \rightarrow Y$ with compact values;
- (3) For every closed subset A of a paracompact domain X and every continuous $g : A \rightarrow Y$, there exists a LSC mapping $G : X \rightarrow Y$ with compact values which extends g ;
- (4) Similar to (2) but for USC selection $H : X \rightarrow Y$ with compact values;
- (5) Similar to (3) but for USC extension $H : X \rightarrow Y$ with compact values.

The implication (5) \Rightarrow (1) is true in a more general case, namely when Y is Čech-complete and X is a paracompact p -space [81, 83].

To finish the section we return once again to comparison of the Dugundji extension theorem and the Michael selection theorem. Arvanitakis [2] proposed a uniform approach to proving both of these theorems. He worked with paracompact k -domains. Recall that a Hausdorff space X is called a k -space if closedness of $A \subset X$ coincides

with closedness of $A \cap K$ for all compact $K \subset X$. Below, completeness of a locally convex vector space E means that the closed convex hull operation preserves the compactness of subsets in E . Next, $C(T; E)$ denotes the vector space of all continuous mappings from a topological space T into E , endowed by the topology of uniform convergence on compact subsets.

Theorem 3.13 [2] *Let X be a paracompact k -space, Y a complete metric space, E a locally convex complete vector space, and $F : X \rightarrow Y$ a LSC mapping. Then there exists a linear continuous operator $S : C(Y; E) \rightarrow C(X; E)$ such that*

$$S(f)(x) \in \text{Clos}(\text{conv}(f(F(x)))) , \quad f \in C(Y; E), \quad x \in X.$$

The proof is based on study of regular *exaves* (extensions/averagings) operators but without any explicit use of probability measures and Milyutin mappings. Recently, Valov [111] suggested a generalization of this theorem to the case of an arbitrary paracompact domain. He extensively exploited in its full force the technique of functors P_β and \widehat{P} , averaging operators, Milyutin mappings, and so on. He also used the universality of the zero-dimensional selection Theorem 1.3, i.e. the fact obtained in [94] that Theorem 1.3 implies both Theorems 1.1 and 1.4.

Theorem 3.14 [111] *Let X be a paracompact space, Y a complete metric space and $F : X \rightarrow Y$ a LSC mapping. Then:*

- (1) *For every locally convex complete vector space E there exists a linear operator $S_b : C_b(Y; E) \rightarrow C_b(X; E)$ such that*

$$S(f)(x) \in \text{Clos}(\text{conv}(f(F(x)))) , \quad f \in C(Y; E), \quad x \in X,$$

and such that S_b is continuous with respect to the uniform topology and the topology of uniform convergence on compact subsets;

- (2) *If X is a k -space and E is a Banach space then S_b can be continuously extended (with respect to both topologies) to a linear operator $S : C(Y; E) \rightarrow C(X; E)$ with the property that $S(f)(x) \in \text{Clos}(\text{conv}(f(F(x))))$.*

Therefore by taking $Y = E = B$ to be a Banach space, F a mapping with closed convex values and $f = id|_Y$, one can see that $S(f)$ is a selection of F : $S(id)(x) \in \text{Clos}(\text{conv}(F(x)))$, $x \in X$.

Next, if A is a completely metrizable closed subspace of X , E a locally convex complete vector space, and $F = F_A$ a mapping defined by $F(x) = \{x\}$, $x \in A$, and $F(x) = A$, $x \in X \setminus A$, then we see that $S_b(f)(x) \in \text{Clos}(\text{conv}(f(F(x))))$, for any $f \in C_b(A; E)$ and hence $S_b(f)(x) = f(x)$, whenever $x \in A$. Therefore $S_b(f)$ is an extension of f . Thus the result is on the one hand stronger than the Dugundji theorem because X can be nonmetrizable, but on the other hand it is weaker because A should be completely metrizable and the result relates to $C_b(A; E)$, not to $C(A; E)$.

As a corollary, the Banach-valued version of the celebrated Milyutin theorem can be obtained:

Theorem 3.15 [111] *Let X be an uncountable compact metric space, K the Cantor set and B a Banach space. Then $C(X; B)$ is isomorphic to $C(K; B)$.*

4 Selection Characterizations of Domains

Theorem 1.1 states that assumptions (1)–(5) imply the existence of selections of a multivalued mapping F . Conversely, assumptions (2)–(5) together with existence of selections imply the condition (1) that a domain X is a paracompact space. In other words, Theorem 1.1 gives a *selection characterization of paracompactness*. By varying the types of the range Banach spaces B , types of families of convex subsets of B , types of continuity of F , etc. one can try to find a selection characterization of some other topological types of domains. Originally, Michael [67] found such types of characterization for normality, collectionwise normality, normality and countable paracompactness, and perfect normality. Below we concentrate on recent results in this direction.

Gutev, Ohta and Yamazaki [41] obtained selection characterizations for three classes of domains inside the class of all λ -collectionwise normal spaces. Recall that this property means that for each discrete family $\{F_\gamma\}_{\gamma \in \Gamma}$ of closed subsets with $|\Gamma| \leq \lambda$ there is a discrete family $\{G_\gamma\}_{\gamma \in \Gamma}$ of open sets such that $F_\gamma \subset G_\gamma$. Note that the equivalence of (1) and (2) in Theorem 4.1 was proposed by Michael [67] (cf. the discussion concerning the proofs in Chap. II of [85]). We also observe that (4) in Theorems 4.1, 4.2 and 4.3 resembles the classical Dowker separation theorem.

Theorem 4.1 [41] *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is λ -collectionwise normal;
- (2) For every Banach space B of the weight less than or equal to λ and every LSC mapping $F : X \rightarrow B$ whose values $F(x)$ are convex compacta, or $F(x) = B$, there exists a continuous singlevalued selection of F ;
- (3) Same as (2) but for the Banach space $B = c_0(\lambda)$;
- (4) For every closed $A \subset X$ and every singlevalued $g, h : A \rightarrow c_0(\lambda)$ such that $g \leq h$, g is upper semicontinuous, and h is lower semicontinuous, there exists a singlevalued continuous $f : X \rightarrow c_0(\lambda)$ such that $f|_A$ separates g and h , i.e. $g \leq f|_A \leq h$.

Theorem 4.2 [41] *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is countably paracompact and λ -collectionwise normal;
- (2) For every generalized $c_0(\lambda)$ -space B and every LSC mapping $F : X \rightarrow B$ with values $F(x)$ being convex compacta, or $F(x) = B$, and with $|F(x)| > 1$, $x \in X$, there exists a continuous singlevalued selection f of F such that $f(x)$ is not an extreme point of $F(x)$, $x \in X$;

- (3) Same as (2) but for the Banach space $B = c_0(\lambda)$;
- (4) Same as (4) in Theorem 4.1 but with strong inequalities $g < h$ and $g < f|_A < h$.

Theorem 4.3 [41] *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is perfectly normal and λ -collectionwise normal;
- (2) For every generalized $c_0(\lambda)$ -space B and every LSC mapping $F : X \rightarrow B$ with values $F(x)$ being convex compacta, or $F(x) = B$, there exists a continuous singlevalued selection f of F such that $f(x)$ is not an extreme point of $F(x)$, whenever $|F(x)| > 1$;
- (3) Same as (2) but for the Banach space $B = c_0(\lambda)$;
- (4) Same as (4) in Theorem 4.1 but with strong inequalities $g(x) < f(x) < h(x)$ for all $x \in A$ with $g(x) < h(x)$.

One of the key ingredients of the proofs is the fact that for a closed-valued and convex-valued mapping F , a selection avoiding all extreme points exists provided that F admits two families of local disjoint selections. Certainly Theorems 4.1, 4.2 and 4.3 constitute a base for Theorems 3.9, 3.10 and 3.11 above.

In [41] authors stated the following question: Do Theorems 4.2 and 4.3 remain valid if in (2) one replaces $c_0(\lambda)$ -space by an arbitrary Banach space B of weight less than or equal to λ ? Yamauchi answered this question in the affirmative.

Theorem 4.4 [117] *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is countably paracompact and λ -collectionwise normal;
- (2) For every Banach space B of weight less than or equal to λ , every LSC mapping $F : X \rightarrow B$ with values $F(x)$ being convex compacta, or $F(x) = B$ and with $|F(x)| > 1, x \in X$, there exists a continuous singlevalued selection f of F such that $f(x)$ is not an extreme point of $F(x), x \in X$.

Passing to λ -paracompactness, the following was proved in [117, Theorem 8]:

Theorem 4.5 *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is normal and λ -paracompact;
- (2) The same as (2) in Theorem 4.4 but with closed values $F(x)$.

Considering $\lambda = \aleph_0$ one observes that in the Michael selection criteria for X being normal and countably paracompact one can assume that a selection f always avoids extreme points of values of multivalued mapping F with $|F(x)| > 1, x \in X$.

Analogously, a domain X is perfectly normal and λ -paracompact if and only if for every Banach space B with $w(B) \leq \lambda$ and every LSC mapping $F : X \rightarrow B$ with convex closed values $F(x)$ (not necessarily with $|F(x)| > 1, x \in X$) there exists a continuous singlevalued selection f of F such that $f(x)$ is not an extreme point of $F(x)$, whenever $|F(x)| > 1$ (cf. [117]).

Before stating one more result recall that *normality of a covering* ω means an existence of a sequence $\omega_1 = \omega, \omega_2, \omega_3, \dots$ of coverings such that each ω_{n+1} is a strong star refinement of ω_n and that a space is called a λ -*PF-normal space* if each its point-finite open coverings is normal. Yamauchi [115] characterized the class of λ -*PF-normal spaces*.

Theorem 4.6 [115] *Let λ be an infinite cardinal. Then for any T_1 -space X the following statements are equivalent:*

- (1) X is λ -*PF-normal*;
- (2) For every simplicial complex K with cardinality less than or equal to λ , every simplex-valued LSC mapping $F : X \rightarrow |K|$ has a continuous singlevalued selection.

An analogue of Theorem 4.6 for dimensional-like properties was also given in [115].

Here $|K|$ stands for a *geometric realization* of K , for example in the Banach space $l_1(\text{Vert}_K)$, where Vert_K is the set of vertices of K , and $|K|$ is endowed with the metric topology, induced by this embedding. In fact, the initial result here was a theorem of Ivanšić and Rubin [51] that every simplex-valued mapping $F : X \rightarrow |K|_w$ admits a selection provided that $F : X \rightarrow |K|_w$ is locally selectionable, where $|K|_w$ denotes $|K|$ endowed with the weak topology.

Yamauchi [114] proposed selection criteria for classes of realcompact spaces, Dieudonné complete spaces and Lindelöf spaces. The starting point was the result of Blum and Swaminatham [12] on selection characterization of realcompactness in terms of the so-called \mathcal{S} -fixed LSC mapping into a locally convex topological vector space. To avoid specific notations we collect here only the results for Lindelöf spaces.

Theorem 4.7 [114] *For any regular space X the following statements are equivalent:*

- (1) X is Lindelöf;
- (2) For every completely metrizable space Y and every closed-valued LSC mapping $F : X \rightarrow Y$ there exist compact-valued USC mapping $H : X \rightarrow Y$ and compact-valued LSC mapping $G : X \rightarrow Y$ such that $G(x) \subset H(x) \subset F(x)$, $x \in X$ and $H(X) = \bigcup\{H(x) : x \in X\}$ is separable;
- (3) For every Banach space B and every LSC mapping $F : X \rightarrow B$ with closed convex values there exists a continuous singlevalued selection f of F with separable image $f(X)$.

Next, we refer to our previous survey on selections to cite the Choban-Gutnev-Nedev conjecture.

Problem 4.8 [88] *For every T_1 space X the following statements are equivalent:*

- (1) X is countably paracompact and collectionwise normal;

- (2) For every Hilbert space H and every LSC mapping $F : X \rightarrow H$ with closed convex values there exists a continuous singlevalued selection f of F .

The implication (2) \Rightarrow (1) is a standard exercise, while (1) \Rightarrow (2) was a hard problem. During the last decade, in a series of papers, Shishkov successfully resolved the problem step by step. Here is a short list of his results. First, he reduced the situation to the case of *bounded* mappings $F : X \rightarrow Y$, i.e. mappings with the bounded in Y image $F(X) = \bigcup\{F(x) : x \in X\}$.

Theorem 4.9 [102] *For every countably paracompact space X and every normed space Y the following statements are equivalent:*

- (1) *For every LSC mapping from X to Y with closed convex values there exists a continuous singlevalued selection;*
- (2) *Same as (1) but for bounded LSC mappings.*

Next, he solved the problem in the case of the domain X a σ -product of metric spaces [103] and extended LSC mappings with normal and countably paracompact domains over the Dieudonné completions of the domains [104]. Then he proved the conjecture for domains which are hereditarily “nice” [105]:

Theorem 4.10 [105] *Let X be a countably paracompact and hereditarily collectionwise normal space, B a reflexive Banach space and $F : X \rightarrow B$ a LSC mapping with convex closed values. Then there exists a continuous singlevalued selection f of F .*

In [106] Shishkov worked with a paracompactness-like restriction on domain.

Theorem 4.11 [106] *Let X be a c -paracompact and collectionwise normal space, B a reflexive Banach space, and $F : X \rightarrow B$ a LSC mapping with convex closed values. Then there exists a continuous singlevalued selection f of F .*

Note that X from the last theorem can be a nonparacompact space, and that collectionwise normality plus (c) -paracompactness of domain is not in general, a necessary restriction for existence of selections (cf. Nedev’s theorem for mappings over ω_1 [80]).

Then Shishkov [107] proved the following fact which, together with Theorem 4.9, finally resolved Choban-Gutsev-Nedev problem.

Theorem 4.12 [107] *Let X be a collectionwise normal space, H a Hilbert space and $F : X \rightarrow H$ a LSC mapping with convex closed and bounded values. Then there exists a continuous singlevalued selection f of F .*

We note that the proof essentially uses the geometric and analytical structure of a Hilbert space. Thus for reflexive range spaces the problem is still open.

We end the section by recent results of Gutsev and Makala [40] who have suggested a characterization for classes of domains by using a controlled local improvement of ε -selections up to a genuine selection.

Theorem 4.13 [40] *Let λ be an infinite cardinal. For any T_1 -space X the following statements are equivalent:*

- (1) X is countably paracompact and λ -collectionwise normal;
- (2) For every Banach space B with $w(B) \leq \lambda$, every LSC mapping $F : X \rightarrow B$ with values $F(x)$ being convex compacta, or $F(x) = B$, every continuous $\varepsilon : X \rightarrow (0; +\infty)$ and every ε -selection $f_\varepsilon : X \rightarrow B$ of F , there exists a continuous singlevalued selection f of F such that $\text{dist}(f(x), f_\varepsilon(x)) < \varepsilon(x)$, $x \in X$.

Similarly we have for normality:

Theorem 4.14 [40] *For any T_1 -space X the following statements are equivalent:*

- (1) X is countably paracompact and normal;
- (2) Same as (2) in Theorem 4.13 but for separable Banach spaces.

Note that the starting point of proofs in [40] was the following:

Theorem 4.15 [40] *For any Banach space B the following statements are equivalent:*

- (1) For every collectionwise normal domain X and every LSC $F : X \rightarrow B$ with values $F(x)$ being convex compacta, or $F(x) = B$ there exists a continuous selection f of F ;
- (2) Same as (1) but without possibility of $F(x) = B$.

5 Generalized Convexities

5.1. Roughly speaking, there exists an entire mathematical “universe” devoted to various generalizations and versions of convexity. In our opinion, even if one simply lists the titles of “generalized convexities” one will find as a minimum, nearly 20 different notions.

As for the specific situation with continuous selections perhaps two principal approaches are really used here. With the *inner* point of view, one starts by introducing some type of “convex hull” operation and defines a convex set as a set which is preserved by such an operation. Typical examples are Menger’s metric convexity [66], Michael’s convex and geodesic structures [70], Mägerl’s paved spaces [62], Bielawski’s simplicial convexity [11], Horvath’s structures [47], Saveliev’s convexity [98], etc.

With respect to *outer* constructions, convex sets are introduced by some list of axioms and then the convex hull $\text{conv}A$ of a set A is defined as the intersection $\bigcap \{C : A \subset C \text{ and } C \text{ is convex}\}$. Among examples are: Levy’s abstract convexity [58], Jamison’s convexity [52], van de Vel’s topological convexity [112], decomposable sets [33, 85], and many others. The following notion was introduced by van de Vel in [112].

Definition 5.1 A family \mathcal{C} of subsets of a set Y is called a *convexity* on Y if it contains \emptyset and Y , is closed with respect to intersections of arbitrary subfamilies and is closed with respect to unions of an arbitrary up-directed subfamilies.

van de Vel [112] proved a Michael’s selection type theorem for LSC mappings into completely metrizable spaces Y endowed with a convexity \mathcal{C} which satisfies a set of assumptions such as compatibility with metric uniformity, compactness and connectedness of *polytopes* (i.e. convex hulls of finite sets), etc. One of the crucial restriction was the so-called *Kakutani S_4 -property* which means that every pair of disjoint convex sets admits extensions up to two complementary convex sets (i.e. half-spaces). In the special issue of “Topology and Applications” entirely dedicated to 50th anniversary of selection theory and to the 80th anniversary of Ernest Michael, Horvath [48] proposed an approach which gives a selection theorem for convexities with the relative S_4 -property.

Theorem 5.2 [48] *Let $(Y; \mathcal{C})$ be a completely metrizable space with convexity, let all polytopes be compact and connected, and let Kakutani S_4 -property hold on polytopes with respect to the induced convexities. Then every LSC mapping $F : X \rightarrow Y$ from a paracompact domain and with closed convex values admits a continuous singlevalued selection.*

He also added facts on selections with results on extensions, approximations and fixed points. In fact, compactness and connectedness of polytopes together with the Kakutani S_4 -property imply homotopical triviality of polytopes and moreover, of all completely metrizable convex sets. Therefore the following Horvath’s theorem generalizes the previous one.

Theorem 5.3 [48] *Let $(Y; \mathcal{C})$ be a completely metrizable space with convexity for which all polytopes are homotopically trivial. Then every LSC mapping $F : X \rightarrow Y$ with paracompact domain and with closed convex values admits a continuous singlevalued selection.*

The key technical ingredient proposed in [48] was the *van de Vel* property, which roughly speaking, fixes the existence of enough reflexive relations (entourages) $R \subset Y \times Y$ such that for every subset $Z \subset Y$ all simplicial complexes

$$S_R(Z) = \{A \text{ is a finite subset of } Z : Z \cap (\bigcap_{a \in A} R(a)) \neq \emptyset\}$$

are homotopically trivial.

In the same issue of *Topology and its Applications*, Gutev [38] presented results on a somewhat similar matter. Briefly, he proposed another approach to proving the van de Vel selection theorem. He incorporated the proof into the technique of the so-called *c-structures* which was suggested around 1990 by Horvath [47]. A *c-structure* χ on a space Y associates to every finite subset $A \subset Y$ some contractible subset $\chi(A) \subset Y$ such that $A \subset B$ implies $\chi(A) \subset \chi(B)$. In the case of the finite subsets A of some prescribed $S \subset Y$ Gutev use the term *c-system on S* .

Of course, as usual $\chi(\{y\}) = \{y\}$). Perhaps the typical statements in [38] are, for example:

Theorem 5.4 [38] *Let X be a paracompact space, Y a space, $\chi : Fin(S) \rightarrow Y$ a c -system on $S \subset Y$, and $G : X \rightarrow S$ a Browder mapping (i.e. a multivalued mapping with all point-preimages open). Then $conv_\chi(G)$ has a continuous singlevalued selection.*

Theorem 5.5 [38] *Let X be a paracompact space, (Y, μ, \mathcal{C}) a uniform space endowed with a S_4 convexity for which all polytopes are compact and all convex sets are connected. Let $\mathcal{V} \in \mu$ be an open convex cover of X and $F : X \rightarrow Y$ a convex-valued LSC mapping. Then F admits a LSC convex-valued selection $\Phi : X \rightarrow Y$ such that the family $\{\Phi(x)\}_{x \in X}$ of its values refines \mathcal{V} .*

By using the latter fact, under the assumptions of the van de Vel theorem, one can construct a decreasing sequence $\Phi_n : X \rightarrow Y$ of convex-valued LSC mappings with $\sup_{x \in X} diam(\Phi_n(x)) \rightarrow 0, n \rightarrow \infty$. Therefore the pointwise passing to $\bigcap_n \{Clos(\Phi_n(x))\}$ gives the desired selection of $F = \Phi_0$.

5.2. Some results on selections appeared for *hyperconvex* range spaces. Recall, that a metric space is *hyperconvex* if and only if it is injective with respect to extensions which preserves the modulus of continuity. In more direct terms, a metric space (Y, d) is hyperconvex if and only if for every family $\{(y_\alpha; r_\alpha) \in Y \times (0; +\infty)\}_{\alpha \in I}$ the inequalities $d(y_\alpha, y_\beta) \leq r_\alpha + r_\beta$ for all $\alpha, \beta \in I$ imply the nonemptiness of the intersection $\bigcap_\alpha D[y_\alpha; r_\alpha]$ of closed balls $D[y_\alpha; r_\alpha] = Clos(D(y_\alpha; r_\alpha))$.

Each hyperconvex space admits the natural *ball convexity* in which polytopes are exactly the sets $\bigcap \{D[y, r] : A \subset D[y, r]\}$ with finite $A \subset Y$. Some authors [64, 121] also use the term *sub-admissible* for sets which are convex with respect to the ball convexity. For example, Wu [121, Theorem 2.4] proved a selection theorem for the so-called *locally uniform weak LSC* mappings into hyperconvex spaces. Note that hyperconvex space equipped with the ball convexity is a uniform convex space with homotopically trivial polytopes, [48]. This is why a selection theorem for LSC mappings [121, Theorem 2.3] is a special case of selection theorems for generalized convexities. Markin [64] generalized Wu’s result to a wider class of multivalued mappings which he named *quasi LSC* although this is exactly the class of *almost LSC* mappings introduced by Deutsch and Kenderov [24].

Another type of selection theorems for hyperconvex range spaces deals with various Lipschitz-type restrictions on a mappings and selections. A subset $Z \subset (Y; d)$ is said to be *externally hyperconvex* if for any points $x_\alpha \in Z$ and any reals r_α with $d(x_\alpha, Z) \leq r_\alpha$ and $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, the intersection $\bigcap_\alpha D[x_\alpha, r_\alpha] \cap Z$ is nonempty. Khamisi, Kirk and Yanez [54] proved the following:

Theorem 5.6 [54] *Let $(Y; d)$ be any hyperconvex space, S any set, and $F : S \rightarrow Y$ any mapping with externally hyperconvex values. Then there exists a singlevalued selection f of F such that*

$$d(f(x), f(y)) \leq Hausd_d(F(x), F(y)) \quad x, y \in S.$$

In particular, for a metric domain $S = (M; \rho)$ and for a nonexpansive $F : S \rightarrow Y$, one can assume a selection f of F to be also nonexpansive, provided that all values $F(x)$ are bounded externally hyperconvex sets.

These results were applied by Askoy and Khamsi [3] for range spaces which are *metric trees*. Briefly, a metric tree is a metric space (Y, d) such that for any $x, y \in Y$ there exists a unique arc joining x and y and such that the arc is isometric to a segment on the real line.

Theorem 5.7 [3] *Let $(Y; d)$ be a metric tree and $F : Y \rightarrow Y$ a mapping all of whose values are bounded closed convex sets. Then there exists a singlevalued selection f of F such that*

$$d(f(x), f(y)) \leq \text{Hausd}_d(F(x), F(y)) \quad x, y \in Y.$$

A somewhat similar result was proved by Markin [65, Theorem 4.3]:

Theorem 5.8 [65] *Let X be a paracompact space, $(Y; d)$ a complete metric tree and $F : X \rightarrow Y$ an almost LSC mapping all of whose values are bounded closed convex sets. Then there exists a singlevalued continuous selection f of F .*

Selection theorems with respect to various types of convexities, L -structures, and G -structures, were obtained in [25, 59, 118], etc. As a rule, all results here are special cases or versions of van de Vel’s convexities, or Horvath’s convexities.

5.3. As for some other “inner convexities”, in a series of papers de Blasi and Pianigiani studied multivalued mappings into so-called α -convex metric spaces $(Y; d)$. This means the existence a continuous mapping $\alpha : Y \times Y \times [0; 1] \rightarrow Y$ with natural restrictions

$$\alpha(y, y, t) = y, \quad \alpha(y, z, 0) = y, \quad \alpha(y, z, 1) = z$$

and with assumption that for some suitable $r = r(\alpha) > 0$ and for every $\varepsilon < r$ there exists $0 < \delta \leq \varepsilon$ such that the inequality

$$\text{Hausd}(\{\alpha(y, z, t) : t \in [0; 1]\}, \{\alpha(y', z', t) : t \in [0; 1]\}) < \varepsilon$$

for Hausdorff distance between curvilinear segments holds for each (y, y') and (z, z') in Y^2 with $d(y, y') < \varepsilon$ and $d(z, z') < \delta$. Clearly, the last assumption reminds one of the estimate for $d(\alpha(y, z, t), \alpha(y', z', t))$ from Michael’s geodesic structure [70]. As usual, $C \subset Y$ is convex (with respect to $\alpha : Y \times Y \times [0; 1] \rightarrow Y$) if $\{\alpha(y, z, t) : t \in [0; 1]\} \subset C$ provided that $y \in C$ and $z \in C$.

Theorem 5.9 [23] *Let X be a paracompact space and Y an α -convex complete metric space. Then every LSC mapping $F : X \rightarrow Y$ with closed convex values admits a continuous singlevalued selection.*

As it was shown earlier in [22] for the case of compact X and Y , and $\dim X < \infty$, Theorem 5.11 is true for every $\alpha : Y \times Y \times [0; 1] \rightarrow Y$ with $\alpha(y, y, t) = y$,

$\alpha(y, z, 0) = y, \alpha(y, z, 1) = z$. A set of applications in fixed point theory and in degree theory are also presented in [21, 23]. Kowalska [57] proved a theorem which unifies selection Theorem 5.9 with a graph-approximation theorems in the spirit of Ben-El-Mechaiekh and Kryszewski [9] who considered the case of classical convexity of a Banach space.

Among others results let us mention the paper of Kisielewicz [55] in which he used a convexity structures in functional Banach spaces $C(S, R^n)$ and $L^\infty(T, R^n)$ of all continuous mappings over a compact Hausdorff domain S and all equivalence classes of almost everywhere bounded mappings over a measure space (T, μ) . Both of these convexities remind of the notion of *decomposable* set of functions [33, 85].

- Definition 5.10** (1) A subset $E \subset L^\infty(T, R^n)$ is said to be *decomposable* if $\chi_A f + \chi_{T \setminus A} g$ belongs to E provided that $f \in E, g \in E$ and A is a measurable subset of T .
- (2) A subset $E \subset L^\infty(T, R^n)$ is said to be *L-convex* if $pf + (1 - p)g$ belongs to E provided that $f \in E, g \in E$ and $p : T \rightarrow [0; 1]$ is a measurable function.
- (3) A subset $E \subset C(S, R^n)$ is said to be *C-convex* if $hf + (1 - h)g$ belongs to E provided that $f \in E, g \in E$ and $h : T \rightarrow [0; 1]$ is a continuous function.

Theorem 5.11 [55]

- (1) Let X be a paracompact space and $F : X \rightarrow C(S, R^n)$ a LSC mapping with closed C-convex values. Then F admits a continuous singlevalued selection if and only if its n th derived mapping $F^{(n)}$ has nonempty values.
- (2) Same as (1) but for $F : X \rightarrow L^\infty(T, R^n)$ with closed L-convex values.

Note that for closed subsets of $L^\infty(T, R^n)$ their L-convexity coincides with decomposability plus usual convexity [55, Proposition 4]. Recall that the *derived mapping* $F^{(1)}(x)$ of a multivalued mapping $F : X \rightarrow Y$ is defined by setting

$$F^{(1)}(x) = \{y \in F(x) : (x' \rightarrow x) \Rightarrow \text{dist}(y, F(x')) \rightarrow 0\} \subset F(x), \quad x \in X$$

and $F^{(k+1)}(x) = (F^{(k)})^{(1)}(x)$. Also, a well-known result of Brown [14] states that for a convex-valued map $F : X \rightarrow R^n$, the nonemptiness of all $F^{(n)}(x), x \in X$, implies that $F^{(n)} : X \rightarrow R^n$ is a LSC selection of F . Hence the standard selection techniques can be applied to $F^{(n)}$. Rather simple examples show the essentiality of the finite-dimensionality of the range space.

5.4. Based on the ingenious idea of Michael who proposed in [71] the notion of a *paraconvex set*, the authors in [84, 86, 87, 89, 99, 100] systematically studied another approach to weakening (or, controlled omission) of convexity. Roughly speaking, to each closed subset $P \subset B$ of a Banach space one associates a numerical function, say $\alpha_P : (0, +\infty) \rightarrow [0, 2)$, the so-called function of non-convexity of P . The identity $\alpha_P \equiv 0$ is equivalent to the convexity of P and the more α_P differs from zero the “less convex” the set P is.

Definition 5.12 The function of non-convexity $\alpha_P(\cdot)$ of the set P associates to each number $r > 0$ the supremum of the set

$$\{\sup\{dist(q, P)/r : q \in conv(P \cap D_r)\}\}$$

over all open balls D_r of the radius r .

For a function $\alpha : (0; +\infty) \rightarrow (0; +\infty)$ a nonempty closed subset P of a Banach space is said to be α -paraconvex provided that function $\alpha(\cdot)$ pointwisely majorates the function of non-convexity $\alpha_P(\cdot)$. Then P is said to be paraconvex provided that $\sup \alpha_P(\cdot) < 1$.

In [89] we proved a paraconvex version of the Ky Fan-Sion minimax theorem.

Theorem 5.13 [89] *Let $\alpha : (0, \infty) \rightarrow (0, 1)$ be a function with the right upper limits less than 1 over the closed ray $[0, \infty)$. Let $f : X \times Y \rightarrow \mathbb{R}$ be a real valued function on Cartesian product of two AR-subcompacta X and Y of a Banach spaces and suppose that:*

- (1) *For each $c \in \mathbb{R}$ and each $x_0 \in X$ the set $\{y \in Y : f(x_0, y) \leq c\}$ is α -paraconvex compact; and*
- (2) *For each $d \in \mathbb{R}$ and each $y_0 \in Y$ the set $\{x \in X : f(x, y_0) \geq d\}$ is α -paraconvex compact for a fixed $\alpha : (0, \infty) \rightarrow [0, 1)$. Then*

$$\max_X(\min_Y f(x, y)) = \min_Y(\max_X f(x, y)).$$

It is interesting to note that our minimax theorem includes cases when finite intersections

$$\bigcap_{i=1}^n \{x \in X : f(x, y_i) \geq c\} : y_i \in Y, \quad \bigcap_{j=1}^k \{y \in Y : f(x_j, y) \leq d\} : x_j \in X$$

of sublevel and uplevel sets are possibly nonconnected: intersection of two paraconvex sets can be nonconnected.

Usually a proof of a minimax theorem for generalized convexities exploits the *intersection* property of convex sets and reduces minimax theorem to some kind of KKM-principle. In our case we used the fact that the closure of *unions* of directly ordered family of arbitrary paraconvex sets are also paraconvex. Therefore as a base for obtaining minimax theorem we have used the selection theory of multivalued mappings instead of versions of the KKM-principle.

In [93] we examined the following natural question: *Does paraconvexity of a set with respect to the classical convexity structure coincide with convexity under some generalized convexity structure?* In other words, is paraconvexity a real non-convexity, or is it maybe a kind of some generalized convexity? It turns out that sometimes the answer is affirmative.

Theorem 5.14 [93] *Let $0 \leq \alpha < 0,5$ and $F : X \rightarrow B$ be a continuous multivalued mapping of a paracompact space X into a Banach space B such that all values $F(x)$ are bounded α -paraconvex sets. Then there exists a continuous singlevalued*

mapping $\mathfrak{F} : X \rightarrow C_b(B, B)$ such that for every $x \in X$ the mapping $\mathfrak{F}_x : B \rightarrow B$ is a continuous retraction of B onto the value $F(x)$ of F .

Here $C_b(B, B)$ denotes the Banach space of all continuous bounded mappings of a Banach space B into itself. The key point of the proof is that the set $URetr_P \subset C_b(B, B)$ of all uniform retractions onto an α -paraconvex set $P \subset B$ is a $\frac{\alpha}{1-\alpha}$ -paraconvex subset of $C_b(B, B)$.

As a corollary, by continuously choosing a retraction onto a paraconvex sets, we showed that if in addition all values $F(x)$, $x \in X$, are pairwise disjoint then the metric subspace $Y = \bigcup_{x \in X} F(x) \subset B$ admits a convex metric structure σ such that each value $F(x)$ is convex with respect to σ .

Finally, let us mentioned the result of Makala [63, Theorem 3.1] who obtained the selection theorem for LSC mappings $F : X \rightarrow Y$ from a collectionwise normal domains X such that each value $F(x)$ equals to Y or, is a compact paraconvex subsets of Y . The main difficulty here was that the class of such values in general, is not closed with respect to intersections with balls.

5.5. Yet another type of a controlled “non-convexity” which is in some sense intermediate between paraconvexity and Menger’s metric convexity goes back to Vial [113] and during the last decade was intensively studied in [5, 49, 50].

For every two points x and y of a normed space $(Y; \|\cdot\|)$ and for every $R \geq 0$, $5\|x - y\|$ denote by $D_R[x; y]$ the intersection of all closed R -balls containing x and y . Clearly, when $R \rightarrow +\infty$, such set $D_R[x; y]$ tends (with respect to the Hausdorff distance) to the usual segment $[x; y]$.

Definition 5.15 [113] A subset A of a normed space $(Y; \|\cdot\|)$ is said to be *weakly convex w.r.t* $R > 0$ if for every $x, y \in A$, with $0 < \|x - y\| < 2R$, there exists a point $z \in A \cap D_R[x; y]$ that differs from x and y .

In a Hilbert space H the metric projection P_A of an R -neighborhood of a weakly convex w.r.t. R set A is singlevalued. The set $\{(x; y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y \geq 0\}$ is $(\sqrt{2}/2-)$ paraconvex but is not weakly convex with respect to arbitrary $R > 0$. The set $\{x \in \mathbb{R}^n : \|x\| \geq R\}$ is not paraconvex and is weakly convex w.r.t R . However, sometimes weak convexity implies paraconvexity. For example, in a Hilbert space H , if $z \in H$ and $0 < r < R$, then every weakly convex (w.r.t. R) subset $A \subset D(z; r)$ is (r/R) -paraconvex [49].

Theorem 5.16 [50] *Let X be a paracompact space and $0 < \varepsilon < R$. Then for every continuous singlevalued ε -selection $f_\varepsilon : X \rightarrow H$ of a LSC mapping $F : X \rightarrow H$ with closed and weakly convex (w.r.t. R) values there exists a continuous singlevalued selection.*

Theorem 5.17 [50] *Let X be a paracompact subset of a topological vector space Z and a uniformly functional contractible subset of Z . Then for every $R > 0$, each Hausdorff uniformly continuous mapping $F : X \rightarrow H$ with closed and weakly convex (w.r.t. R) values admits a continuous singlevalued selection.*

Both contractibility and uniform continuity restrictions are essential as Examples 1–3 from [50] show.

5.6. To complete this section we mention one more nontrivial convexity structure. Namely, the so called *tropical (or, max-plus)* convexity in $(\mathbb{R} \cup \{-\infty\})^n$. It is very intensively studied area with many various applications in abstract convex analysis, algebraic geometry, combinatorics, phylogenetic analysis, etc. For a survey and details cf. [120].

Definition 5.18 For an ordered N -tuple $t = (t_j)$ of “numbers” $t_j \in [-\infty; 0]$ with $\max\{t_j\} = 0$ and for a points $A_1(x_1^1, x_2^1, \dots, x_n^1), \dots, A_N(x_1^N, x_2^N, \dots, x_n^N)$ from $(\mathbb{R} \cup \{-\infty\})^n$ their max-plus t -combination is defined as the point

$$\left(\max_j \{x_1^j + t_j\}, \max_j \{x_2^j + t_j\}, \dots, \max_j \{x_n^j + t_j\} \right).$$

A subset $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is said to be *max-plus convex* if it contains all max-plus t -combinations of all of its points.

Zarichnyi proved a selection theorem for max-plus convex-valued mappings.

Theorem 5.19 [122] *Let X and Y be compact metrizable spaces and $Y \subset \mathbb{R}^n$. Then every LSC mapping $F : X \rightarrow Y$ with max-plus convex values admits a continuous singlevalued selection.*

It is interesting to observe that the proof never uses any sequential procedure of approximation. Instead, Zarichnyi constructs a version of Milyutin surjection $M : Z \rightarrow X$ of a zero-dimensional compact space Z onto X and associating map $m : X \rightarrow I(Z)$ with values in *idempotent* probability measures. Next, exactly as in [94], the desired selection f of F is defined as the idempotent barycenter $f(x) = \int_{M^{-1}(x)} s(t) dm(x)$ for a suitable selection s of the composition $F \circ M$ (such a selection exists due to Theorem 1.3).

6 Multivalued Selections

The foundation for results of this section is the compact-valued selection Theorem 1.4. Historically there were various ways to prove this result or its variants: the original Michael’s approach [72] via pointwise closures of limit point sets of certain “tree” of 2^{-n} -singlevalued selections, Choban’s method of coverings [16] which axiomatized and transformed Michael’s construction into a maximally possible general form, approach based on the notion of (complete) *sieves* [18], and a proof via the 0-dimensional selection theorem [85].

In a series of papers Gutev recently proposed a more advanced point of view for sieves on a set X . Recall that a *tree* is a partially ordered set $(T; \leq)$ with all well-ordered sublevel sets $\{a : a \leq b, a \neq b\}_{b \in T}$. Roughly speaking, in [37] a

sieve on X is defined as some kind of multivalued mapping $(T; \preceq) \rightarrow X$ which is order-preserving with respect to inverse inclusion. In particular, as a corollary of his techniques, Gutev [37, Corollary 7.3] obtained the following generalization of compact-valued selection Theorem 1.4, which was proposed earlier in [1].

Theorem 6.1 [37] *A multivalued mapping $F : X \rightarrow Y$ admits a compact-valued USC selection $H : X \rightarrow Y$, which, in turn, admits a compact-valued LSC selection $G : X \rightarrow Y$, provided that the following conditions are satisfied:*

- (1) X is a paracompact space;
- (2) Y is a monotonically developable and sieve-complete space;
- (3) F is a LSC mapping; and
- (4) For every $x \in X$, $F(x)$ is a closed subset of Y .

Here, in comparison with compact-valued selection Theorem 1.4, only the restriction (2) is changed. Monotonically developable spaces are a natural generalization of Moore spaces. Note that Y is monotonically developable and sieve-complete space if and only if Y is the image of a completely metrizable non-Archimedean space under some open surjection [119]. If one omits in (2) the assumption that Y is a monotonically developable then by [37, Corollary 7.2] it is possible to guarantee only the existence an USC compact-valued selection $H : X \rightarrow Y$.

If we equip a paracompact domain X in Theorem 6.1 by a sequence $\{X_n\}$ of its finite-dimensional subspaces $\dim X_n \leq n$, then by [36, Corollary 7.7], we always obtain an USC compact-valued selection $H : X \rightarrow Y$ with $|H(x)| \leq n + 1$, $x \in X_n$. If we equip a paracompact domain X by a sequence $\{X_n\}$ of its finite-dimensional subspaces, $\dim X_n \leq n$, then we always obtain an USC compact-valued selection $H : X \rightarrow Y$ with $|H(x)| \leq n + 1$, $x \in X_n$.

As a rule, all selections in [37] are constructed as a composition of two suitable multivalued mappings. The first one is related to completeness and the other one arises from a system (tree) of various coverings of the domain and their refinements.

Applying the same “trees-sieves” technique in [39] upper semicontinuity of a selection was replaced by closedness of its graph. Below are two typical examples.

Theorem 6.2 [39] *For a T_1 -space X the following statements are equivalent:*

- (1) X is normal;
- (2) If Y is a metrizable space and $F : X \rightarrow Y$ is a compact-valued LSC mapping then there are compact-valued mappings $G : X \rightarrow Y$ and $H : X \rightarrow Y$ such that $G(x) \subset H(x) \subset F(x)$, $x \in X$, G is LSC and the graph of H is a closed subset of $X \times Y$;
- (3) If Y is a metrizable space and $F : X \rightarrow Y$ is a compact-valued LSC mapping then there exists a compact-valued selection of F with a closed graph.

Theorem 6.3 [39] *For a T_1 -space X the following statements are equivalent:*

- (1) X is countably paracompact and normal;
- (2) Same as (2) in theorem above, but for closed-valued mappings into a separable range space.

Choban, Mihaylova and Nedev [19] collected various types of selection characterizations of classes of topological spaces formulated in terms of multivalued selections. Recall that the n th image of a set is defined inductively by setting $F^1(A) = F(A)$, $F^{n+1}(A) = F(F^{-1}(F^n(A)))$, and that the *largest image* is defined as the union of all n th images, $n \in \mathbb{N}$.

Theorem 6.4 [19] *For a T_1 -space X the following statements are equivalent:*

- (1) X is strongly paracompact (i.e. Hausdorff and each open cover admits a star-finite refinement);
- (2) For every LSC mapping $F : X \rightarrow Y$ into a discrete space Y there exists a discrete space Z , a singlevalued map $g : Z \rightarrow Y$, a LSC mapping $G : X \rightarrow Z$, and an USC finite-valued mapping $H : X \rightarrow Z$ such that $g(G(x)) \subset g(H(x)) \subset F(x)$, $x \in X$, and all sets $H^\infty(x)$ are countable;
- (3) Same as (2) but without LSC mapping G and without finiteness of the values $H(x)$;
- (4) Same as (2) but with a regular X , without USC mapping H , and with a countable $G^\infty(x)$.

For a space X , let $c\omega(X)$ denote the *cozero dimensional kernel* of X , i.e. the complement of the union of all open zero-dimensional subsets of X .

Theorem 6.5 [19] *For a T_1 -space X the following statements are equivalent:*

- (1) X is strongly paracompact and $c\omega(X)$ is Lindelöf;
- (2) See (2) in previous theorem with $Y = Z$ and $g = id|_Z$;
- (3) Same as (2) but without LSC mapping G and without finiteness of values $H(x)$;
- (4) Same as (2) but with a regular X , without USC mapping H and with a countable $G^\infty(x)$.

In the next theorem $l(X)$ denotes the Lindelöf number of the space X and singlevalued selections are not assumed to be continuous.

Theorem 6.6 [19] *For any regular space X and any cardinal number τ the following statements are equivalent:*

- (1) $l(X) \leq \tau$;
- (2) For every LSC closed-valued mapping $F : X \rightarrow Y$ into a complete metrizable space Y there exists a LSC closed-valued selection G of F such that $l(G(X)) \leq \tau$;
- (3) For every LSC mapping $F : X \rightarrow Y$ into a complete metrizable space Y there exists a singlevalued selection g of $Clos(F)$ such that $l(g(X)) \leq \tau$;
- (4) For every LSC mapping $F : X \rightarrow Y$ into a discrete space Y there exists a singlevalued selection g of $Clos(F)$ such that $|g(X)| \leq \tau$;
- (5) Every open cover of X admits a refinement of cardinality $\leq \tau$.

Similar characterization was obtained in [19] for the degree of compactness of a space. Gutev and Yamauchi in [46], using once again the “trees-sieves” technique,

presented a generalizations of results [19] to arbitrary complete metric range spaces. For example, in comparison with Theorem 6.4 they proved the following:

Theorem 6.7 [46] *For a T_1 -space X the following statements are equivalent:*

- (1) X is strongly paracompact;
- (2) For every LSC closed-valued mapping $F : X \rightarrow Y$ into a complete metric space $(Y; \rho)$ there exist a complete ultrametric space $(Z; d)$, a uniformly continuous map $g : Z \rightarrow Y$, and a USC compact-valued mapping $H : X \rightarrow Z$ such that $g(H(x)) \subset F(x)$, $x \in X$, and the set $H(H^{-1}(S))$ is totally ε -bounded whenever $\varepsilon > 0$ and $S \subset Z$ is totally ε -bounded;
- (3) For every LSC closed-valued mapping $F : X \rightarrow Y$ into a discrete space Y there exist a discrete space Z , a singlevalued map $g : Z \rightarrow Y$, and a USC compact-valued mapping $H : X \rightarrow Z$ such that $g(H(x)) \subset F(x)$, $x \in X$ and the set $H(H^{-1}(S))$ is finite whenever $S \subset Z$ is finite.

Similarly, [46, Corollaries 6.2 and 6.3] a space X is strongly paracompact and $c\omega(X)$ is Lindelöf (resp., compact) if and only if for every LSC closed-valued mapping $F : X \rightarrow Y$ into a completely metrizable space Y there exists a USC compact-valued selection H of F such that set $H(H^{-1}(S))$ is separable (resp., compact), whenever $S \subset Z$ is separable (resp., compact).

Yamauchi [116] gave a selection characterization of the class which unifies compact spaces and finite-dimensional paracompact spaces. A topological space is said to be *finitistic* (another term is *boundedly metacompact*) if any of its open covers admits an open refinement of finite order, or equivalently for paracompact spaces, if and only if it contains a compact subset K such that each closed subset of the complement of K is finite-dimensional. Below is a typical statement.

Theorem 6.8 [116] *For a T_1 -space X the following statements are equivalent:*

- (1) X is paracompact and finitistic;
- (2) Each LSC closed-valued mapping $F : X \rightarrow Y$ into a completely metrizable space Y admits a USC compact-valued selection $H : X \rightarrow Y$ of F with the property that for every open cover ν of Y there exists a natural number N such that every value $H(x)$, $x \in X$, can be covered by some $\nu_0 \subset \nu$ with $\text{Card}(\nu_0) \leq N$.

Finally, let us mentioned the survey paper by Choban [17] on reduction principles in selection theory. Briefly, he discussed questions concerning extensions of LSC mappings F with nonparacompact domains onto paracompact ones and a notion of (complete) metrizable of family $\{F(x)\}$ of values rather than (complete) metrizable of a range space.

7 Miscellaneous Results

7.1. Tymchatyn and Zarichnyi [109] applied the selection theorem of Fryszkowski [33] to decomposable-valued mappings $F : X \rightarrow L_1([0; 1], B)$ in order to construct

a continuous linear regular operator which extends partially defined pseudometrics to pseudometrics defined on the whole domain. Denote by $\mathcal{P}\mathcal{M}(X)$ the set of all continuous pseudometrics over metrizable compact X and $\mathcal{P}\mathcal{M}$ the subset of $\mathcal{P}\mathcal{M}(X)$ of all continuous pseudometrics ρ with compact domains $dom(\rho) \subset X$. Identifying a pseudometric with its graph we can consider both of these sets endowed with the topology induced from the compact exponent $exp(X \times X \times [0; \infty))$.

Theorem 7.1 [109] *There exists a continuous linear regular extension operator $u : \mathcal{P}\mathcal{M} \rightarrow \mathcal{P}\mathcal{M}(X)$, $u(\rho)|_{dom(\rho) \times dom(\rho)} \equiv \rho$.*

Here, regularity of an operator means that it preserves the *norm* of pseudometrics, i.e. their maximal values. As it typically arises for extensions, the answer is given by some *formula*. Namely, under some isometric embedding X into a separable Banach B , the desired operator u can be defined as

$$u(\rho)(x, y) = \int_0^1 \rho(f(dom(\rho), x)(t), f(dom(\rho), y)(t)) dt,$$

where $f : expX \times X \rightarrow L_1([0; 1], B)$ is a continuous singlevalued selection of the decomposable-valued LSC mapping $F : expX \times X \rightarrow L_1([0; 1], B)$ defined by

$$F(A, x) = L_1([0; 1], \{x\}) = \{x\}, x \in A; \quad F(A, x) = L_1([0; 1], A), x \notin A.$$

Metrizability of a compact space X is a strongly essential assumption [109, Theorem 6.1].

Theorem 7.2 [109] *For a compact Hausdorff space X the following statements are equivalent:*

- (1) X is metrizable;
- (2) There exists a continuous extension operator $u : \mathcal{P}\mathcal{M} \rightarrow \mathcal{P}\mathcal{M}(X)$.

7.2. Gutev and Valov [45] applied selection theory to obtain a new proof of Prokhorov’s theorem and its generalization outside the class of Polish spaces. Recall that a probability measure μ on a $T_{3,5}$ -space X is a countably additive mapping $\mu : \mathcal{B}(X) \rightarrow [0, 1]$ with $\mu(X) = 1$ and with regularity (or, the Radon) property that

$$\mu(B) = \sup\{\mu(K) : K \subset B, K \text{ is compact}\}$$

for every Borel set $B \in \mathcal{B}(X)$. Roughly speaking, values of measure are realized over subcompacta with any precision.

The set $P(X)$ of all probability measures can naturally be considered as the subset of the conjugate space $C^*(X)$ of the Banach space $C(X)$ and is endowed with the induced topology. Thus Prokhorov’s theorem states that for a Polish space X , the Radon property holds not for a unique measure but for an arbitrary compact set of measures. Namely, for a compact $M \subset P(X)$ and for any $\varepsilon > 0$, there exists a

compact $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$, for all $\mu \in M$. For more general domains the following result holds [45]:

Theorem 7.3 [45] *For a sieve-complete space X , a paracompact space $S \subset P(X)$ and any $\varepsilon > 0$ there exists a USC compact-valued mapping $H : S \rightarrow P(X)$ such that $\mu(X \setminus H(\mu)) < \varepsilon$, for every $\mu \in S$.*

Note that for a paracompact space X , its sieve-completeness coincides with its Čech-completeness. Returning to the original case of Polish space X , the outline of the proof looks as follows. First, for each $\mu \in P(X)$ the set $G_\varepsilon(\mu) = \{K - \text{compact} : \mu(X \setminus K) < 0, 5\varepsilon\} \subset P(X)$ is nonempty simply due to the Radon property. Denote by $\text{exp}(X)$ the completely metrizable space of all subcompacta of X endowed with the Vietoris topology, or with Hausdorff distance metric. It turns out that the multivalued mapping $G_\varepsilon : P(X) \rightarrow \text{exp}(X)$ is a LSC mapping. Hence its pointwise closure F_ε is also LSC and moreover, $\mu(X \setminus K) \leq 0, 5\varepsilon$, for every $\mu \in P(X)$ and every $K \in F_\varepsilon(\mu)$. By the compact-valued selection Theorem 1.4, the mapping F_ε admits an USC compact-valued selection, say $H : P(X) \rightarrow \text{exp}(X)$. Finally, the union $\cup\{K' : K' \in H(\mu)\}$ yields the desired compact subset $K \subset X$.

7.3. Zippin [123] considered the convex-valued selection Theorem 1.1 as the base for resolving the extension problem for operators from a linear subspaces E of c_0 into the spaces $C(K)$, where K is a compact Hausdorff space. For any $\varepsilon > 0$, he considered the multivalued mapping $F : \text{Ball}(E^*) \rightarrow (1 + \varepsilon)\text{Ball}(c_0^*)$ by setting $F(e^*)$ equal to $\{0\}$ if $e^* = 0$ and

$$F(e^*) = \{x^* \in (1 + \varepsilon)\text{Ball}(c_0^*) : x^* \text{ extends } e^* \text{ and } \|x^*\| < (1 + \varepsilon)\|e^*\| \}$$

otherwise. Under the weak-star topology, all values of F are convex metrizable compacta. After (a nontrivial) verification of lower semicontinuity of F and applying Theorem 1.4, one finds a singlevalued weak-star continuous mapping $f : \text{Ball}(E^*) \rightarrow (1 + \varepsilon)\text{Ball}(c_0^*)$ such that $f(e^*)$ extends e^* and $\|f(e^*)\| \leq (1 + \varepsilon)\|e^*\|$.

Hence for an operator $T : E \rightarrow C(K)$ with the norm $\|T\| = 1$, let $f_T : K \rightarrow \text{Ball}(E^*)$ be defined by $f_T(k)(e) = T(e)(k)$, for $k \in K$. Then f_T is weak-star continuous and hence the composition $f \circ f_T : K \rightarrow (1 + \varepsilon)\text{Ball}(c_0^*)$ with the above selection f is also continuous. Defining $\widehat{T} : c_0 \rightarrow C(K)$ by $\widehat{T}(x)(k) = (f \circ f_T)(k)(x)$, $x \in c_0$, we see that \widehat{T} extends T because $f(e^*)$ extends e^* , \widehat{T} is linear and well-defined, since $(f \circ f_T)(k)$ is a linear functional and $\|\widehat{T}\| = \sup\{\|\widehat{T}(x)(k) : \|x\| \leq 1, k \in K\} \leq \sup\{\|(f \circ f_T)(k)\| \|x\| : \|x\| \leq 1, k \in K\} \leq 1 + \varepsilon$.

7.4. For two multivalued mappings $F_1 : X \rightarrow Y_1, F_2 : X \rightarrow Y_2$ and a singlevalued mapping $L : Y_1 \times Y_2 \rightarrow Y$ denote by $L(F_1; F_2)$ the composite mapping, which associates to each $x \in X$ the set

$$\{y \in Y : y = L(y_1; y_2), y_1 \in F_1(x), y_2 \in F_2(x)\}.$$

Definition 7.4 Let f be a selection of the composite mapping $L(F_1; F_2)$. Then pair (f_1, f_2) is said to be a *splitting* of f if f_1 is a selection of F_1 , f_2 is a selection of F_2 and $f = L(f_1; f_2)$.

Therefore the *splitting problem* [90] for the triple (F_1, F_2, L) is the problem of finding continuous selections f_1 and f_2 which split a continuous selection f of the composite mapping $L(F_1; F_2)$. As a special case of a constant mappings $F_1 \equiv A \subset Y, F_2 \equiv B \subset Y$ and $L(x_1, x_2) = x_1 + x_2$ we have the following:

Problem 7.5 [90] Let A and B be closed convex subsets of a Banach space Y and $C = A + B$ their Minkowski sum. Is it possible to find continuous singlevalued mappings $a : C \rightarrow A$ and $b : C \rightarrow B$ such that $c = a(c) + b(c)$ for all $c \in C$?

The answer is positive for strictly convex and finite-dimensional A and B (cf. [92, Corollary 3.6]), and for finite-dimensional A and B with $C = A + B$ being of a special kind, the so-called P -set (cf. [6, Theorem 2.6]). A collection of various examples and affirmative results on approximative splittings, uniformly continuous (or Lipschitz) splittings can be found in [4, 6, 7]. Note that under the replacement of the sum-operator $L(x_1, x_2) = x_1 + x_2$ by an arbitrary linear operator L , the problem has a negative solution even in a rather low-dimensional situation [92, Example 3.2].

Theorem 7.6 [92] For any 2-dimensional cell D there exist:

- (a) Constant multivalued mappings $F_1 : D \rightarrow R^3$ and $F_2 : D \rightarrow R$ with convex compact values;
- (b) A linear surjection $L : R^3 \oplus R \rightarrow R^2$; and
- (c) A continuous selection f of the composite mapping $F = L(F_1, F_2)$, such that $f \neq L(f_1, f_2)$ for any continuous selections f_i of $F_i, i=1,2$.

The construction uses the convex hull C of one full rotation of the spiral $K = \{(\cos t, \sin t, t) : 0 \leq t \leq 2\pi\}$ and the fact that its projection onto the xy -plane admits no continuous singlevalued selections.

7.5. As for the finite-dimensional selection Theorem 1.5, during the discussed period three voluminous papers were devoted to its versions, generalizations or applications. In [44] Gutev and Valov proved the following result on the density of selections.

Theorem 7.7 [44] Let for a mapping $F : X \rightarrow Y$ all assumptions of Theorem 1.5 be true. Let in addition $\Psi : X \rightarrow Y$ be a mapping with an F_σ -graph such that for each $x \in X$ the intersection $F(x) \cap \Psi(x)$ is a σZ_{n+1} -subset of the value $F(x)$. Then in the set $Sel(F)$ of all continuous singlevalued selections of F endowed with the fine topology the subset of those selections of F which pointwisely avoid values $\Psi(x)$ constitutes a dense G_δ -subspace.

Recall that for a metric range space $(Y; d)$ the *fine* topology in $C(X, Y)$ is defined by its local base

$$V(f, \varepsilon(\cdot)) = \{g \in C(X, Y) : d(g(x), f(x)) < \varepsilon(x), x \in X\},$$

when $\varepsilon(\cdot)$ runs over the set of all positive continuous functions on X . Also, for metric space $(B; \rho)$, a subset $A \subset B$ is said to be a σZ_{n+1} -subset if it is the union of countably many sets $A_i \subset B$ such that each continuous mapping from the $(n + 1)$ -cell to B can be approximated (with respect to the uniform topology) by a sequence of continuous mappings to $B \setminus A_i$. The sequential process of proving this theorem is based on the following result.

Theorem 7.8 [44] *Let all assumptions of Theorem 7.7 be true with exception that Ψ is a closed-graph mapping. Let f be a continuous singlevalued selection of F , $\varepsilon(\cdot)$ a positive continuous function on X and ρ a compatible metric on Y . Then F admits a continuous singlevalued selection g such that $g(x) \notin \Psi(x)$ and $\rho(g(x), f(x)) < \varepsilon(x)$, for every $x \in X$.*

Next, recall that Shchepin and Brodsky [101] proved that for any paracompact space X with $\dim X \leq n + 1$, a completely metrizable space Y , and for any L -filtration $\{F_i\}_{i=0}^{n+1}$ of maps $F_i : X \rightarrow Y$, the ending mapping F_{n+1} admits a continuous singlevalued selection. One of the points in the definition of L -filtration (cf. [88]), is the property that graph-values $\{x\} \times F_i(x)$ are closed in some prescribed G_δ -subset of the product $X \times Y$. In [36, Corollary 7.10] Gutev proved a generalization of this result to the case when the graph of mapping F_{n+1} is a G_δ -subset of $X \times Y$. Roughly speaking, Gutev proposed his own version of the Shchepin-Brodsky L -filtrations technique. In particular, he generalized the previous theorem [36, Corollary 7.12].

Theorem 7.9 [36] *Theorem 7.7 is true under the change of the assumption “the family $\{F(x)\}_{x \in X}$ is ELC^n ” with the restriction that F be an ELC^n -mapping, i.e. $\{x\} \times \{F(x)\}_{x \in X}$ is ELC^n in the product $X \times Y$.*

As an example of a result on improvement of near-selections in [36, Corollary 7.15] we quote the following:

Theorem 7.10 [36] *Let X be a countably paracompact and normal space, $(Y; d)$ a complete metric space and $F : X \rightarrow Y$ a Hausdorff continuous closed-valued mapping all of whose values $F(x)$ are uniformly LC^n -subsets of Y . Then for every positive lower semicontinuous numerical function $\varepsilon(\cdot)$ on X there exists a positive lower semicontinuous numerical function $\delta(\cdot)$ on X with the following property:*

If $g : X \rightarrow Y$ is a continuous singlevalued δ -selection of F then there exists a continuous singlevalued selection f of F such that $d(g(x), f(x)) < \varepsilon(x)$, $x \in X$.

Moreover, if all values $F(x)$ are n -connected then F admits a continuous single-valued selection.

During preparation of the previous survey [88], Brodsky, Shchepin and Chigodize announced results on problem of local triviality of Serre fibrations with two-dimensional fibers. Their paper [13] appeared in 2008.

Theorem 7.11 *Let $p : E \rightarrow B$ be a Serre fibration of a locally connected compact space E onto a compact ANR-space B . Let all fibers $p^{-1}(x)$ be homeomorphic to a fixed two-dimensional manifold M which differs from the sphere and the projective*

plane. Then p admits a continuous section (i.e. $p^{-1} : B \rightarrow E$ admits a continuous singlevalued selection) provided that one of the following restrictions holds:

- (1) $\pi_1(M)$ is Abelian and $H^2(B; \pi_1(M)) = 0$;
- (2) $\pi_1(M)$ is nonabelian, M is not the Klein bottle and $\pi_1(B) = 0$;
- (3) M is the Klein bottle and $\pi_1(B) = \pi_2(B) = 0$.

The proof follows the strategy of [101] and is based on selection results for L -filtrations which, roughly speaking, are realized by the interplay between L -theory and U -theory of multivalued mappings (cf. [88]).

7.6. Various results exist in continuous selection theory with some additional restriction of algebraic nature. For example, a multivalued mapping $F : X \rightarrow Y$ between (as a rule, locally convex topological) vector spaces is called *additive* if $F(x_1 + x_2) = F(x_1) + F(x_2)$, $x_1, x_2 \in X$, i.e. the image of the sum of two points coincides with the Minkowski sum of the images of these points. Next, $F : X \rightarrow Y$ is called *subadditive* (resp., *superadditive*, resp., *convex*) if $F(x_1 + x_2) \subset F(x_1) + F(x_2)$, (resp., if $F(x_1 + x_2) \supset F(x_1) + F(x_2)$, resp., if $F(tx_1 + (1 - t)x_2) \subset tF(x_1) + (1 - t)F(x_2)$, $t \in [0, 1]$). The existence of linear selections was proved for several such types of mappings with compact convex values in locally convex spaces. In particular, additive mappings always have linear selections [34], and every superadditive mapping possesses a linear selection [108]. Recently, Protasov [82] obtained criteria on X and Y for the affirmative answer on existence of linear selections for arbitrary subadditive mappings.

- Theorem 7.12** (1) Any subadditive mapping $F : X \rightarrow Y$ with compact convex values has a continuous linear selection if and only if $\dim X = 1$ or $\dim Y = 1$.
- (2) Any convex mapping $F : X \rightarrow Y$ with compact convex values has a continuous affine selection if and only if $\dim Y = 1$.

Moreover, in “only if” parts of (1) and (2) one can omit the continuity restriction: if $\dim Y \geq 2$ then there exists a convex mapping $F : X \rightarrow Y$ without affine selections. Applications in Lipschitz stability problem for linear operators in Banach spaces are presented in [82] as well.

7.7. We end our survey by some selected results from metric projection theory. For more detailed information cf. [15, 29].

Let us recall that the *operator of almost best approximation*, or ε -*projection*, of a real Banach space $(X, \|\cdot\|_X)$ to a set $M \subset X$ is defined as the multivalued map

$$x \mapsto P_{M,\varepsilon}(x) = \{z \in M : \|z - x\|_X \leq \rho(x, M) + \varepsilon\},$$

where $\rho(x, M) = \inf_{y \in M} \|x - y\|_X$ is the distance from x to M . If $\varepsilon = 0$, then $P_M = P_{M,0}$ is the metric projection operator. Clearly, all sets $P_{M,\varepsilon}(x)$ are nonempty, whereas the equality $P_M(x) = \emptyset$ is in general, possible.

If $\|z - x\|_X \leq \rho(x, M) + \varepsilon$ is replaced by $\|z - x\|_X \leq (1 + \varepsilon)\rho(x, M)$ then such multivalued mapping is called a *multiplicative ε -projection onto M* .

Recall the Konyagin theorem [56] which states that the metric ε -projection operator admits a continuous singlevalued selection in the case $X = C[0; 1]$ with the standard norm and where $M = R = \left\{ \frac{f}{g} : f \in U, g \in W \right\}$ is the set of all generalized fractions with U and W linear closed subspaces of X . Note that this result is not true for $X = L_p[0; 1]$.

Ryutin [95, 96] considered the set

$$R_{U,W} = \text{Clos} \left\{ \frac{f}{g} : f \in U, g \in W \text{ and } g > 0 \right\}$$

of generalized fractions in the space $X = C(K)$, where K is connected metric compact, or in the the space $X = L_1[0; 1]$, and with finite-dimensional U and W .

Theorem 7.13 [96] *Let the intersection of the set $R_{U,W}$ with the closed unit ball $D \subset C(K)$ be compact. Then for every $\varepsilon > 0$ the multiplicative ε -projection of D onto $R_{U,W}$ admits a uniformly continuous singlevalued selection.*

In the space $X = L_1[0; 1]$ the situation is more complicated. Namely, in [95] a wide class of pairs (U, W) of finite-dimensional subspaces in $L_1[0; 1]$ were found with the property that there exists $\varepsilon_0 = \varepsilon_0(U, W) > 0$ such that the multiplicative ε -projection onto $R_{U,W}$ admits a singlevalued continuous selection only if $\varepsilon \geq \varepsilon_0$.

Livshits [60, 61] considered $X = C[0; 1]$ with the standard norm and continuous selections of the metric ε -projection operator onto the set of all *splines* (i.e. piecewise polynomials) with non-fixed nodes. Namely, for a fixed $n, d \in \mathbb{N}$, denote by $S_n^d[0, 1]$ the set of all functions $f \in C[0; 1]$ such that for some (depending on f) nodes $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$, each restriction $f|_{[x_{k-1}, x_k]}$ is a polynomial of degree $\leq d$.

Theorem 7.14 [60, 61]

- (1) *A continuous singlevalued selection of the metric projection onto the set $S_n^1[0, 1]$ exists if and only if $n \leq 2$;*
- (2) *For any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists a continuous singlevalued selection of the metric ε -projection onto the set $S_n^1[0, 1]$;*
- (3) *For any $n > 1$ and any $d > 1$ there exists $\varepsilon = \varepsilon(n, d)$ such that there is no continuous singlevalued selection of the metric ε -projection onto the set $S_n^d[0, 1]$.*

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