CHAPTER 16

Continuous Selections of Multivalued Mappings

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RECENT PROGRESS IN GENERAL TOPOLOGY II
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In this paper we have collected a selection of recent results of theory of continuous selections of multivalued mappings. We have also considered some important applications of these results to other areas of mathematics. The first three parts of the paper are devoted to convex-valued mappings, to selectors on hyperspaces, and to links between selection theory for LSC mappings and approximation theory for USC mappings, respectively. The fourth part includes various other results.

Since our recent book REPOVŠ and SEMENOV [1998a] comprehensively covers most important work in this area approximately until the mid 1990's, we have therefore decided to focus in this survey on results which have appeared since then. As is often the case with surveys, due to the limitations of space, one has to make a selection. Therefore we apologize to all those authors whose results could not be included in this paper.

1. Solution of Michael's problem for *C*-domains

A singlevalued mapping $f : X \to Y$ between sets is said to be a *selection* of a given multivalued mapping $F : X \to Y$ if $f(x) \in F(x)$, for each $x \in X$. Note that by the Axiom of Choice selections always exist. We shall be working in the category of *topological* spaces and *continuous* singlevalued mappings. There exist many selection theorems in this category. However, the citation index of one of them is by an order of magnitude higher than for any other. This is the Michael selection theorem for convex-valued mappings:

1.1. THEOREM (MICHAEL [1956a]). *A multivalued mapping $F : X \to Y$ admits a continuous singlevalued selection, provided that the following conditions are satisfied:*

(1) *X is a paracompact space;*

(2) *Y is a Banach space;*

(3) *F is a lower semicontinuous* (*LSC*) *mapping;*

(4) *For every $x \in X$, $F(x)$ is a nonempty convex subset of Y; and*

(5) *For every $x \in X$, $F(x)$ is a closed subset of Y.*

A natural question arises concerning the necessity (essentiality) of any of the conditions (1)-(5). Here is a summary of known results:

Ad 1. With fixed conditions (2)-(5), condition (1) turned out to be necessary. This is a characterization of paracompactness in MICHAEL [1956a].

Ad 2. With fixed conditions (1), (3)-(5), condition (2) can easily be weakened to the following condition:

(2') Y is a Fréchet space.

However, the question about the necessity of condition (2') is in general still open. In many special cases (which cover the most important situations), the problem of complete metrizability of the space Y in which the images lie has already been solved in the affirmative.

MÄGERL [1978] has provided an affirmative answer in the case when Y is a compact subset of a topological linear space E, by proving that Y must be metrizable if every closed- and convex-valued LSC mapping from a paracompact domain X to Y admits a continuous singlevalued selection. Moreover, it suffices to take for the domain X only zero-dimensional compact spaces (in the sense of the Lebesgue covering dimension $\dim$).

Nedev and Valov have shown that in Mägerl's theorem it suffices to require instead of a singlevalued continuous selection that there exists a multivalued USC selection. They also proved that Y must be completely metrizable if Y is a normal space (see NEDEV and VALOV [1984]).

VAN MILL, PELANT and POL [1996] have proved, without the convexity condition (4), that a metrizable range Y must be completely metrizable if for every 0-dimensional domain X, each closed-valued LSC mapping $F : X \to Y$ admits a singlevalued continuous selection.

Ad 3. Recall, that lower semicontinuity of a multivalued mapping $F : X \to Y$ between topological spaces X and Y means that for each $x \in X$ and $y \in F(x)$, and each open neighborhood $U(y)$, there exists an open neighborhood $V(x)$ such that $F(x') \cap U(y) \neq \emptyset$, whenever $x' \in V(x)$. Applying the Axiom of Choice to the family of nonempty intersections $F(x') \cap U(y)$, $x' \in V(x)$, we see that LSC mappings are exactly those, which admit local (noncontinuous) selections. In other words, the notion of lower semicontinuity is by definition very close to the notion of a selection.

Clearly, one can consider a mapping F which has an LSC selection G and then apply Theorem 1.1 to the mapping $\overline{conv}\ G \subset F$. For a metric space X, one of the the largest classes of such mappings was introduced by GUTEV [1993] under the name *quasi lower semicontinuous maps* (for more details see §3 of Part B in REPOVŠ and SEMENOV [1998a]).

Ad 4. This is essentially the only nontopological and nonmetric condition in (1)–(5). For $\dim X = n + 1 < \infty$ and Y completely metrizable it is possible (by MICHAEL [1956b]) to weaken the convexity restriction to the following purely topological condition:

(4') $F(x) \in C^n$ and $\{F(x)\}_{x \in X} \in ELC^n$.

In the infinite-dimensional case, it follows from the work PIXLEY [1974] and MICHAEL [1992] that there does not exist any purely topological analogue of condition (4) which would be sufficient for a selection theorem for an arbitrary paracompact domain.

In REPOVŠ and SEMENOV [1995, 1998b, 1998c, 1999] various possibilities were investigated to avoid convexity in metric terms. We exploited Michael's idea of paraconvexity in MICHAEL [1959a]. To every closed nonempty subset P of the Banach space B, a numerical function $\alpha_P : (0, \infty) \to [0, \infty)$ was associated. The identity $\alpha_P = 0$ is equivalent to convexity of P. Then all main selection theorems for convex-valued mappings remain valid if one replaces the condition $\alpha_{F(x)} = 0$ with the condition of the type $\alpha_{F(x)} < 1$, uniformly for all $x \in X$.

Ad 5. In general, one cannot entirely omit the condition of closedness of values of $F(x)$. However, if it is strongly needed then it can be done. For example, by MICHAEL [1989], in the finite-dimensional selection theorem, the closedness of $F(x)$ in Y can be replaced by the closedness of all $\{x\} \times F(x)$ in some G_δ-subset of the product $X \times Y$. Or, by MICHAEL [1956a], if X is perfectly normal and Y is separable, then it suffices to assume in Theorem 1.1 that the convex set $F(x)$ contains all interior (in the convex sense) points of its closure.

Around 1970 Michael and Choban independently showed that one can drop the closedness of $F(x)$ on any countable subset of the domain (for more details see Part B in RE-

POVŠ and SEMENOV [1998a]). Michael proposed the following way of uniform omission of closedness:

1.2. PROPOSITION (MICHAEL [1990]). *Let Y be any completely metrizable subset of a Banach space B, with the following property:*

(*) $K \subset C \subset Y \Longrightarrow \overline{conv}\ K \subset C$, *where K is a compactum and C is convex and closed (in Y).*

Then every LSC mapping $F : X \to Y$ defined on a paracompact space X with closed (in Y) and convex images has a continuous selection.

□ The compact-valued selection theorem guarantees, due to complete metrizability of Y, the existence of a compact-valued LSC selection $H : X \to Y$ of the mapping F. It remains to apply Theorem 1.1 to the multivalued selection $\overline{conv}\ H$ of the given mapping F. □

By the Aleksandrov theorem, such an Y must be a G_δ-subset of B. Property (*) is satisfied by any intersection of a countable number of open convex sets: it suffices to consider the corresponding Minkowski functionals.

However, there exist convex G_δ-sets which are not intersection of any countable number of open convex sets. For example, in the compactum $P[0,1]$ of all probability measures on the segment $[0,1]$ such is the convex complement of any absolutely continuous measure.

Hence at present, one of the central problems of selection theory is the following problem No. 396 from VAN MILL and REED [1990]:

1.3. PROBLEM (MICHAEL [1990]). Let Y be a G_δ-subset of a Banach space B. Does then every LSC mapping $F : X \to Y$ of a paracompact space X with convex closed values in Y have a continuous selection?

GUTEV [1994] proved that the answer is affirmative when X is a countably dimensional metric spaces or a strongly countably dimensional paracompact space. Problem 1.3 has recently been answered in the affirmative for domains having the so-called C-property:

1.4. THEOREM (GUTEV and VALOV [2002]). *The answer to Problem 1.3 above is affirmative for C-spaces X.*

□ We present an adaptation of the original Gutev-Valov argument. Of the C-property we shall need only the part of a theorem of USPENSKII [1998], to the effect that every mapping of such an X into a Banach space with open graph and aspherical values has a selection.

Hence let A_n be closed subsets of a Banach space B and

$$F : X \to Y = B \setminus (\bigcup_{n=1}^{\infty} A_n)$$

a convex-valued LSC mapping with values that are closed in Y. Let $\Phi(x) = Cl_B(F(x))$, $x \in X$. Apply Theorem 1.1 to the mapping $\Phi : X \to B$. Let $\mathbb{S}_\Phi$ be the set of all selections of Φ, endowed with the topology defined by the following local basis (fine topology):

$$\mathbb{O}(f, \varepsilon(\cdot)) = \{g : \|f(x) - g(x)\| < \varepsilon(x)\},$$

where $\varepsilon : X \to (0,\infty)$ runs through the set of all continuous mappings.

It is well-known that the space $C(X,B)$ of all singlevalued continuous mappings from X to B, endowed with such fine topology is a Baire space. Moreover, it contains the uniform topology. Clearly, $\mathbb{S}_\Phi$ is a uniformly closed subset of $C(X,B)$. Hence $\mathbb{S}_\Phi$ is also a Baire space.

With each closed $A_n \subset B$ one can naturally associate the set of selections, which avoid the set A_n. Namely, let $\mathbb{A}_n = \{f \in \mathbb{S}_\Phi \ : \ f(x) \notin A_n, \text{ for all } x \in X\}$. If $f \in \bigcap_n \mathbb{A}_n$ then $f : X \to B \setminus (\ \bigcup_n A_n) = Y$, i.e for every $x \in X$,

$$f(x) \in Y \cap \Phi(x) = Y \cap Cl_B(F(x)) = F(x),$$

because $F(x)$ is closed in Y.

It remains to verify that for every $n \in \mathbb{N}$, the families $\mathbb{A}_n$ of functions are open, nonempty and dense in $\mathbb{S}_\Phi$, and then apply the Baire property of $\mathbb{S}_\Phi$. Since we are dealing with a unique A_n, it is possible to simply delete the index n.

That $\mathbb{A} = \{f \in \mathbb{S}_\Phi \ : \ f(x) \notin A, \text{ for all } x \in X\}$ is open in $\mathbb{S}_\Phi$ for a closed $A \subset B$, is clear: if $f(X) \subset B \setminus A$, then for $\varepsilon(x) = \frac{1}{2}\,\mathrm{dist}(f(x), A)$, the inclusion $g \in \mathbb{O}(f, \varepsilon(\cdot)) \cap \mathbb{S}_\Phi$ implies $g(X) \subset B \setminus A$.

So far all proofs have been a repetition of the argument from MICHAEL [1988]. Formally speaking, that $\mathbb{A}$ is nonempty follows from density of $\mathbb{A}$ in $\mathbb{S}_\Phi$. However, we shall proceed in reverse order since it is more convenient to begin with the nonemptiness of $\mathbb{A}$.

Let us define a mapping $[\Phi < A] : X \to B$ as follows:

$$[\Phi < A]\,(x) = \{y \in B \ : \ y \text{ is closer to } \Phi(x) \text{ than to } A\}.$$

Clearly $F(x) \subset [\Phi < A](x)$. Hence our new mapping assumes nonempty values. Since the set A is closed and the mapping Φ is LSC, it follows that the graph $Gr[\Phi < A]$ is open.

Let us prove asphericity of each set $[\Phi < A]\ (x),\ x \in X$. To this end we first deform $[\Phi < A]\ (x)$ into $\Phi(x) \setminus A$, and then we check the asphericity of the latter difference. For $y \in [\Phi < A]\ (x)$ we choose $r(y) > 0$ such that the closed ball $\overline{D}(y, r(y))$ intersects $\Phi(x)$ but does not intersect A. A simple selection (or separation in Dowker's spirit) arguments show that one can assume that $r(\cdot)$ is continuous. We apply Theorem 1.1 to the mapping $y \mapsto \Phi(x) \cap \overline{D}(y, r(y))$, i.e. we pick one of its continuous selections, say $z(\cdot)$.

It is geometrically evident that the entire segment $[z(y), y]$ lies in $[\Phi < A]\ (x)$ and it thus simply linear homotopy deforms $[\Phi < A]\ (x)$ into $\Phi(x) \setminus A$ (see Fig. 1).

Let us now verify that $\Phi(x) \cap A$ is a Z-subset of $\Phi(x)$ with respect to finite-dimensional domains. To this end, let us consider any mapping $\gamma : K \to \Phi(x)$ of a finite-dimensional K and for any $\delta > 0$ we associate to it the multivalued mapping $\Gamma : K \to Y = B \setminus (\ \bigcup_n A_n)$ given by:

$$\Gamma(k) = Cl_Y(F(x) \cap D(\gamma(k), \delta)).$$

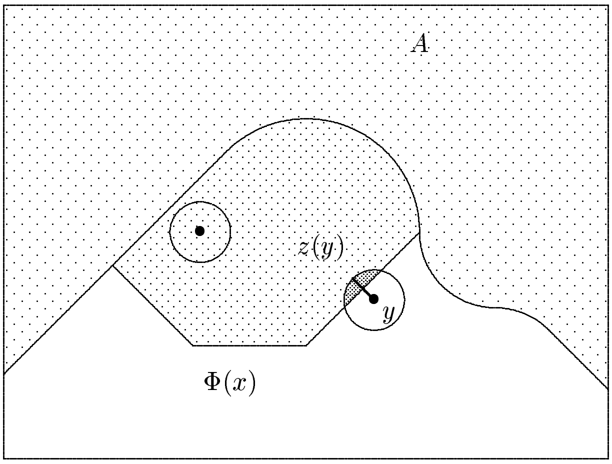

Figure 1

The finite-dimensional selection theorem applies to mapping Γ, due to complete metrizability of Y, convexity of $F(x)$ and continuity of γ. Hence the resulting selection γ' is δ-close to γ and avoids A. The asphericity of the difference $\Phi(x) \setminus A$ now follows by a standard argument (see USPENSKII [1998]).

Thus we can apply the Uspenskii selection theorem to the mapping $[\Phi < A] : X \to Y$ defined on the C-space X. Let g be a selection of $[\Phi < A]$. We repeat the previous proof, choosing closed balls $\overline{D}(g(x), r(x))$ intersecting $\Phi(x)$ but avoiding the set A, such that $r(\cdot) : X \to (0, \infty)$ is a continuous mapping. Then a selection (one more application of Theorem 1.1) of the mapping $x \mapsto \Phi \cap \overline{D}(g(x), r(x))$ is the desired selection of the mapping Φ, avoiding A. Therefore we have proved the nonemptiness of the set $\mathbb{A} \subset C(X, B)$.

In order to prove the density of $\mathbb{A} \subset \mathbb{S}_\Phi$ we pick $\phi \in \mathbb{S}_\Phi$ and a continuous mapping $\varepsilon : X \to (0, \infty)$. Then one can repeat the above argument on nonemptiness of the set of selections avoiding $\mathbb{A}$ for the mapping

$$\Psi(x) = \Phi(x) \cap \overline{D}(\varphi(x), \frac{\varepsilon(x)}{2}).$$

In other words, there is an element in $\mathbb{A}$ which is ε-close to φ. This proves the density of $\mathbb{A}$ in $\mathbb{S}_\Phi$. □

For the sake of completeness we reproduce here the complete statement of the result of Gutev and Valov.

1.5. THEOREM (GUTEV and VALOV [2002]). *For any paracompact space X the following conditions are equivalent:*

(a) *X is a C-space;*

(b) *Let Y be a Banach space and $F : X \to Y$ an LSC mapping with closed convex values. Then, for every sequence of closed-valued mappings $\Psi_n : X \to Y$ such that each Ψ_n has a closed graph and $\Psi_n(x) \cap F(x)$ is a Z_∞-set in $F(x)$ for every*

$x \in X$ and $n \in \mathbb{N}$, there exists a singlevalued continuous mapping $f : X \to Y$ with $f(x) \in F(x) \setminus \bigcup\{\Psi_n(x) : n \in \mathbb{N}\}$, for each $x \in X$; and

(c) *Let Y be a Banach space and $F : X \to Y$ be an LSC mapping with closed convex values. Then, for every closed $A \subset Y$ there exists a singlevalued continuous selection for F avoiding A, provided that $A \cap F(x)$ is a Z_∞-set in $F(x)$, for each $x \in X$.*

At present it is reasonable to expect that an affirmative solution of Problem 1.3 would yield a characterization of C-property of the domain X.

1.6. PROBLEM. Are the conditions $(a) - (c)$ from Theorem 1.5 equivalent to the following condition:

(d) Let Y be any G_δ-subset of a Banach space and $F : X \to Y$ an LSC mapping with convex values which are closed in Y. Then F admits a singlevalued continuous selection.

As a continuation of the above technique, Valov has given a selectional characterization of paracompact spaces, having the so-called *finite* C-property. To introduce it we do not use the original definition given by BORST [200?], but its characterization via the C-property. Namely, a paracompact space X has finite C-property if there exists a C-subcompact $K \subset X$ such that $\dim A < \infty$, for each closed $A \subset X \setminus K$.

1.7. THEOREM (VALOV [2002]). *For any paracompact space X the following conditions are equivalent:*

(a) *X has finite C-property;*

(b) *For any space Y and any infinite aspherical filtration $\{F_n : X \to Y\}_{n=1}^{\infty}$ of strongly LSC mappings there exists $m \in \mathbb{N}$ such that F_m admits a singlevalued continuous selection; and*

(c) *For any space Y and any infinite aspherical filtration $\{F_n : X \to Y\}_{n=1}^{\infty}$ of open-graph mappings there exists $m \in \mathbb{N}$ such that F_m admits a singlevalued continuous selection.*

Here, *strong lower semicontinuity* of a mapping $F : X \to Y$ means that the set $\{x \in X : K \subset F(x)\}$ is open for each subcompactum $K \subset X$. As for a filtration of mappings, we have that for each $x \in X$ and for each natural $n \in \mathbb{N}$,

$$F_1(x) \subset F_2(x) \subset F_3(x) \subset \ldots$$

and the inclusion $F_n(x) \subset F_{n+1}(x)$ is homotopically trivial up to dimension n (compare with SHCHEPIN and BRODSKY [1996]).

The coincidence of the class of spaces having finite C-property with the class of weakly infinitely dimensional spaces (in the sense of Smirnov) is a necessary condition for the affirmative solution of one of the main problems of infinite dimensional theory: Does every weakly infinite-dimensional compact metric space have the C-property?

2. Selectors for hyperspaces

2.A. Given a Hausdorff space X and the family $\mathcal{F}(X)$ of all nonempty closed subsets of X, we say that a singlevalued mapping $s : \mathcal{F}(X) \to X$ is a *selector* on $\mathcal{F}(X)$, provided that $s(A) \in A$, for every $A \in \mathcal{F}(X)$. From the formal point of view, a selector is simply a selection of the multivalued evaluation mapping, which associates to each $A \in \mathcal{F}(X)$ the same A, but as a subset of X. However, historically the situation was converse. Fifty years ago, in his fundamental paper MICHAEL [1951] proposed a splitting of the problem about existence of a selection $g : Y \to 2^X$ into two separate problems: first, to check that g is continuous and second, to prove that there exists a selector on 2^X. Hence, the selection problem was originally reduced to a certain selector problem.

Subsequently, the situation has stabilized to the present state. Namely, selectors are a special case of selections, but with an important exception: as a rule, no general selection theorem can be directly applied for resolving a specific problem on selectors. Specific tasks require specific techniques. Well-known papers ENGELKING, HEATH and MICHAEL [1968], CHOBAN [1970], and NADLER and WARD [1970] illustrate the point.

From early 1970's to mid 1990's the best result on continuous selectors was due to VAN MILL and WATTEL [1981], who characterized the orderable Hausdorff compacta as the compacta having a continuous selector for the family of at most two-points subsets (hence it was the extension of the similar result for the class of continua MICHAEL [1951]).

In the last five years the interest in theory of selectors has sharply increased – perhaps the monograph of BEER [1993] was one of the reasons. Over thirty papers have been published or are currently in print. We have chosen the results of HATTORI and NOGURA [1995] and VAN MILL, PELANT and POL [1996] as the starting point of this part of the survey.

2.B. For a subset $\mathbb{S} \subset \mathcal{F}(X)$ a *selector* is a mapping $s : \mathbb{S} \to X$ which selects a point $s(A) \in A$ for each $A \in \mathbb{S}$. Here, hyperspaces $\mathcal{F}(X)$ and their subsets are endowed with the *Vietoris* topology τ_V which is generated by all families of the type

$$\{A \in \mathcal{F}(X) : A \subset \bigcup_{i=1}^{n} V_i, \;\; A \cap V_i \neq \emptyset \;\},$$

over all finite collections of open subsets V_i of X. It is well-known that for metric spaces the Vietoris topology and the Hausdorff distance topology coincide if and only if the space X is compact.

By HUREWICZ [1928], for each metrizable space X the absence of a closed subspace of X homeomorphic to the rationals $\mathbb{Q}$ is equivalent to X being a *hereditarily Baire* space, i.e. every nonempty closed subspace of X is a Baire space. Due to the absence of continuous selectors for $\mathcal{F}(\mathbb{Q})$ (see ENGELKING, HEATH and MICHAEL [1968]), every metrizable space admitting a continuous selector is hereditarily Baire. This implication holds in the class of all regular spaces.

2.1. THEOREM (HATTORI and NOGURA [1995]). *Let X be a regular space having a continuous selector for $\mathcal{F}(X)$. Then X is a hereditarily Baire space.*

□ For a representation of a first Baire category space $X = \bigcup_{n=1}^{\infty} X_n$ as a union of closed nowhere dense subsets $X_n \subset X$ and for any continuous selector $s : \mathcal{F}(X) \to X$ it is possible to inductively construct a sequence of pairs $\{(A_n, F_n)\}_{n=1}^{\infty}$ such that for each n:

(0) A_n is a regular closed subset of X and F_n is a finite subset of the interior of A_n;

(1) $F_{n-1} \subset F_n \subset IntA_n \subset A_n \subset A_{n-1}$;

(2) $s(A) \notin X_n$, whenever $F_n \subset A \subset A_n$; and

(3) $s(A_n) \notin F_n$.

Having done such a construction, we see that $s(\bigcap_{n=1}^{\infty} A_n) \notin X_n$. This contradicts the fact that $X = \bigcup_{n=1}^{\infty} X_n$. □

Note that GUTEV, NEDEV, PELANT and VALOV [1992] proved (in a somewhat similar manner) that a metric space X is hereditarily Baire whenever every LSC mapping from the Cantor set to $\mathcal{F}(X)$ admits a USC compact-valued selection. Moreover, they showed that under such hypotheses either X is *scattered* (i.e. every closed subset has an isolated point) or X contains a homeomorphic copy of the Cantor set.

Thus we have the following facts for hyperspaces of the rationals:

2.2. THEOREM (ENGELKING, HEATH and MICHAEL [1968] HATTORI and NOGURA [1995]). *There exist no continuous selectors for $\mathcal{F}(\mathbb{Q})$, for the family of all closed nowhere dense subsets of $\mathbb{Q}$ or for the family of all clopen subsets of $\mathbb{Q}$. There exists a continuous selector on the family of subsets of $\mathbb{Q}$ of the form $C \cap \mathbb{Q}$, where C is connected subset of the real line.*

A natural question concerning existence of selectors for the family of all discrete closed subsets of $\mathbb{Q}$ arises immediately. A negative answer is a direct corollary of the following theorem in which $\mathcal{C}(M)$ denotes the family of all discrete closed subsets of a metric space M, which admits a representation as the value of some Cauchy sequence having no limit.

2.3. THEOREM (VAN MILL, PELANT and POL [1996]). *Let $\mathcal{C}(M)$ have a singlevalued continuous selector s. Then M is a completely metrizable space. Moreover, one can assume that every selector s is USC and finite-valued.*

Nogura and Shakhmatov investigated spaces with a "small" number of different continuous selectors. Recall that *orderability* of a topological space X means the existence of a linear order, say $<$, on X such that the family of intervals and rays with respect to $<$ constitutes a base for topology of X.

2.4. THEOREM (NOGURA and SHAKHMATOV [1997a]). *Let X be an infinite connected Hausdorff space. Then there are exactly two continuous selectors for $\mathcal{F}(X)$ if and only if X is compact and orderable.*

As a corollary, the topological (up to a homeomorphism) description of the interval is as follows: this is an infinite, separable, connected, Hausdorff space, admitting exactly two selectors for $\mathcal{F}(X)$. They also proved the following result:

2.5. THEOREM (NOGURA and SHAKHMATOV [1997b]). *A Hausdorff space X has finitely many selectors if and only if X has finitely many components of connectedness and there exists a compatible linear order $<$ on X such that every closed subset A of X has a minimal element with respect to $<$.*

Moreover, the total numbers of different selectors over all X in Theorem 2.5 constitutes a sufficiently scattered subsequence of the natural numbers $\mathbb{N}$. The first seven members are: $1, 2, 4, 24, 576, 720$ and 4096. This total number is a function of two natural parameters: the number n of all components of connectedness and the number m of all compact, nonsingleton components of connectedness (see NOGURA and SHAKHMATOV [1997b] for the precise formula).

2.C. There exist different topologies on the set $\mathcal{F}(X)$ whose restrictions on $X \subset \mathcal{F}(X)$ are compatible with the original topology of X. The Vietoris topology τ_V is only one of them. Hence different topologies on hyperspaces give different problems on selectors, continuous with respect to these topologies. Gutev was the first to systematically study this subject. For a metric space (X, d) the *d-proximal topology* $\tau_{\delta(d)}$ on $\mathcal{F}(X)$ is defined as the Vietoris topology τ_V but with the following additional "boundary" restriction

$$\{A \in \mathcal{F}(X) : A \subset \bigcup_{i=1}^{n} V_i,\ \ A \cap V_i \neq \emptyset,\ \ dist(A, X \setminus \bigcup_{i=1}^{n} V_i) > 0\ \},$$

over the all finite collections of open subsets V_i of X.

Thus $\tau_{\delta(d)} \subset \tau_V$ and it can easily be seen that $\tau_{\delta(d)} \subset \tau_{H(d)}$, where $\tau_{H(d)}$ is the topology on $\mathcal{F}(X)$ generated by the Hausdorff metric.

2.6. THEOREM (GUTEV [1996]). *For every complete nonarchimedean metric d on the space X there exists a $\tau_{\delta(d)}$-continuous selector for $\mathcal{F}(X)$.*

This theorem improves earlier results in ENGELKING, HEATH and MICHAEL [1968] and CHOBAN [1970] because each completely metrizable space X with $\dim X = 0$ admits a complete nonarchimedean metric d. The assumption in Theorem 2.6 that the metric d is nonarchimedean cannot be simply replaced by $\dim X = 0$. Namely, on the space $\mathbb{I}$ of irrationals there exists a complete compatible metric d such that $(\mathcal{F}(\mathbb{I}), \tau_{\delta(d)})$ has no continuous selectors – see COSTANTINI and GUTEV [200?].

Theorem 2.6 holds for separable spaces X for the so-called *d-ball proximal* topology $\tau_{\delta B(d)} \subset \tau_{\delta(d)}$, which is defined as topology $\tau_{\delta(d)}$ with the additional restriction that the union $\bigcup_{i=1}^{n} V_i$ can be represented as a union of a finite number of closed balls of (X, d) (for details see GUTEV [1996], GUTEV and NOGURA [2000]).

Bertacchi and Costantini unified separability of the domain with the nonarchimedean restriction on metric d.

2.7. THEOREM (BERTACCHI and COSTANTINI [1998]). *Let (X, d) be a separable complete metric space with a nonarchimedean metric d. Then the hyperspace $(\mathcal{F}(X), \tau_W(d))$ admits a selector if and only if it is totally disconnected.*

Here, $\tau_{W(d)}$ in Theorem 2.7 stands for the *Wijsman* topology which is the weakest topology on $\mathcal{F}(X)$ with continuous distance functions $dist(x, \cdot) : \mathcal{F}(X) \to \mathbb{R},\ x \in X$. Note, that $\tau_{W(d)} \subset \tau_{\delta B(d)}$.

To end this list of the most important recent hyperspace topology results, recall that the *Fell* topology τ_F has the base:

$$\{A \in \mathcal{F}(X) : A \subset \bigcup_{i=1}^{n} V_i, \;\; A \cap V_i \neq \emptyset\}$$

over all finite collections of open subsets V_i of X with compact complement of the union $\cup V_i$.

Gutev and Nogura presented the most comprehensive up-to-date view of selectors for Vietoris-like topologies.

2.8. THEOREM (GUTEV and NOGURA [2000]). *Let X be a completely metrizable space which has a clopen $\mathbb{D}$-orderable base for some $\mathbb{D} \subset \mathcal{F}(X)$. Then $\mathcal{F}(X)$ admits a $\tau_{V(\mathbb{D})}$-continuous selector.*

Here, the base of the topology $\tau_{V(\mathbb{D})}$, which is called a $\mathbb{D}$-*modification* of the Vietoris topology τ_V, constitutes the base of the Vietoris topology neighborhoods:

$$\{A \in \mathcal{F}(X) : A \subset \bigcup_{i=1}^{n} V_i, \;\; A \cap V_i \neq \emptyset\},$$

with the additional property that the complement $X \setminus \cup V_i$ can be represented as a union of a finite number of elements of the family $\mathbb{D}$.

The following examples show, that $\tau_{V(\mathbb{D})}$-continuous selectors with respect to various $\mathbb{D} \subset \mathcal{F}(X)$ are continuous selectors with respect to Vietoris-like topologies $\tau_{\delta(d)}$, $\tau_{\delta B(d)}$, $\tau_{B(d)}$, $\tau_{W(d)}$, and τ_F. Below, the *d-clopeness* of $A \subset X$ means that $dist(A, X \setminus A) > 0$ and *d-strongly clopeness* of $A \subset X$ means the existence of a finite $F \subset A$ and a positive number $d(x) < dist(x, X \setminus A)$, for every $x \in F$, such that A is the union of closed balls of radii $d(x)$, centered at $x \in F$.

2.9. THEOREM (GUTEV and NOGURA [2000]). *Let (X, d) be a metric space.*

(a) *If $\mathbb{D}$ is a family of d-clopen subsets of (X, d) then $\tau_{V(\mathbb{D})} \subset \tau_{\delta(d)}$;*

(b) *If $\mathbb{D}$ is a family of d-clopen subsets of (X, d) which are finite unions of closed balls then $\tau_{V(\mathbb{D})} \subset \tau_{B(d)}$;*

(c) *If $\mathbb{D}$ is a family consisting of finite unions of closed balls then $\tau_{V(\mathbb{D})} \subset \tau_{B(d)}$;*

(d) *If $\mathbb{D}$ is a family of strongly d-clopen subsets of (X, d) then $\tau_{V(d)} \subset \tau_{W(d)}$;*

(e) *If $\mathbb{D}$ is a family of compact subsets of X then $\tau_{V(\mathbb{D})} \subset \tau_F$.*

The proof of Theorem 2.9 looks like a sophisticated modification (or "$\mathbb{D}$-modification") of the well-known method of coverings based on the existence of a suitable sieve (p, γ) on X.

Here, a pair (p, γ) is called a *sieve* on X if $p = \{(p_n, A_n\}$ is a countable spectrum of a discrete pairwise disjoint index sets A_n and surjections $p_n : A_{n+1} \to A_n$ and $\gamma = \{\gamma_n\}$ is a sequence of open coverings $\gamma_n = \{V_{\alpha,n} : \alpha \in A_n\}$ which are linked together with the property that $V_\alpha = \bigcup\{V_\beta : \beta \in p_n^{-1}(\alpha)\}, \; \alpha \in A_n$.

So the values of a $\tau_{V(\mathbb{D})}$-continuous selector on $\mathcal{F}(X)$ are constructed as the kernels of p-chains of some precise and $\mathbb{D}$-orderable sieve on X. A series of results on selectors follows from Theorems 2.8 and 2.9. For instance, as an improvement of Theorem 2.6, one can see that every completely metric space (X, d) admits a $\tau_{\delta(d)}$-continuous selector, whenever d-clopen subsets of X constitute a base of topology of X. An interesting approach was proposed for the Fell topology and ball proximal topology. Thus Theorems 2.10 and 2.11 below are a part of a result of Gutev and Nogura.

For the moment, let us say that a surjection $l : (X, d) \to (Y, p)$ between metric spaces is a *fiber-isometry* if $d(x, y) = p(l(x), l(y))$, for all x, y with different $l(x)$ and $l(y)$.

2.10. THEOREM (GUTEV and NOGURA [2000]). *Let Y be the hedgehog of the countable weight over a convergent sequence of reals and X a strongly zero-dimensional metrizable nonlocally compact space. Then there exists a surjection $l : X \to Y$ such that one can associate to each compatible metric p on Y a compatible metric d on X such that:*

(a) *l is a fiber-isometry with respect to d and p; and*

(b) *X has a clopen base of closed d-balls.*

In particular, the associated mapping $l_{\mathcal{F}} : \mathcal{F}(Y) \to \mathcal{F}(X)$ is continuous with respect to ball proximal topologies. By virtue of Theorem 2.10 and the absence of ball proximal-continuous selectors for hyperspaces of hedgehog Y, the following characterization theorem can be proved:

2.11. THEOREM (GUTEV and NOGURA [2000]). *For a strongly zero-dimensional metrizable space X the following conditions are equivalent:*

(a) *X is locally compact and separable; and*

(b) *There exists a τ_F-continuous selector on $\mathcal{F}(X)$.*

As a continuation of this result and in the spirit of the van Mill-Wattel theorem, GUTEV and NOGURA [2001, 200?a] have recently proved that a Hausdorff space X is topologically well-orderable if and only if $\mathcal{F}(X)$ admits a τ_F-continuous selector. ARTICO and MARCONI [2001] and GUTEV [2001] have generalized this characterization to $\mathcal{F}_2(X) = \{A \in \mathcal{F}(X) : |A| \leq 2\}$.

2.D. Let us return to the Vietoris topology on $\mathcal{F}(X)$. Another characterization of the van Mill-Wattel type was obtained by Fujii and Nogura. See also MIYAZAKI [2001a] for extending van Mill-Wattel result to the class of almost compact spaces.

2.12. THEOREM (FUJII and NOGURA [1999]). *Let X be a compact Hausdorff space. The following two conditions are equivalent:*

(a) *X is homeomorphic to an ordinal space; and*

(b) *There exists a continuous selector $s : \mathcal{F}(X) \to X$, whose values are isolated points of a closed subset of X.*

Artico, Marconi, Moresco and Pelant proposed one more selector description of certain topological properties. A topological space is said to be *nonarchimedean* if for some base of open sets for an arbitrary pair of nondisjoint members of the base one of the members is a subset of the other one. A nonarchimedean space is a P-space if and only if all of its countable subsets are closed (and hence discrete).

2.13. THEOREM (ARTICO, MARCONI, MORESCO and PELANT [2001]). *Let X be a nonarchimedean P-space. Then the following conditions are equivalent:*

(a) *X is scattered;*

(b) *X is topologically well-orderable space; and*

(c) *There exists a continuous selector on $\mathcal{F}(X)$.*

FUJII, MIYAZAKI and NOGURA [2002] have recently showed that any countable regular space X admits a continuous selector if and only if it is scattered.

Now, let $s : \mathcal{F}(X) \to X$ be a continuous selector and let $G \subset X$ be nonempty and clopen. One can define another selector, say s_G which associates to each A outside of G exactly $s(A)$ and to each A meeting G, the value $s(A \cap G)$. It easy to see that s_G is also continuous and that $s_G(X) \in G$ (see Fig. 2).

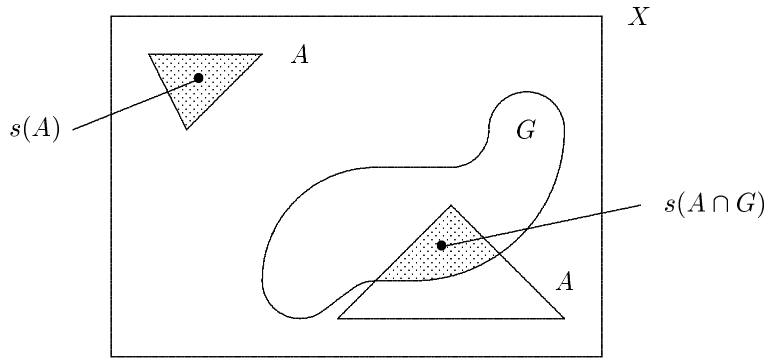

Figure 2

This observation allowed GUTEV and NOGURA [2001] to prove that if a zero-dimensional (in the ind sense) Hausdorff space X admits a continuous selector then the set $\{f(X) : f$ is a continuous selector$\}$ is dense in X. Conversely, if the set $\{f(X) : f$ is a continuous selector$\}$ is dense in X, then X is totally disconnected. A local "countable" version of such observation yields the following characterization:

2.14. THEOREM (GUTEV and NOGURA [2001]). *For any first countable Hausdorff space X admitting a continuous selector the following conditions are equivalent:*

(a) $\operatorname{ind} X = 0$*; and*

(b) *For each point $x \in X$ there exists a selector $f_x : \mathcal{F}(X) \to X$ which is "maximal" with respect to x, i.e. which selects the point x for every closed $A \subset X$, containing x.*

As a corollary, a locally compact Hausdorff space X admitting a continuous selector is zero-dimensional in the ind sense if and only if the set $\{f(X) : f$ is a continuous selector$\}$ is dense in X.

Subsequently, Garcia-Ferreira, Gutev, Nogura, Sanchis and Tomita examined more closely the notion of x-maximal and minimal selectors. Hence in the spirit of Michael theory they proved that for a Hausdorff space X, admitting a continuous selector every such selector is maximal with respect to some point if and only if X has at most two different continuous selectors on $\mathcal{F}(X)$:

2.15. THEOREM (GARCIA-FERREIRA ET AL. [200?]). *For any countable space X the following two conditions are equivalent:*

(a) *X is a scattered metrizable space; and*

(b) *For every point $x \in X$ there exists a continuous x-maximal selector on $\mathcal{F}(X)$.*

3. Relations between *U*- and *L*-theories

3.A. Everyone understands the notion of continuity of a singlevalued mapping between topological spaces in a unique sense. For a multivalued mapping the term "continuity" has a "multivalued" interpretation, because of many different topologies on the hyperspace $\mathcal{F}(X)$ which coincide on $X \subset \mathcal{F}(X)$ with the original topology. In this part we consider links between two most useful types of continuity - the upper semicontinuity and the lower semicontinuity of multivalued mappings. In both cases one can try to find some suitable relations between multivalued and singlevalued mappings.

In the first case (upper semicontinuity) the notion of *approximation* by singlevalued mappings arises naturally. For lower semicontinuity the notion of a *selection* is a starting point. Hence we shall for the moment talk about U*-theory* and about L*-theory*, respectively.

On the one hand, the main techniques and facts of U-theory and L-theory look very similar. For example, UV^n-properties of values $F(x)$ and $ELC^n\&C^n$-properties in the case $\dim X < \infty$ (or the nontopological restriction of convexity of values for an arbitrary domain, which occurs as a paracompact space in both theories).

Moreover, in both cases there are principal obstructions to purely topological passage from finite dimensional to the infinite dimensional cases - see examples of PIXLEY [1974], TAYLOR [1975] and DRANISHNIKOV [1993].

On the other hand, no theorems of U-theory follow directly from theorems of L-theory and vice versa. For example, clearly an ε-approximation f of F can be defined as a selection of a double ε-enlargement $F_\varepsilon : x \mapsto D_Y(F(D_X(x,\varepsilon)),\varepsilon)$. Many authors have observed this fact. But in general, there is no information about topological or convexity-like properties of values of such an enlargement. Hence the selection theory cannot be applied directly. In brief, we had two closed but "parallel" theories for multivalued mappings.

3.B. Shchepin and Brodsky have proposed a unified approach of simultaneously using U- and L-theories in order to find a new proof of the finite-dimensional Michael selection theorem together with its recent generalization in MICHAEL [1989]. The key ingredient of the their considerations is the notion of a *filtration* of a multivalued mapping. Two different kinds of filtrations were used.

For a topological spaces X and Y a finite sequence $\{F_i\}_{i=0}^n$ of a multivalued mappings $F_i : X \to Y$ is said to be an *L-filtration* if: (1) F_i is a selection of F_{i+1}, $0 \leq i < n$; (2) the identity inclusions $F_i(x) \subset F_{i+1}(x)$ are *i-apolyhedral* for all $x \in X$, i.e. for each polyhedron P with $\dim P \leq i$ every continuous mapping $g : P \to F_i(x)$ is null-homotopic in $F_{i+1}(x)$; (3) the families $\{\{x\} \times F_i(x)\}_{x \in X}$ are ELC^{i-1} families of subsets of the Cartesian product $X \times Y$, $0 \leq i \leq n$; (4) for every $0 \leq i \leq n$ there exists a G_δ-subset of $X \times Y$ such that $\{x\} \times F_i(x)$, $x \in X$, are closed in this G_δ-subset.

3.1. THEOREM (SHCHEPIN and BRODSKY [1996]). *Let X be a paracompact space with $\dim X \leq n$, Y a complete metric space and $\{F_i\}_{i=0}^n$ an L-filtration of maps $F_i : X \to Y$. Then the mapping $F_n : X \to Y$ admits a singlevalued continuous selection.*

It is easy to see that Theorem 3.1 applies to the constant filtration $F_i = F$ and hence we obtain the finite-dimensional selection theorem as a corollary of the filtered Theorem 3.1. Moreover, for a constant filtration $F_i = F$ with F having the properties (3) and (4) from the definition of L-filtration above, we obtain the generalization of the finite-dimensional selection theorem which was proposed in MICHAEL [1989]. Note, that almost the same filtered approach for open-graph mappings was earlier proposed by BIELAWSKI [1989], but with no detailed argumentation.

Next, we define the notion of a *U-filtration*. For topological spaces X and Y a finite sequence $\{H_i\}_{i=0}^n$ of compact-valued upper semicontinuous mappings is said to be a *U-filtration* if: (1') H_i is a selection of H_{i+1}, $0 < i < n$; (2') the identity inclusions $H_i(x) \subset H_{i+1}(x)$ are *UV^i-aspherical* for all $x \in X$, i.e. for every open $U \subset H_{i+1}(x)$ there exists a smaller open $V \subset H_i(x)$ such that every continuous mapping $g : S^i \to V$ is null-homotopic in U; where S^i is the standard i-dimensional sphere. Clearly, condition (2') in the definition of a U-filtration looks like an approximate version of condition (2) in the definition of an L-filtration.

We shall now formulate the notion of (graphic) approximations of multivalued mappings $F : X \to Y$. Let $\mathcal{V} = \{\mathcal{V}_\alpha\}_{\alpha \in \mathcal{A}}$ be a covering of X and $\mathcal{W} = \{W_\gamma\}_{\gamma \in \Gamma}$ a covering of Y. A singlevalued mapping $f\colon X \to Y$ is said to be a $(\mathcal{V} \times \mathcal{W})$-approximation of F if for every $x \in X$ there exist $\alpha \in A$ and $\gamma \in \Gamma$ and points $x' \in X$, $y' \in F(x')$ such that x and x' belong to V_α, $f(x)$ and y' belongs to W_γ. In other words, the graph of f lies in the neighborhood of the graph F with respect to the covering $\{V_\alpha \times W_\gamma\}_{\alpha \in A, \gamma \in \Gamma}$ of the Cartesian product $X \times Y$.

For metric spaces X and Y and for coverings $\mathcal{V}$ and $\mathcal{W}$ of X and Y by $\varepsilon/2$-open balls we obtain the more usual notion of (graphic) ε-approximations AUBIN and CELLINA [1984]. Namely, a singlevalued mapping $f_\varepsilon : X \to Y$ between metric spaces (X, ρ) and (Y, d) is said to be *ε-approximation* of a given multivalued mapping $F : X \to Y$ if for every $x \in X$ there exist points $x' \in X$ and $y' \in F(x')$ such that $\rho(x, x') < \varepsilon$ and $d(y', f_\varepsilon(x)) < \varepsilon$.

We recall the well-known Cellina approximation theorem (see AUBIN and CELLINA [1984]) which states that each convex-valued USC $F : X \to Y$ from a metric space (X, ρ) into a normed space $(Y, ||\cdot||)$ with convex values $F(x)$, $x \in X$, is *approximable*, i.e. that for every $\varepsilon > 0$ there exists a singlevalued continuous ε-approximation of F. The following theorem is a natural finite-dimensional version of Cellina's theorem:

3.2. THEOREM (SHCHEPIN and BRODSKY [1996]). *Let X be a paracompact space with $\dim X \leq n$, Y an ANE for the class of all paracompacta and $\{H_i\}_{i=0}^n$ a U-filtration of* *mappings $H_i : X \to Y$. Then for every open in $X \times Y$ neighborhood G of the graph $\Gamma(H_n)$ of the mapping H_n there exists a continuous singlevalued mapping $h : X \to Y$ such that the graph $\Gamma(h)$ lies in G.*

The following theorem gives an intimate relation between L-filtrations and U-filtrations. This theorem can also be regarded as a *filtered* analogue of the compact-valued selection theorem:

3.3. THEOREM (SHCHEPIN and BRODSKY [1996]). *Let X be a paracompact space, Y be a complete metric space and $\{F_i\}_{i=0}^n$ an L-filtration of maps $F_i : X \to Y$. Then there exists a U-filtration $\{H_i\}_{i=0}^n$ such that H_n is a selection of F_n. Moreover each H_i is a selection of F_i, $0 \le i \le n$ and the inclusions $H_i(x) \subset H_{i+1}(x)$ are UV^i-apolyhedral.*

We point out some discordance: in the definition of an L-filtration we talked about *apolyhedrality* and in the definition of a U-filtration about *asphericity*. In view of Theorem 3.3, the UV^i-asphericity of $H_i(x) \subset H_{i+1}(x)$ cannot be directly derived from the UV^i-asphericity of inclusions $F_i(x) \subset F_{i+1}(x)$, $x \in X$, of given L-filtration $\{F_i\}_{i=0}^n$. Moreover, there is a gap in the original proof of Theorem 3.1 — the authors in fact need an L-filtration $\{F_i\}_{i=0}^{n^2}$ of length n^2, not n. BRODSKY [2000] later partially filled this gap by considering *singular* filtrations (see below). Recently, BRODSKY, CHIGOGIDZE and KARASEV [2002] have completely solved the problem (see Theorem 4.22 below).

We now formulate the crucial technical ingredient of the whole procedure. The following theorem asserts the existence of another U-filtration $\{H_i'\}_{i=0}^n$ accompanying a given L-filtration $\{F_i\}_{i=0}^n$. Here, we drop the conclusions that $H_0(x) \subset F_0$(x), …, $H_{n-1}(x) \subset F_{n-1}(x)$ and add the property that the sizes of values $H_n(x)$ can be chosen to be less than arbitrary given $\varepsilon > 0$.

3.4. THEOREM (SHCHEPIN and BRODSKY [1996]). *Let X be a paracompact space with $\dim X \le n$, Y a Banach space and $\varepsilon > 0$. Then for every L-filtration $\{F_i\}_{i=0}^n$, every U-filtration $\{H_i\}_{i=0}^n$ with H_n being a selection of F_n and every open in $X \times Y$ neighborhood G of the graph $\Gamma(H_n)$ of the mapping H_n there exists another U-filtration $\{H_i'\}_{i=0}^n$ such that:*

(1) *H_n' is a selection of F_n;*

(2) *The graph $\Gamma(H_n')$ lies in G; and*

(3) diam $H_n'(x) < \varepsilon$*, for each $x \in X$.*

The proof of Theorem 3.4 is divided, roughly speaking into two steps. One can begin by the application of Theorem 3.2 to the given a U-filtration $\{H_i\}_{i=0}^n$ with $H_n \subset F_n$. Hence we obtain some singlevalued continuous mapping $h\colon X \to Y$ which is an approximation of H_n. Then we perform a "thickening" procedure with h, in order to obtain a new L-filtration $\{F_i'\}_{i=0}^n$ with small sizes of values $F_n'(x)$, $x \in X$. Such an L-filtration $\{F_i'\}_{i=0}^n$ naturally arises from the ELC^{n-1} properties of the values of the final mapping F_n of a given L-filtration $\{F_i\}_{i=0}^n$. Finally, we use the "filtered" compact-valued selection Theorem 3.3 exactly for the new L-filtration $\{F_i'\}_{i=0}^n$. The result of such an application gives the desired U-filtration $\{H_i'\}_{i=0}^n$ with small sizes of values $H_n'(x)$, $x \in X$. This is the strategy of the proof of Theorem 3.4. The inductive repetition for some series $\sum_k \varepsilon_k < \varepsilon$ involves in each fiber $F_n(x)$ a sequence of subcompacta $H_n^k(x)$ which turns out to be fundamental with respect to the Hausdorff metric. Its limit point gives the value $f(x)$ of the desired selection f of F.

3.C. For resolving some difficulties with the proof of Theorem 3.1 and in order to find new applications of the filtered approach, BRODSKY [2002, 200?] introduced the notion of a *singular* filtration of a multivalued mapping.

For two multivalued mappings $\Phi : X \to Y$ and $\Psi : X \to Z$ with the same base, their *fiberwise transformation* is defined as a singlevalued continuous mapping $\hat{f} : \Gamma_\Phi \to \Gamma_\Psi$ between their graphs such that $\hat{f}(\{x\} \times \Phi(x)) \subset \{x\} \times \Psi(x), x \in X$.

For a multivalued mapping $F : X \to Y$ its *singular* n-length filtration is defined as a triple $\mathbb{F} = \{\hat{f}_i, F_i, f_i\}$ where $F_i : X \to Y_i$ are multivalued mappings, $f_i : F_i \to F$ and $\hat{f}_i : F_i \to F_{i+1}$ are fiberwise transformations such that $f_i = f_{i+1} \circ \hat{f}_i$ (see Fig. 3).

A singular filtration $\mathbb{F}$ is said to be:

(1) *simple* if all fiberwise transformations are fiberwise inclusions;

(2) *complete* if all spaces Y_i are completely metrizable and all fibers $\{x\} \times F_i(x)$ are closed in some G_δ-subset of $X \times Y_i$;

(3) *contractible* if inclusions $\hat{f}_i(\{x\} \times F_i(x)) \subset \{x\} \times F_{i+1}(x), x \in X$ are homotopically trivial;

(4) *connected* if inclusions $\hat{f}_i(\{x\} \times F_i(x)) \subset \{x\} \times F_{i+1}(x), x \in X$ are i-aspherical for all i;

(5) *lower continuous* if all mappings F_i are LSC and family $\{\{x\} \times F_i(x)\}_{x \in X}$ is ELC^{i-1}; and

(6) *compact* if all mappings F_i all compact-valued and USC.

3.5. THEOREM (BRODSKY [2000]). *For each complete, connected and lower continuous n-length filtration $\mathbb{F} = \{\hat{f}_i, F_i, f_i\}$ of mappings from a metrizable space X with $\dim X \leq n$ into a completely metrizable space Y there exists a singlevalued continuous selections of F_n.*

Theorem 3.5 is based on the following theorem which, briefly speaking, reduces a singular filtration to some simple filtration with nice topological properties. Note, that in the following theorem n can be equal to infinity.

3.6. THEOREM (BRODSKY [2000]). *For each complete, connected and lower continuous n-length filtration $\mathbb{F} = \{\hat{f}_i, F_i, f_i\}$ of mappings from a metrizable space X with $\dim X \leq n$ into a completely metrizable space Y there exist compact, contractible, simple n-filtration $\mathbb{G} = \{\hat{g}_i, G_i, g_i\}$ and fiberwise transformation $\hat{h} : \mathbb{F} \to \mathbb{G}$.*

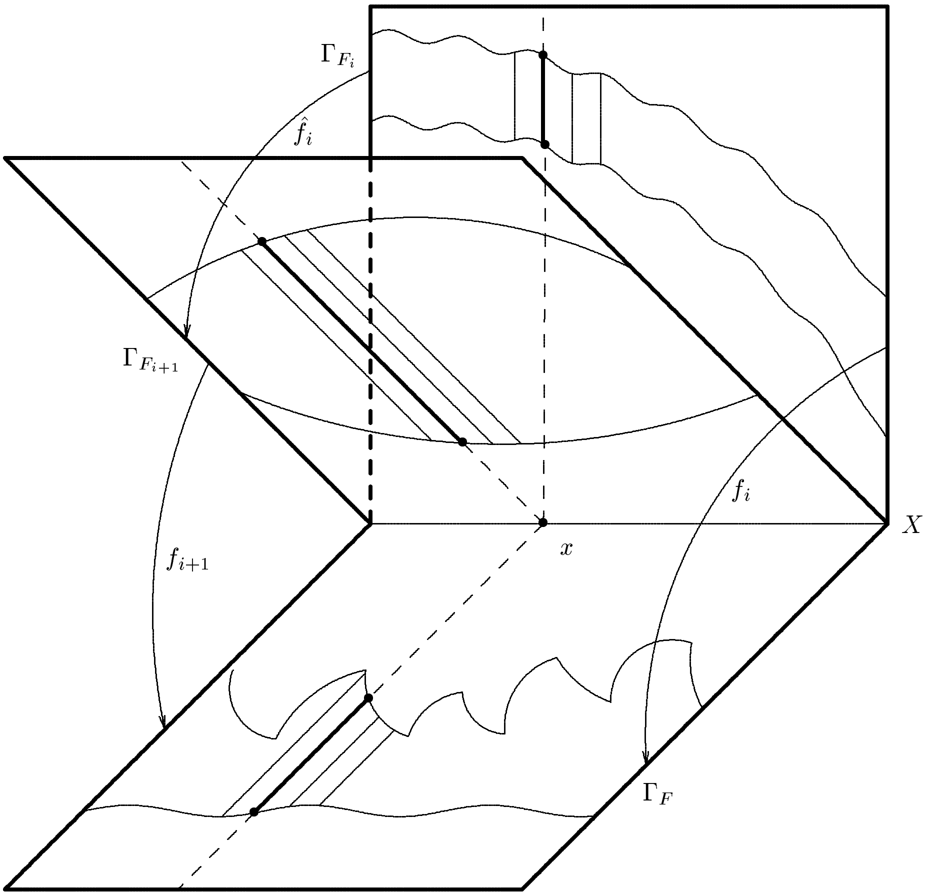

Figure 3

Clearly, Theorem 3.6 is a singular version of Theorem 3.3. Some words about its proof are in order. Metrizable domain X is the image of some zero-dimensional metric space Z under some perfect mapping $p : Z \to X$. The inductive procedure of extensions of $\hat{h}$ reduces to a selection problem for a suitable multivalued mapping from Z to a space of continuous singlevalued mappings from the graph of fiberwise join $\Gamma_{F*p^{-1}}$. The latter functional space is endowed by some asymmetric (not metric) and the analogue of standard zero-dimensional selection theorem done here "by hands", following known covering technique. In Theorem 3.6 metrizability of X is needed because of necessity of certain asymmetry in a suitable functional space.

We formulate two possibilities for applications which give the first known positive step towards a solution of the two-dimensional Serre fibration problem (see Problem 5.12 below).

3.7. THEOREM (BRODSKY [2002]). *Let $f : X \to Y$ be a mapping from a metrizable space onto an ANR metrizable space with all preimages homeomorphic to a fixed compact two-dimensional manifold. Then each partial section of f over closed subsets $A \subset X$ admits a local continuous extension, whenever:*

(1) *f is homotopically 0-regular; or*

(2) *f is a Serre fibration and X and Y are locally connected.*

The intermediate result between Theorems 3.5 and 3.7 states that a connected complete lower continuous 2-length singular filtration admits a continuous selection whenever it maps an ANR metric space X into a complete metric space Y and all values of the last member of the filtration are hereditary aspherical. Note, that any two-dimensional manifold is a locally hereditary aspherical space.

There are many applications of the filtered approach in the approximation theory which we shall omit here for the lack of space (see BRODSKY [1999, 2002]).

3.D. In Sections 1 and 2 above problems from L-theory were reduced to problems in U-theory. Here we consider a converse reduction which was proposed in REPOVŠ and SEMENOV [200?].

A family $\mathcal{L}$ of nonempty subsets of a topological space Y is said to be *selectable* with respect to a pair (X, A) if for each LSC mapping $F : X \to Y$ with values from $\mathcal{L}$ (i.e. $F(x) \in \mathcal{L}$ for every $x \in X$) and each selection $s : A \to Y$ of the restriction $F|_A$ there exists a selection $f : X \to Y$ of F which extends s (shortly, $\mathcal{L} \in \mathbb{S}(X, A)$).

For a positive r and for a family $\mathcal{L}$ of nonempty subsets of a metric space Y we denote by $\mathcal{L}_r$ the family of all subsets of Y which are r-close (with respect to the Hausdorff distance) to the elements of the family.

A family $\mathcal{L}$ of a nonempty subsets of a metric space Y is said to be *nearly selectable* with respect to a pair (X, A) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each LSC mapping $F : X \to Y$ with values from $\mathcal{L}_\delta$ and for each selection $s : A \to Y$ of the restriction $F|_A$ there exists an ε-selection $f : X \to Y$ of F which extends s. Shortly, $\mathcal{L} \in \mathbb{NS}(X, A)$. Below, we shall consider *only* hereditary families of sets.

3.8. THEOREM (REPOVŠ and SEMENOV [200?]). *Let $H : X \to Y$ be a USC mapping between metric spaces and A a closed subset of X, and let all values of H be in some family $\mathcal{L}$ which is nearly selectable with respect to the pair (X, A). Then for every covering ω of X, every $\varepsilon > 0$ and every selection $s : A \to Y$ of the restriction $H|_A$ there exists an $(\omega \times \varepsilon)$-approximation of H which extends s.*

□ For a given $\varepsilon > 0$ we choose $\varepsilon > \delta > 0$ with respect to the definition of near selectability of the family $\mathcal{L}$. Next we construct a new multivalued mapping $F : X \to Y$ such that:

(a) F is an LSC mapping and $H(x) \subset F(x)$, for every $x \in X$; and

(b) For every $x \in X$, there exists $z_x \in X$ such that $H(z_x) \subset F(x) \subset D(H(z_x), \delta)$.

In particular, $F(x) \in \mathcal{L}_\delta$. By paracompactness of the domain X, one can find a locally finite open star-refinement ν of the given covering of X. For each $x \in X$, let V_x be an arbitrary element of the covering ν such that $x \in V_x$. Now the covering of the domain arises naturally. Namely,

$$U_x = H_{-1}(D(H(x), \delta)) \bigcap V_x, \; x \in X.$$

We shall need the following lemma:

3.9. LEMMA (SEMENOV [2000b]). *For each positive τ and each open covering $\{U_x\}_{x\in X}$, $x \in U_x$, of a metric space X there exists a lower semicontinuous numerical function $l : X \to (0, \tau/2]$ with the following property: for every $x \in X$ there exists $z \in X$ such that $z \in D(x, l_x) \subset U_z$.*

We now apply this lemma to the covering $\{U_x\}_{x\in X}$ chosen above, for $\tau = \varepsilon$, and we define $F : X \to Y$, by setting $F(x) = H(D(x, l_x))$. □

A typical example of a nearly selectable family with respect to arbitrary paracompact domains and their closed subsets is the family of all nonempty convex subsets of a normed space. As a special case we have for the empty set A:

3.10. COROLLARY (REPOVŠ and SEMENOV [200?]). *Let $H : X \to Y$ be an USC mapping between metric spaces and let all values of H belong to some family $\mathcal{L}$ which is nearly selectable with respect to X. Then H is approximable.*

As a concrete application we have the following relative approximation fact:

3.11. COROLLARY (REPOVŠ and SEMENOV [200?]). *Let $H : X \to Y$ be a convex-valued USC mapping from a metric space X into a normed space Y and $A \subset X$ a closed subset. Let $\varepsilon > 0$ be given. Then:*

(1) *Each $(\varepsilon/4)$-selection $s : A \to Y$ of the restriction $H|_A$ can be extended to some ε-approximation $h : X \to Y$ of H; and*

(2) *There exists a continuous function $\delta : X \to (0, \infty)$ such that each $\delta(\cdot)$-approximation $\alpha : A \to Y$ of H can be extended to an ε-approximation $h : X \to Y$ of H.*

In the nonconvex situation the same technique was applied in SEMENOV [2000b] for resolving approximation problem for paraconvex-valued mappings. For a nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ and for an open ball $D \subset Y$ of radius r, one defines:

$$\delta(P, D) = \sup\{dist(q, P)/r \mid q \in conv(P \cap D)\},$$

and the value of its *function of nonconvexity* α_P at a point $r > 0$ is defined as $\alpha_P(r) = \sup\{\delta(P, D)\}$, where sup is taken over the set of all open balls of radius r. Next, a subset of a Banach space is said to be *α-paraconvex* if its function of nonconvexity majorates by the preassigned constant $\alpha \in [0, 1)$.

A direct calculation shows that for every $\alpha < 1$ and $R > 0$ the family $\mathcal{P}_{\alpha,R}$ of all α-paraconvex subsets P of a Banach space Y with $diam\, P < R$ is nearly selectable with respect to paracompact spaces. Moreover, one can check that in this case $\delta = \frac{\varepsilon}{12(1+\alpha)R}$ is a suitable answer for $\delta = \delta(\varepsilon)$ in the definitions above. Hence for any $\alpha \in [0, 1)$ and any USC mapping $F : X \to Y$ from a metric space to a normed space we see that F is approximable if all values $F(x)$, $x \in X$, are α-paraconvex in Y.

As for other unified U- and L- facts we conclude by the following result.

3.12. THEOREM (BEN-EL-MECHAIEKH and KRYZSZEWSKI [1997]). *Let $F : X \to Y$ be a LSC mapping and $H : X \to Y$ a USC mapping from a paracompact space X into a Banach space Y. Suppose that both mappings are convex-valued, F is closed-valued and $F(x) \cap H(x) \neq \emptyset, x \in X$. Then for every $\varepsilon > 0$ there exists a continuous singlevalued mapping $f : X \to Y$ which is a selection of F and ε-approximation of H.*

4. Miscellaneous results

4.A. Problem 1.3 deals with generalized ranges of multivalued mappings: substitution of a G_δ-subset instead of a Banach space. The following problem is related to generalized domains (as a variation, one can consider reflexive Banach spaces instead of Hilbert space).

4.1. PROBLEM (CHOBAN, GUTEV and NEDEV). Does every LSC closed- and convex-valued mapping from a collectionwise normal and countably paracompact domain into a Hilbert space admit a singlevalued continuous selection?

As a continuation of the result in NEDEV [1987], Choban and Nedev considered more complicated, in general *nonparacompact* domains of an LSC mappings. They extended a given LSC mapping to some paracompactification (Dieudonné completion) of an original domain and then applied Theorem 1.1. Recall that GO-spaces are precisely the subspaces of linearly ordered spaces. Their result is a step towards resolving (still open) Problem 4.1.

4.2. THEOREM (CHOBAN and NEDEV [1997]). *Every LSC closed- and convex-valued mapping $F : X \to Y$ from a generalized ordered space X to a reflexive Banach space Y has a singlevalued continuous selection.*

Shishkov obtained similar results for domains which are σ-products of metric spaces. Such a product of uncountably many copies of reals is collectionwise normal and countably paracompact but not pseudoparacompact.

4.3. THEOREM (SHISHKOV [2001]). *Every closed- and convex-valued LSC mapping of a σ-product of a metric spaces into a Hilbert space has a singlevalued continuous selection.*

Initially, Shishkov result dealt with separable metric spaces. He had earlier proved that the same selection result holds for any reflexive range and any collectionwise normal, countably paracompact and pseudoparacompact domain. Recently Shishkov has strengthened the Choban-Nedev theorem above because of paracompactness of the Dieudonné completition of GO-spaces.

4.4. THEOREM (SHISHKOV [2002]). *Each LSC closed- and convex-valued mapping of a normal and countably paracompact domain into a reflexive Banach space admits a LSC closed- and convex-valued extension over the Dieudonne completition of the domain.*

It is interesting that the property of the domain to be collectionwise normal and countably paracompact admits a characterization via multivalued selections. It turns out that for such purpose it suffices to consider in the assumption of the classical compact-valued Michael's selection theorem (see MICHAEL [1959c]) not only an LSC mapping, but such a mapping together with its a USC selection.

4.5. THEOREM (MIYAZAKI [2001b]). *For a T_1-space X the following conditions are equivalent:*

(a) *X is a collectionwise normal and countably paracompact space; and*

(b) *For every completely metrizable space Y, every LSC mapping $F : X \to Y$ with $F(x)$ either compact or $F(x) = Y$ for all $x \in X$ and every compact-valued USC selection $H : X \to Y$ of F there exist a compact-valued USC mapping $\Phi : X \to Y$ and a compact-valued LSC mapping $G : X \to Y$ such that $H(x) \subset G(x) \subset \Phi(x) \subset F(x), \quad x \in X$.*

See MIYAZAKI [2001b] for other conditions which are equivalent to $(a), (b)$. Inside the class of all normal spaces the collectionwise normality property has the following multivalued extension-type description.

4.6. THEOREM (MIYAZAKI [2001b]). *For a normal space X the following conditions are equivalent:*

(a) *X is collectionwise normal; and*

(b) *For every finite-dimensional completely metrizable space Y and every USC mapping $H : X \to Y$ with values consisting of at most n points there exist a compact-valued USC mapping $\Phi : X \to Y$ and a compact-valued LSC mapping $G : X \to Y$ such that $H(x) \subset G(x) \subset \Phi(x),\ x \in X$.*

Recall also the following generalization of the compact-valued Michael's selection theorem to the class of Čech-complete spaces.

4.7. THEOREM (CALBRIX and ALLECHE [1996]). *For each paracompact space X, each regular AF-complete space Y admitting a weak k-development and each closed-valued LSC mapping $F : X \to Y$ there exist a compact-valued USC mapping $\Phi : X \to Y$ and a compact-valued LSC mapping $G : X \to Y$ such that $G(x) \subset \Phi(x) \subset F(x), \quad x \in X$.*

Note that every AF-complete submetrizable space X has a weak k-development. A space is called *AF-complete* if it is Hausdorff and has a sequence of open coverings which is complete. The class of all Čech-complete spaces coincides with the class of all completely regular AF-complete spaces and completely metrizable spaces are exactly metrizable AF-complete spaces.

4.B. Künzi and Shapiro used Theorem 1.1 to prove the uniform version of the Dugundji extension theorem for partially defined mappings:

4.8. THEOREM (KÜNZI and SHAPIRO [1997]). *For each metrizable space X there exists a continuous mapping $E : C_{vc}(X) \to C_b(X)$ such that $E(f)|_{dom\ f} = f$ for all maps $f \in C_{vc}(X)$ and for every $K \in exp_c(X)$ the restriction $E|_{p^{-1}(K)}$ is a linear positive operator with $E(id_K) = id_X$.*

Here $C_{vc}(X)$ and $C_b(X)$ are sets of all continuous numerical mappings f with compact domain $dom\ f \subset X$ and all continuous bounded numerical mappings on the whole space X. Elements of $C_{vc}(X)$ are identified with their graphs and topology is induced by the Vietoris topology on $\mathcal{F}(X \times \mathbb{R})$, where $C_b(X)$ is endowed with the usual sup-norm topology. One can associate to each $f \in C_{vc}(X)$ its domain and obtain the projection p onto $exp_c(X)$ - the compact exponent of X.

A sketch of the proof goes as follows. For a Banach space B and every $K \in exp_c(B)$ one must consider the subset $R(K) \subset C_b(B, B)$ consisting of all $r : B \to B$ with

$r(B) \subset \overline{conv}(K)$ and $r|_K = id_K$. Clearly, $R(K)$ is a closed convex subset of $C_b(B, B)$ which is nonempty, due to the Dugundji theorem. It turns out that Theorem 1.1 is applicable to the mapping $R : exp_c(B) \to C_b(B, B)$. Hence the desired operator of simultaneous extension can be given by the formula

$$E(f)(x) = \int f d\mu(dom\ f)(x),$$

where X is embedded into the conjugate space of the Banach space $B = BL(X, d)$ of all bounded Lipshitz numerical mappings on the metric space (X, d).

Moreover, the above formula works for mappings not only to reals, but to Banach spaces and Cartesian products of Banach spaces. Metrizability of the domain X can be weakened to the restriction that X is one-to-one continuous preimage of a metric space. Note that the one-point-Lindelöfication of an uncountable discrete space admits such an operator E, although it is not a submetrizable space.

STEPANOVA [1993] had earlier characterized preimages of metric spaces under perfect mappings as spaces X for which a continuous mapping $E : C_{vc}(X) \to C_b(X)$ exists with $E(f)|_{dom\ f} = f$ and $\sup_{x \in dom\ f} |f(x)| = \sup_{x \in X} |E(f)(x)|$.

4.C. Filippov and Drozdovsky introduced new types of semicontinuity which unify lower and upper semicontinuity.

4.9. DEFINITION. A multivalued mapping $F : X \to Y$ is said to be *mixed semicontinuous* at a point $x \in X$ if for each open sets U and V with $F(x) \subset U$ and $F(x) \cap V \neq \emptyset$, respectively, there exists an open neighborhood W of x such that for every $x' \in W$ one of the following holds:

$$F(x') \subset U \qquad \text{or} \qquad F(x') \cap V \neq \emptyset.$$

They proved the following theorem concerning USC selections for mappings which are mixed semicontinuous at each point of domain:

4.10. THEOREM (FILIPPOV and DROZDOVSKY [1998, 2000]). *Let X be a hereditary normal paracompact space and Y a completely metrizable space. Then every compact-valued mixed semicontinuous mapping $F : X \to Y$ has a USC compact-valued selection.*

Considering the case $Y = \{0; 1\}$, it easy to see that the domain X must be hereditarily normal whenever Theorem 4.10 holds for each mixed semicontinuous mapping. Theorem 4.10 is useful in theory of differential equations with multivalued right-hand sides because of the well-known conditions in DAVY [1972] for the original mapping F imply the same conditions for USC selection of F. Hence, the inclusion $y' \in F(t, y)$ admits a solutions for a mixed semicontinuous right-hand side.

In their proof the authors used the idea of universality of the zero-dimensional selection theorem (see Part A of REPOVŠ and SEMENOV [1998a]): they considered the projection $\pi_X : A(X) \to X$ of the absolute of the domain over the domain. This is a perfect mapping and $A(X)$ is a paracompact space, because X is such. The hereditary normality of domain, extremal disconnectedness of the absolute and mixed continuity of F show that the mapping $G : A(X) \to Y$ defined by $G(x) = \liminf_{y \to x} F(\pi_X(y))$, is an LSC selection of the composition $F \circ \pi_X$ with nonempty closed values. By the compact-valued selection theorem we find an USC compact-valued selection H of G and finally $H \circ \pi_X^{-1}$ gives the desired selection of F.

A simple example of a mixed continuous mapping $F : X \to Y$ is given by the mapping $F(x) = \Psi(x), x \in A; F(x) = \Phi(x), x \in X \setminus A$ where $\Phi : X \to Y$ is compact-valued LSC mapping and $\Psi : A \to Y$ is its USC compact-valued selection over closed subset $A \subset X$. Hence, as a corollary of Theorem 4.7 we see that the selection Ψ admits an USC extension over whole X.

FRYSZKOWSKI and GORNIEWICZ [2000] introduced somewhat different type of mixed continuity. They considered mappings which are lower semicontinuous at some points of the domain and upper semicontinuous at all remaining points of the domain. General theorems on multivalued selections are proved together with various applications in theory of differential inclusions.

Finally, we mention here one more ”unified” selection result, which has recently been proved by Arutyunov. The following theorem looks like a mixture of theorems of Kuratowski–Ryll-Nardzewski and Michael-Pixley:

4.11. THEOREM (ARUTYUNOV [2001]). *Let $F : X \to Y$ be a measurable mapping from a metric space X endowed by a σ-additive, regular measure, $dim_X Z \leq 0$ and $A \subset X$ such that all values $F(x), x \in A$, are convex and F is LSC over $Z \cup Cl(A)$. Then F admits a singlevalued measurable selection which is continuous over $Z \cup A$.*

Moreover as usual, the set of all such selections is pointwise dense in values of multivalued mapping F. For applications in the optimal control theory see ARUTYUNOV [2000].

4.D. Cauchy problem for differential inclusions $x' \in F(t, x)$, $x(0) = 0$ was first reduced in ANTOSIEWICZ and CELLINA [1975] to a selection problem for some multivalued mapping $\hat{F} : K \to L_1(I, \mathbb{R}^n)$. Here I is a segment of reals, K is some suitable convex compactum of continuous functions $u : I \to \mathbb{R}^n$ and

$$\hat{F}(u) = \{v \in L_1(I, \mathbb{R}^n) | v(t) \in F(t, u(t)) \text{ a.e. in } I\}.$$

The mapping $\hat{F}$ is LSC whenever F is also such. But the values of $\hat{F}$ are in general nonconvex. They are decomposable subsets of $L_1(I, \mathbb{R}^n)$.

4.12. DEFINITION. A set Z of a measurable mappings from a measurable space $(T, \mathbb{A}, \mu)$ into a topological space E is said to be *decomposable* if for every $f, g \in Z$ and for every $A \in \mathbb{A}$, the mapping defined by $h(t) = f(t)$, when $t \in A$ and $h(t) = g(t)$, when $t \notin A$, belongs to Z.

The intersection of all decomposable sets, containing a given set S, is called the *decomposable hull* $Dec(S)$ of the set. For spaces of numerical functions on nonatomic domains, the decomposable hull of the two-point set is homeomorphic to the Hilbert space. Hence, it is a very unusual convexity-like property. Thus it is impossible to adapt the proof of the convex-valued selection theorem directly to decomposable-valued mappings. One of the reasons is a big difference between the mapping which associates to each set its convex hull (it is continuous in the Hausdorff metric on subsets), and the one which associates to each set its decomposable hull (it fails to be continuous). For example, the decomposable hull $Dec(D(f,a))$ of any ball $B(f,a)$ coincides with the entire space $L_1(T,E)$.

FRYSZKOWSKI [1983] (resp. BRESSAN and COLOMBO [1988]) proved selection theorems for a decomposable-valued LSC mappings with compact metric domains (resp. with separable metric domains). In both cases the so-called Lyapunov convexity theorem or its generalizations were used. The principal obstruction for a similar proof of the selection theorem for any paracompact domains is that the Lyapunov theorem fails for infinite number nonatomic real-valued measures.

In an attempt to return to the original idea of the Theorem 1.1, AGEEV and REPOVŠ [2000] introduced the notions of *dispersibly decomposable* sets and *dispersibly decomposable* hulls $Disp(A) \subset Dec(A)$. All decomposable sets are dispersibly decomposable sets and also (which is more important) all open and closed balls are dispersibly decomposable. Precisely the latter fact enables one to apply the usual techniques developed for the Michael convex-valued selection Theorem 1.1. Thus they proved the following selection theorem for the multivalued mappings with uniformly dispersed values (the so called *dispersible* multivalued mappings):

4.13. THEOREM (AGEEV and REPOVŠ [2000]). *Let $(T, \mathbb{A}, \mu)$ be a separable measurable space, E a Banach space, X a paracompact space and $L_1(T,E)$ the space of all Bochner integrable functions. Then each dispersible closed-valued mapping $F : X \to L_1(T,E)$ admits a continuous selection.*

The main technical step was the following lemma on a dividing of segment onto disjoint measurable subsets.

4.14. LEMMA. *For every $\sigma > 0$ and every point $s = (s_0, s_1, ..., s_n) \in \Delta^n$ of the standard n-dimensional simplex Δ^n there exists a partition $P = \{P_i\}_{i=0}^n$ of the interval I such that*

$$|m(P_i \cap J) - s_i \cdot m(J)| < \sigma,$$

for each $0 \leq i \leq n$ and each subinterval $J \subset I$.

The partitions from Lemma 4.14 are called σ-approximatively s-dispersible.

4.15. DEFINITION. A multivalued mapping $F : X \to L_1(I,E)$ is said to be *dispersible* if for each $x_0 \in X$, $\varepsilon > 0$, $s \in \Delta^n$ and each functions $u_0, u_1, \ldots, u_n \in F(x_0)$ there exist a neighborhood $V(x_0)$ of the point x_0 and a number $\sigma > 0$ such that for any σ-approximatively s-dispersible partition $P = \{P_i\}_{i=0}^n$, the function $\sum_{i=0}^{n} u_i \cdot \chi_{P_i}$ is contained in $D(F(x), \varepsilon)$, for every point $x \in V(x_0)$.

After checking that a LSC mapping F is dispersible whenever for each point $x \in X$ the value $F(x)$ is a decomposable set, one can obtain the generalization of the Fryszkowski, Bressan and Colombo theorems to arbitrary paracompact domains.

The following theorem substantially generalizes GONCHAROV and TOLSTONOGOV [1994] to paracompact domains:

4.16. THEOREM (AGEEV and REPOVŠ [2000]). *Let $(T, \mathbb{A}, \mu)$ be a separable measurable space and X be a paracompact space. Let $F : X \to L_1(T, E)$ be a dispersible closed-valued mapping and $\{ G_i : X \to L_1(T, E)\}_{i \in \mathbb{N}}$ be a sequence of dispersible multivalued mappings with open graphs such that $D(G_i(x); \varepsilon_i) \subset G_{i+1}(x)$, where the sequence $\{\varepsilon_i\}$ does not depend on $x \in X$. If for every point $x \in X$ the intersection $\Phi(x) = F(x) \cap G(x)$ is nonempty, where $G(x) = \bigcup_{i=1}^{\infty} G_i(x)$, then the multivalued mapping $\Phi : X \to L_1(T, E)$, $x \mapsto \Phi(x)$, admits a continuous selection.*

In a series of papers TOLSTONOGOV [1999a, 1999b, 1999c] studied selections passing through fixed points of multivalued contractions, depending on a parameter, with decomposable values. In particular, such parametric fixed points sets are absolute retracts and the sets of such selections are dense in the set of all continuous selections of the convexified mappings. Earlier, GORNIEWICZ and MARANO [1996] proposed unified approach for proving such nonparametric facts as for convex-valued contraction and also for decomposable-valued contractions (see also GORNIEWICZ, MARANO and SLOSARSKI [1996]).

4.E. Continuing the subject of unusual convexities, let us say something about some papers which are related to different kinds of such structures. Saveliev proposed a relaxation of Michael's axiomatic structure $\{(M_n, k_n)\}$ on a metric space M (see MICHAEL [1959b]) in the following three directions. First, he assumed M to be uniform. This reminds one of the approaches of GEILER [1970] and VAN DE VEL [1993b]. Second, the convex combination functions k_n were assumed to be multivalued. Recall that we have met such a situation earlier for decomposable-valued mappings. Moreover the sequence of mappings k_n was replaced by a multivalued (and partially defined) mapping C from the set $\Delta(M)$ of all formal convex combinations of elements of M into M. This repeats the approach of HORVATH [1991]. Briefly, a convexity on M is defined as a triple (M, C, Z) where Z is a topology on M which may be different from the uniform topology of M.

4.17. THEOREM (SAVELIEV [2000]). *Let X be a normal space, M a complete uniform space, and (M, C, Z) a continuous convexity with a countable convex uniform base and with uniform topology of M which is finer than the topology Z. Then every LSC closed-valued and convex-valued mapping from X to Z admits a selection whenever $p(X) \geq l_u(M)$.*

Here $l_u(M)$ denotes the Lindelöf number of the uniformity and $p(X)$ is the largest cardinal number $\mu \leq \kappa$ (where κ is a cardinal much bigger than cardinalities of all sets considered) such that each open cover of X whose cardinality is less than μ has a locally finite open refinement. Note that $p(X) = \kappa$ for a paracompact space X. Hence by putting $Z = M$ in Theorem 4.17 one can obtain the selection theorem for paracompact domains. Such a theorem includes as a special cases the convex-valued selection theorems of MICHAEL [1959b], CURTIS [1985], HORVATH [1991], and VAN DE VEL [1993a, 1993b].

JI-CHENG HOU [2001] proved a selection theorem for mappings into spaces having H-structure (in the sense of Horvath) which are ball-locally-uniformly LSC, but in general not LSC (see Part B of REPOVŠ and SEMENOV [1998a]).

Colombo and Goncharov considered a specific type of convexity in Hilbert spaces.

4.18. DEFINITION. A closed subset K of a Hilbert space is called *ϕ-convex* if there exists a continuous function $\phi : K \to [0, \infty)$ such that

$$\langle v, y - x \rangle \ \leq \ \phi(x) \|v\| \cdot \|y - x\|^2$$

for all $y, x \in K$ and all v proximally normal to K at x.

All convex sets as well as sets with sufficiently smooth boundary are ϕ-convex. In such sets one can obtain a kind of geodesic between two points which allows a convexity type structure.

4.19. THEOREM (COLOMBO and GONCHAROV [2001]). *Each continuous mapping from a metric space into a finite-dimensional Euclidean space admitting as values closed simply connected C^2-manifolds with negative sectional curvature, uniformly bounded from below, has a dense family of continuous selections.*

Continuous singlevalued selections f of a given multivalued mapping F are usually constructed as uniform limits of sequences of certain approximations $\{f_n\}$ of F. Practically all known selection results have been obtained by using one of the following two approaches for a construction of $\{f_n\}$. In the first (and the most popular) one, the method of outside approximations, mappings f_n are continuous ε_n-selections of F, i.e. $f_n(x)$ all lie *near* the set $F(x)$ and all mappings f_n are *continuous*. In the second one, the method of inside approximations, f_n are δ_n-continuous selections of F, i.e. $f_n(x)$ all lie *in* the set $F(x)$, however f_n are discontinuous.

In REPOVŠ and SEMENOV [1999] continuous selections were constructed as uniform limits of a sequence of δ-continuous ε-selections. Such a method was needed in order to unify different kinds of selection theorems. Namely, one forgets about closedness of values $F(x)$ over a countable subset C of domain and restricts nonconvexity of values $F(x)$ outside a zero-dimensional subset of domain. The density theorem holds as well.

4.20. THEOREM (REPOVŠ and SEMENOV [1999]). *Let $\beta : (0, \infty) \to (0, \infty)$ be a weakly g-summable function and $F : X \to Y$ a lower semicontinuous mapping from a paracompact space X into a Banach space Y. Suppose that $C \subset X$ is a countable subset of the domain such that values $F(x)$ are closed for all $x \in X \setminus C$ and that $Z \subset X$ with $dim_X Z \leq 0$. Then F has a singlevalued continuous selection, whenever $\beta(\cdot)$ is a pointwise strong majorant of the function* $(\sup\{\alpha_{Cl(F(x))}(\cdot) \,|\, x \in X \setminus Z\})^+$.

In REPOVŠ and SEMENOV [2001] we compared the nonconvexity of the set and nonconvexity of its ε-neighborhoods. The answers depend on smoothness properties of a unit sphere of a Banach range space. Hence on the one hand there exist a 4-dimensional Banach space B, its 1-dimensional subset P and a sequences ε_n, t_n of positive reals tending to zero such that the function of nonconvexity $\alpha_P(\cdot)$ always is less than some $q \in [0, 1)$ while $\alpha_{D(P,\varepsilon_n)}$ are identically equal to 1 on the intervals $(0, t_n)$ (i.e., nonconvexity of neighborhoods in principle differs from nonconvexity of the set). On the other hand, for uniformly convex Banach spaces, the inequality $\alpha_P(\cdot) < q < 1$ always implies the inequality $\alpha_{D(P,\varepsilon)}(\cdot) < p < 1$. Note, that even on the Euclidean plane there are examples with $q < p$.

4.F. Recently a new series of papers on continuous selections has appeared. All of them, are related to the substitution in selection theory of Lebesgue dimension $\dim$ by *extension dimension*. The extension dimension of a topological space equals to a class of CW-complexes, not a natural number. This notion was introduced by DRANISHNIKOV [1995] (see also DRANISHNIKOV and DYDAK [1996]). A comprehensive survey of the subject can be found in CHIGOGIDZE [2002].

KARASEV [200?] proved an $extdim$-analogue of Michael's finite-dimensional theorem. BRODSKY, CHIGOGIDZE and KARASEV [2002] found a unified "filtered" approach to both selection and approximation results with respect to $extdim$-theory.

For a CW-complexes L and K we say that $L \leq K$ if

$$(L \in AE(X)) \quad \Longrightarrow \quad (K \in AE(X))$$

for each X from a suitable class of spaces. First, authors were interested in the separable and metrizable situation. It now seems that proofs remain valid in general position, i.e. for paracompact domains and completely metrizable ranges. Thus $L \sim K$ if $L \leq K$ and $K \leq L$ and $[L]$ denotes the equivalence class.

4.21. DEFINITION. A space X is said to have *extension dimension* $\leq [L]$ (notation: $ed(X) \leq [L]$) if $L \in AE(X)$.

Clearly, $\dim X \leq n$ is equivalent to $ed(X) \leq [S^n]$ and $\dim_G X \leq n$ is equivalent to $ed(X) \leq [K(G,n)]$, where $K(G,n)$ is the Eilenberg-MacLane complex. One can develop homotopy and shape theories specifically designed to work for at most $[L]$-dimensional spaces (see CHIGOGIDZE [2002]). The theories were developed mostly for *finitely* dominated complexes L. Absolute extensors for at most $[L]$-dimensional spaces in a category of continuous maps are precisely $[L]$-soft mappings. And compacta of trivial $[L]$-shape are precisely $UV^{[L]}$-compacta.

As for selection theorems, they are presently known to be true for *finite* complexes L only. All notions used in the filtered approach to selection theorem (see Subsection 3.3 above) admit natural extensional analogues. For example, a pair of spaces $V \subset U$ is said to be $[L]$-*connected* if for every paracompact space X of extension dimension $ed(X) \leq [L]$ and for every closed subspace $A \subset X$ any mapping of A into V can be extended to a mapping of X into U. Or, a multivalued mapping $F : X \to Y$ is called $[L]$-*continuous at a point* $(x, y) \in \Gamma_F$ of its graph if for every neighborhood Oy of the point $y \in Y$, there is a neighborhood $O'y$ of the point y and a neighborhood Ox of the point $x \in X$ such that for all $x' \in Ox$, the pair $F(x') \cap O'y \subset F(x') \cap Oy$ is $[L]$-connected. Hence the above filtered Theorems 3.1 and 3.5 admit the following generalization.

4.22. THEOREM (BRODSKY, CHIGOGIDZE and KARASEV [2002]). *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension $ed(X) \leq [L]$. Let a complete lower $[L]$-continuous multivalued mapping Φ of X into a complete metric space Y contain an n-$UV^{[L]}$-filtered compact submapping Ψ which is singlevalued on some closed subset $A \subset X$. Then any neighborhood U of the graph Γ_Ψ in the product $X \times Y$ contains the graph of a singlevalued continuous selection s of the mapping Φ which coincides with $\Psi\big|_A$ on the set A.*

In particular, under the assumptions of Theorem 4.22 each complete lower $[L]$-continuous multivalued mapping $F : X \to Y$ into a complete metric space has a continuous selection.

There exists an extension dimensional version of Uspenskii's selection theorem for C-domains which we used in Section 1 above.

4.23. THEOREM (BRODSKY and CHIGOGIDZE [200?]). *Let L be a finite CW-complex and $F : X \to Y$ a multivalued mapping of a paracompact C-space X of extension dimension $ed(X) \leq [L]$ to a topological space Y. If F admits infinite fiberwise $[L]_c$-connected filtration of strongly LSC multivalued mappings, then F has a singlevalued continuous selection.*

Recall, that a multivalued mapping F is said to be *strongly lower semicontinuous* if for any point $x \in X$ and any compact set $K \subset F(x)$ there exists a neighborhood V of x such that $K \subset F(z)$ for every $z \in V$.

As for an applications, we mention two facts concerning the so-called *Bundle problem* (see Problem 5.12. below):

4.24. THEOREM (BRODSKY, CHIGOGIDZE and SHCHEPIN [200?]). *Let $p : E \to B$ be a Serre fibration of LC^0-compacta with a constant fiber which is a compact two-dimensional manifold. If $B \in ANR$, then any section of p over closed subset $A \subset B$ can be extended to a section of p over some neighborhood of A.*

4.25. THEOREM (BRODSKY, CHIGOGIDZE and SHCHEPIN [200?]). *Let $p : E \to B$ be a topologically regular mapping of compacta with fibers homeomorphic to a 3-dimensional manifold. If $B \in ANR$, then any section of p over closed subset $A \subset B$ can be extended to a section of p over some neighborhood of A.*

5. Open problems

5.1. PROBLEM (MICHAEL). Let Y be a G_δ-subset of a Banach space B. Does then every LSC mapping $F : X \to Y$ of a paracompact space X with convex closed (in Y) values have a continuous selection?

We wish to emphasize that this problem is infinite-dimensional by its nature. Indeed, for $\dim X < \infty$ one can simply apply the finite-dimensional selection theorem. For C-domains X see proof in Section 1. Gutev has observed that using so-called selection-factorization technique of Choban and Nedev the problem for $\dim B < \infty$ reduces to metrizable domains and then one can apply the Michael selection theorem for perfectly normal domains and nonclosed-valued mappings into a separable range space. Maybe, one should first attempt it for Hilbert spaces B or for reflexive spaces B.

5.2. PROBLEM. Is it true that the affirmative answer to Problem 5.1 for an arbitrary Banach space B characterizes C-property of the domain?

5.3. PROBLEM (CHOBAN, GUTEV and NEDEV). Does each LSC closed- and convex-valued mapping from a collectionwise normal and countably paracompact domain into a Hilbert space admit a singlevalued continuous selection?

The following problems due to van de Vel relate to an axiomatic definition of the convexity notion. For detailed discussion see VAN DE VEL [1993a, 1993b].

5.4. PROBLEM (VAN DE VEL). Does every LSC compact- and convex-valued mapping from a normal domain into a metric space Y endowed by an uniform convex system admit a singlevalued continuous selection?

5.5. PROBLEM (VAN DE VEL). Problem 5.4 for paracompact domains and closed-valued mappings.

Recall, that a *convex system* on a set Y means a collection of subsets of Y which is closed for intersection and for chain union. The difference from convex structures is that the set Y itself needs not be convex. Subsets of convex sets are called admissible - they are the sets which have a convex hull. This definition includes the structures defined earlier by MICHAEL [1959b] and CURTIS[1985]. A *polytope* is the convex hull of an admissible finite set. Of course, singletons are assumed to be convex. The uniformity of the convex system means that all polytopes are compact, all convex sets are connected and the (partial) convex hull operator is uniformly continuous i.e. for all $\varepsilon > 0$ there is $\delta > 0$ such that if two finite sets A and B are δ-close in the Hausdorff metric, then $hull(A)$ and $hull(B)$ are ε-close. The answers are affirmative for uniform convex systems with real parameters, as defined by Michael and Curtis. It also holds for uniform metric convex systems which can be extended to uniform metric convex structures. Such an extension problem is in general also unsolved.

RICCERI [1987] proved that a multivalued mapping G from an interval $I \subset \mathbb{R}$ into a topological space Y admits an LSC multivalued selection H whenever the graph of G is connected and locally connected and for every open set $\Omega \subset Y$, the set $G^{-}(\Omega)\cap \mathrm{int}(I)$ has no isolated points. He stated the following factorization problem.

5.6. PROBLEM (RICCERI). Let X and Y be any topological spaces and let $F : X \to 2^Y$. Find suitable conditions under which there exist an interval $I \subset \mathbb{R}$, a continuous function $h : X \to I$ and a mapping $G : I \to 2^Y$, satisfying the following properties:

(1) $G(h(x)) \subset F(x)$ for all $x \in X$;

(2) The graph of G is connected and locally connected; and

(3) For every open set $\Omega \subset Y$, the set $G^{-}(\Omega)\cap \mathrm{int}(I)$ has no isolated points.

The motivation for this problem comes from the fact that each time it has a positive answer, the multifunction F admits a lower semicontinuous multiselection with nonempty values.

Next, we reproduce some problems proposed by GUTEV and NOGURA [200?b]. Below, $Sel(X)$ and $Sel_n(X)$ means the set of all continuous (with respect to Vietoris topology) selectors for closed subsets of X and for subsets, consisting of $\leq n$ elements. And a space X is *zero-dimensional* if it has a base of clopen sets, i.e. if $\mathrm{ind}\, X = 0$.

5.7. PROBLEM. Does there exist a space X such that $Sel_2(X) \neq \emptyset$ but $Sel_n(X) = \emptyset$ for some $n > 2$?

Recall that $Sel_2(X) \neq \emptyset \implies X$ is linearly ordered topological space $\implies Sel(X) \neq \emptyset$ for compact space X, but $Sel_2(\mathbb{R}) \neq \emptyset$ while $Sel(\mathbb{R}) = \emptyset$. Having also $Sel_2(X) \neq \emptyset \implies \mathrm{ind}\, X \leq 1$ for compact X it is naturally ask the following:

5.8. PROBLEM. Does there exist a space X such that $Sel_2(X) \neq \emptyset$ and ind $X > 1$?

As mentioned above (see Section 2), $(Sel(X) \neq \emptyset \ \& \ \text{ind}\, X = 0) \Longrightarrow \{f(X) : f$ is a continuous selector$\}$ is dense in X. In the other direction X is totally disconnected whenever $\{f(X) : f$ - continuous selector$\}$ is dense in X.

5.9. PROBLEM. Does there exist a space X which is not zero-dimensional but $\{f(X) :$ f is a continuous selector$\}$ is dense in X?

In comparison with results in ENGELKING, HEATH and MICHAEL [1968] and CHOBAN [1970], the following question seems to be interesting:

5.10. PROBLEM (GUTEV and NOGURA). Does there exist a zero-dimensional metrizable space X such that $\mathcal{F}(X)$ has a continuous selector but $\dim X \neq 0$?

The following *Bundle problem* has a negative answer for $n > 4$, a positive answer for $n = 1$ and partially positive solution for $n = 2$ (see Theorem 3.7 above). That for mappings between finite-dimensional compacta and for $n > 4$ the answer is affirmative.

5.11. PROBLEM (SHCHEPIN). Let $p : E \to B$ be a Serre fibration with a constant fiber which is an n-dimensional manifold. Is p a locally trivial fibration?

Shchepin has proposed Problems 5.12-5.15 related to the Bundle problem below:

5.12. PROBLEM. Does every open mapping of a locally connected continuum onto arc have a continuous section?

5.13. PROBLEM. Is any piecewise linear n-soft mapping of compact polyhedra a Serre n-fibration?

5.14. PROBLEM. Does every Serre fibration with a compact locally connected base have a global section if all of its fibers are contractible compact 4-manifolds with boundary?

5.15. PROBLEM. Is the complex-valued mapping $z_1^3 + z_2^3$ of $\mathbb{C}^2$ (2-dimensional complex space) onto $\mathbb{C}^1$ a Serre 1-fibration?

One of the approaches to a possible solution of the Bundle problem for $n = 2$ relates to a convexity-like structures in the space $H_0(D^2)$ of all autohomeomorphisms of two-dimensional disk which act identically over the boundary of the disk. Hence we can ask the following problems concerning the function of nonconvexity of H_0 considered as a subset of the Banach space $C(D^2; \mathbb{R}^2)$. Recall, that H_0 is a contractible ANR and moreover it is homeomorphic to the Hilbert space (Mason's theorem). The following three problems concerning paraconvexity of the set $H_0(D^2)$ are due to Shchepin and Semenov.

5.16. PROBLEM. Let $f \in H_0(D^2)$ and $dist(f, id|_{D^2}) = 2r$. Estimate the distance $dist(\frac{f+id}{2}; H_0)$. Is $0,5r$ the correct answer?

Passing to higher dimensional simplices we obtain:

5.17. PROBLEM. Let $f_1, f_2, ..., f_n \in H_0(D^2)$ and $f \in conv\{f_1, f_2, ..., f_n\}$. Is it true that $dist(f; H_0) \leq 0,5r$ where r is minimal radius of a ball which covers all $f_1, f_2, ..., f_n$?

As a preliminary step in attacking Problem 5.17 one can change autohomeomorphisms of disk to embeddings of a segment into the plane.

5.18. PROBLEM. Let f and g be two embeddings of the segment $[0, 1]$ into the Euclidean plane and $dist(f, g) = 2r$. Estimate the distance between the mapping $\frac{f+g}{2}$ and the set of all embeddings of this segment to the plane.

The two following problems are due to Semenov. The Nash's embedding theorem asserts that for each Riemannian metric ρ on a smooth compact manifold M^n there exists an isometric embedding of $(M^n; \rho)$ to $\mathbb{R}^N$ where $N \sim n^2$. Let us consider embeddings into the infinite-dimensional Hilbert space H.

5.19. PROBLEM. For each Riemannian metric ρ on M define $F(\rho)$ as the set of all isometric embeddings of $(M, \rho) \hookrightarrow H$. Does then the multivalued mapping F admit a continuous selection?

Note, that each value $F(\rho)$ is invariant under the action of the orthogonal group $O(H)$ which is homotopically trivial (due to the Kuiper theorem).

The following problem arises as a possible unification of proofs of Michael selection Theorem 1.1 and Fryszkowski selection theorem. For a Banach space B consider a subset S of all continuous linear operators on B with the property that $A \in S \implies (id - A) \in S$ and define S-convex subsets $C \subset B$ as the sets with the property that $x \in C, y \in C, A \in S \implies (id - A)x + Ay \in C$.

5.20. PROBLEM. Find a suitable axiomatic restrictions for S under which Theorem 1.1 holds for mappings with S-convex values.

Observe, that for $S = \{t \cdot id | t \in [0, 1]\}$ we obtain the standard convexity in Banach space B and for $B = L_1(T, \mu)$ and $S =$ the set of all operators of multiplications on characteristic functions of measurable sets S-convexity coincides with decomposability.

Recall that a subset M of a Banach space B is said to be *proximinal* if for each $x \in B$ the set $P_M(x)$ of points of M which are nearest to x is non-empty.

5.21. PROBLEM (DEUTSCH). Is there a semicontinuity condition on the metric projection P_M onto a proximinal subspace M in a Banach space that is both necessary and sufficient for the metric projection to admit a continuous selection?

FISCHER [1988] proved that for a Banach space of continuous functions on compacta and for any finite-dimensional subspace M an affirmative answer is given by the so-called *almost lower semicontinuity*. This last notion was introduced for general multivalued mappings by DEUTSCH and KENDEROV [1983] and characterizes the property of a multivalued mapping to have continuous ε-selections for any positive ε. It should be mentioned that LI [1991] has given an *intrinsic* characterization of those finite-dimensional subspaces M of $C_0(T)$ such that P_M admits a continuous selection. (These are the so-called *weakly regularly interpolating* subspaces.)

Recently BROWN, DEUTSCH, INDUMATHI AND KENDEROV [2002] have established a geometric characterization of those Banach spaces X in which P_M admits a continuous selection for any one-dimensional subspace M.

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