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PRESERVATION OF γ-SPACES AND COVERING PROPERTIES OF PRODUCTS

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Abstract. We prove that the Hurewicz property is not preserved by finite products in the Miller model. This is a consequence of the fact that Miller forcing preserves ground model γ-spaces.

1. Introduction

When trying to describe σ-compactness in terms of open covers, Hurewicz [5] introduced the following property, nowadays called *the Menger property*: A topological space X is said to have this property if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup\mathcal{V}_n : n \in \omega\}$ is a cover of X. The current name (the Menger property) has been adopted because Hurewicz proved in [5] that for metrizable spaces his property is equivalent to a certain basis property considered by Menger in [10]. If in the definition above we additionally require that $\{\cup\mathcal{V}_n : n \in \omega\}$ be a *γ-cover* of X (this means that the set $\{n \in \omega : x \notin \cup\mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the Hurewicz covering property introduced in [6]. Contrary to a conjecture of Hurewicz, the class of metrizable spaces having the Hurewicz property turned out to be wider than the class of σ-compact spaces [7, Theorem 5.1].

As for most of the topological properties, it is interesting to ask whether the Hurewicz property is preserved by finite products. One of the motivations behind this question comes from spaces of continuous functions; see [8, Theorem 21]. In the case of general topological spaces there are ZFC examples of Hurewicz spaces whose product is not even Menger; see [18, Section 3] and the discussion in the introduction of [14]. That is why *we concentrate in what follows on subspaces of the Cantor space* 2^ω. (Let us note that the preservation of the Hurewicz property by finite products of metrizable spaces reduces to subspaces of 2^ω; see the end of the proof of [14, Theorem 1.1] on p. 331 of that paper.) The covering properties of products of subspaces of 2^ω with the Hurewicz property turned out to be sensitive to the ambient set-theoretic universe: Under CH there exists a Hurewicz space whose

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square is not Menger; see [7, Theorem 2.12]. Later, a similar construction was carried out under a much weaker assumption; see [17, Theorem 43]. In particular, under the Martin Axiom there are Hurewicz subspaces of the Cantor space whose product is not Menger.

On the other hand, the product of any two Hurewicz subspaces of 2^ω is Menger in the Laver and Miller models; see [14] and [19], respectively. In the Miller model we actually know that the product of finitely many Hurewicz subspaces of 2^ω is Menger (for the Laver model this is unknown even for three Hurewicz subspaces), because in this model the Menger property is preserved by products of subspaces of 2^ω; see [19]. This is why the Miller model seemed to be the best candidate for a model where the Hurewicz property is preserved by finite products of metrizable spaces. Our next theorem refutes this expectation, and hence the question whether one can find ZFC examples of Hurewicz subspaces X, Y of 2^ω with non-Hurewicz product remains open.

Standardly, by the Miller model we mean a forcing extension of a ground model of GCH by adding a generic for the forcing obtained by an iteration of length ω_2 with countable supports of the poset defined by Miller in [11]. We recall the definition of this poset in the proof of Lemma 2.5.

Theorem 1.1. *In the Miller model there are two γ-subspaces X, Y of 2^ω such that $X \times Y$ is not Hurewicz. In particular, in this model the Hurewicz property is not preserved by finite products of metrizable spaces.*

A family $\mathcal{U}$ of subsets of a set X is called an *ω-cover* of X if $X \notin \mathcal{U}$ and for every finite subset F of X there exists $U \in \mathcal{U}$ such that $F \subset U$. A space X is called a *γ-set* if every open ω-cover of X contains a γ-subcover. This notion was introduced in [4], where it was proved that a Tychonoff space X is a γ-space if and only if the space $C_p(X)$ of all continuous functions from X to $\mathbb{R}$ with the topology of the pointwise convergence has the Fréchet-Urysohn property; i.e., for every $f \in C_p(X)$ and $A \subset C_p(X)$ with $f \in \bar{A}$ there exists a sequence $\langle f_n : n \in \omega \rangle \in A^\omega$ converging to f.

It is well-known that γ-spaces have the Hurewicz property in all finite powers; see, e.g., [7, Theorem 3.6 and Figure 2] and references therein. This follows from the following characterization proved in [4]: X is a γ-space if and only if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open ω-covers of X there exists a sequence $\langle U_n \in \mathcal{U}_n : n \in \omega \rangle$ such that $\{U_n : n \in \omega\}$ is a γ-cover of X.

Our proof of Theorem 1.1 is based on the fact that if $X \subset 2^\omega$, $X \in V$, and X is a γ-space in V, then X remains a γ-space in the forcing extension by a countable support iteration of posets satisfying property (†) introduced in Definition 2.1 below. This seems to be the first attempt to find iterable properties of forcing posets guaranteeing the preservation of ground model γ-spaces. Previously, only specific posets were treated: By [13] and [16] γ-spaces are preserved by Cohen and random forcing, respectively, whereas the Hechler forcing kills all ground model uncountable γ-spaces; see [12].

Let us note that Cohen forcing satisfies (†) but fails to preserve Hurewicz spaces; see the discussion in [13] after Problem 4.1 therein. This is why our proof of Theorem 1.1 leaves open the following question.

Question 1.2. Does Miller forcing preserve the Hurewicz property of ground model metrizable spaces containing no topological copies of 2^ω? What about Sierpiński spaces?

2. Proof of Theorem 1.1

Theorem 1.1 is a direct consequence of Lemmata 2.2, 2.3, 2.4, and 2.5 proved below, combined with one of the main results of [13]. We shall consider only posets $\mathbb{P}$ such that below any $p \in \mathbb{P}$ there exist incompatible r, q. This is not an essential restriction because most of the posets considered in literature have this property. First we need to introduce some auxiliary notions.

Definition 2.1.

- A poset $\mathbb{P}$ is said to have the property (†) if for every countable elementary submodel $M \ni \mathbb{P}$ of $H(\theta)$ for big enough θ, every $p \in \mathbb{P} \cap M$, and $\phi_i : \mathbb{P} \cap M \to \mathbb{P} \cap M$ for all $i \in \omega$ such that $\phi_i(p) \leq p$ for all $p \in \mathbb{P} \cap M$ and $i \in \omega$, there exists an $(M, \mathbb{P})$-generic $q \leq p$ forcing
$$\dot{G} \cap \{\phi_i(p) : p \in M \cap \mathbb{P}\} \quad \text{is infinite for all } i \in \omega,$$
where $\dot{G}$ is the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filter.
- Let $\mathcal{B} = \{B_n : n \in \omega\}$ be a bijective enumeration of the standard clopen base of the topology on 2^ω; i.e., $\mathcal{B}$ consists of finite unions of elements of the family $\{[s] = \{x \in 2^\omega : x \restriction |s| = s\} : s \in 2^{<\omega}\}$. Let $X \subset 2^\omega$ and $M \ni X$ be as above. $\mathcal{W} \subset \mathcal{B}$ is called $\langle X, M, \omega\rangle$-hitting if $\mathcal{W} \cap \mathcal{U}$ is infinite for every ω-cover $\mathcal{U}$ of X such that $\mathcal{U} \in M$ and $\mathcal{U} \subset \mathcal{B}$.
- The poset $\mathbb{P}$ is called $\langle X, \gamma\rangle$*-preserving* if for every countable elementary submodel M such that $X, \mathbb{P} \in M$, $\langle X, M, \omega\rangle$-hitting $\mathcal{W} \subset \mathcal{B}$, and $p \in \mathbb{P} \cap M$ there exists an $(M, \mathbb{P})$-generic condition $q \leq p$ forcing $\mathcal{W}$ to be $\langle X, M[\dot{G}], \omega\rangle$-hitting.

In what follows we shall denote by $\Omega(X)$ and $\Gamma(X)$ the family of all open ω- and γ-covers of a topological space X, respectively. The following lemma justifies our terminology.

Lemma 2.2. *If $\mathbb{P}$ is $\langle X, \gamma\rangle$-preserving and $X \subset 2^\omega$ is a γ-set, then X remains a γ-set in $V^{\mathbb{P}}$.*

Proof. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for an ω-cover of X by elements of $\mathcal{B}$, $p \in \mathbb{P}$, and let $M \ni \dot{\mathcal{U}}, p$ be a countable elementary submodel. Let $\{\mathcal{U}_i : i \in \omega\}$ be an enumeration of $\Omega(X) \cap M \cap \mathcal{P}(\mathcal{B})$ and let $U_i \in \mathcal{U}_i$ be such that $\mathcal{W} = \{U_i : i \in \omega\} \in \Gamma(X)$. Then $\mathcal{W}$ is $\langle X, M, \omega\rangle$-hitting, and hence there exists an $(M, \mathbb{P})$-generic $q \leq p$ forcing $\mathcal{W} \cap \dot{\mathcal{U}}$ to be infinite. Thus q forces $\mathcal{W} \cap \dot{\mathcal{U}}$ to be a γ-subcover of $\dot{\mathcal{U}}$. □

Lemma 2.3. *If $\mathbb{P}$ satisfies (†), then it is $\langle X, \gamma\rangle$-preserving for every $X \subset 2^\omega$.*

Proof. Let us enumerate $V^{\mathbb{P}} \cap M$ as $\{\dot{\mathcal{U}}_i : i \in \omega\}$. For every $p \in \mathbb{P} \cap M$ and $i \in \omega$, if p does not force $\dot{\mathcal{U}}_i$ to be an ω-cover of X consisting of elements of $\mathcal{B}$, we can find $r_{i,p} \leq p$ which forces $\dot{\mathcal{U}}_i$ to not be an ω-cover of X by elements of $\mathcal{B}$. Otherwise we set $\mathcal{U}_{i,p} = \{B \in \mathcal{B} : \exists r \leq p (r \Vdash B \in \dot{\mathcal{U}}_i)\}$ and note that $\mathcal{U}_{i,p} \in \Omega(X) \cap M$. Furthermore, by elementarity we have that for every $B \in \mathcal{U}_{i,p}$ there exists $M \ni r \leq p$ such that $r \Vdash B \in \dot{\mathcal{U}}_i$. Let $\{p_n : n \in \omega\}$ be an enumeration of $M \cap \mathbb{P}$, and for every n, i set $\mathcal{U}'_{i,p_n} = \mathcal{U}_{i,p_n} \setminus \{B_k : k \leq n\}$. Since $\mathcal{W}$ is $\langle X, M, \omega\rangle$-hitting, $|\mathcal{W} \cap \mathcal{U}'_{i,p}| = \omega$ for every

$p \in M \cap \mathbb{P}$ and $i \in \omega$. For every p, i as above pick $U_{i,p} \in \mathcal{W} \cap \mathcal{U}'_{i,p}$ and $r_{i,p} \leq p$ such that $r_{i,p} \in M$ and $r_{i,p} \Vdash U_{i,p} \in \dot{\mathcal{U}}_i$.

Now let us fix $p_* \in \mathbb{P} \cap M$ and consider maps $\phi_i : p \mapsto r_{i,p}$, $i \in \omega$. It follows that there exists an $(M, \mathbb{P})$-generic $q \leq p_*$ forcing the set $\dot{G} \cap \{r_{i,p} : p \in \mathbb{P} \cap M\}$ to be infinite for all $i \in \omega$. Let $G \ni q$ be $\mathbb{P}$-generic and let $i \in \omega$. If $\dot{\mathcal{U}}_i^G$ is an ω-cover of X by elements of $\mathcal{B}$, then no $r_{i,p} \in G$ can force the negation thereof, and hence for each such $r_{i,p}$ we have $U_{i,p} \in \mathcal{W} \cap \dot{\mathcal{U}}_i^G$. Therefore $|\mathcal{W} \cap \dot{\mathcal{U}}_i^G| = \omega$ since no $B \in \mathcal{B}$ can belong to $\mathcal{U}'_{i,p}$ for infinitely many $p \in M \cap \mathbb{P}$. □

Remark. It is a simple exercise to check that if in the definition of (†) we restrict ourselves to only one $\phi : M \cap \mathbb{P} \to M \cap \mathbb{P}$, then we get an equivalent statement. The longer formulation which we have chosen seems to be easier to apply, though. □

By the definition we have that for every $X \subset 2^\omega$ finite iterations of $\langle X, \gamma\rangle$-preserving posets are again $\langle X, \gamma\rangle$-preserving. The proof of the next fact is modelled after that of [1, Lemma 2.8]. In fact, we just "add an ϵ" to it, using ideas from [3].

Lemma 2.4. *Let $X \subset 2^\omega$. Then countable support iterations of $\langle X, \gamma\rangle$-preserving posets are again $\langle X, \gamma\rangle$-preserving.*

Proof. We shall inductively prove the following formally stronger statement:

> Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ be a countable support iteration of $\langle X, \gamma\rangle$-preserving posets, let M be a countable elementary submodel of $H(\lambda)$ for a sufficiently large regular cardinal λ such that $\delta, \mathbb{P}_\delta \in M$, and let $\mathcal{W} \subset \mathcal{B}$ be $\langle X, M, \omega\rangle$-hitting. For any $\delta_0 \in \delta \cap M$ and $(M, \mathbb{P}_{\delta_0})$-generic condition q_0 forcing $\mathcal{W}$ to be $\langle X, M[\dot{G}_{\delta_0}], \omega\rangle$-hitting, the following holds: If $\dot{p}_0 \in V^{\mathbb{P}_{\delta_0}}$ is such that
>
> $$q_0 \Vdash_{\mathbb{P}_{\delta_0}} \dot{p}_0 \in \mathbb{P}_\delta \cap M \quad \text{and} \quad \dot{p}_0 \restriction \delta_0 \in \dot{G}_{\delta_0},$$
>
> where $\dot{G}_{\delta_0}$ is the canonical name for the $\mathbb{P}_{\delta_0}$-generic, then there is an $(M, \mathbb{P}_\delta)$-generic condition q such that
>
> $$q \restriction \delta_0 = q_0 \quad \text{and} \quad q \Vdash_{\mathbb{P}_\delta} \text{"}\dot{p}_0 \in \dot{G}_\delta \wedge \mathcal{W} \text{ is } \langle X, M[\dot{G}_\delta], \omega\rangle\text{-hitting."}$$

We are going to prove this statement by induction on δ; the only non-trivial case (modulo [1, Lemma 2.6] and the proof thereof) is when δ is a limit ordinal. Fix a strictly increasing sequence $\langle \delta_n : n \in \omega\rangle$ of ordinals in M cofinal in $M \cap \delta$. For every $\nu < \mu < \delta$ let us denote by $\mathbb{P}_{[\nu,\mu)}$ a $\mathbb{P}_\nu$-name for the iteration of $\dot{\mathbb{Q}}_\beta$, $\beta \in \mu \setminus \nu$, in $V^{\mathbb{P}_\nu}$. As usual (see, e.g., [9]) we shall identify $\mathbb{P}_{[\nu,\mu)}$ with the set of all functions p with domain $\mu \setminus \nu$ such that $1_{\mathbb{P}_\nu}{}^\frown p \in \mathbb{P}_\mu$, ordered as follows: Given a $\mathbb{P}_\nu$-generic G and $p_0, p_1 \in \mathbb{P}_{[\nu,\mu)}$, $p_1^G \leq p_0^G$ in $\mathbb{P}_{[\nu,\mu)}^G$ if there exists an $s \in G$ such that $s^\frown p_1 \leq s^\frown p_0$ in $\mathbb{P}_\mu$.

Set $D_0 = \mathbb{P}_\delta$ and let $\{D_i : i \geq 1\}$ be the set of all open dense subsets of $\mathbb{P}_\delta$ which belong to M and $\{\dot{\mathcal{U}}_i : i \geq 1\}$ an enumeration of $V^{\mathbb{P}_\delta} \cap M$ such that each $\tau \in V^{\mathbb{P}_\delta} \cap M$ equals $\dot{\mathcal{U}}_i$ for infinitely many i. We shall define by induction on $n \in \omega$ a condition $q_n \in \mathbb{P}_{\delta_n}$ and a name $\dot{p}_n \in V^{\mathbb{P}_{\delta_n}}$ such that:

(1) q_0 and $\dot{p}_0$ are as in the quoted claim at the beginning of the proof; q_n is $(M, \mathbb{P}_{\delta_n})$-generic; $q_{n+1} \restriction \delta_n = q_n$;

(2) $\dot p_n$ is a $\mathbb{P}_{\delta_n}$-name such that

$$q_n \Vdash_{\mathbb{P}_{\delta_n}} \text{“}\dot p_n \text{ is a condition in } \mathbb{P}_\delta \cap M \text{ such that}$$

(a) $\dot p_n \restriction \delta_n \in \dot G_{\delta_n}$;
(b) $\dot p_n \le \dot p_{n-1}$;
(c) $\dot p_n \in D_n$; and
(d) if $n \ge 1$, then $\dot p_n$ decides whether $\dot{\mathcal{U}}_n$ is an ω-cover of X by elements of $\mathcal{B}$, and in the case when decided to be such a cover, $\dot p_n$ forces, in addition, that $\exists m \ge n (B_m \in \dot{\mathcal{U}}_n \cap \mathcal{W})$.”

Assume that q_n and $\dot p_n$ have already been constructed. For a while we shall work in $V[G]$, where $G \ni q_n$ is $\mathbb{P}_{\delta_n}$-generic. Then $p_n := \dot p_n^G \in D_n \cap M$ and $p_n \restriction \delta_n \in G$. Find $p_n' \le p_n$ such that $p_n' \restriction \delta_n \in G$ and $p_n' \in D_{n+1} \cap M$. It exists because the set

$$D' = \{p' \in \mathbb{P}_{\delta_n} : (p' \perp p_n \restriction \delta_n) \vee (\exists p_n' \in D_{n+1}(p_n' \le p_n \wedge p' = p_n' \restriction \delta_n))\}$$

is dense in $\mathbb{P}_{\delta_n}$ and belongs to M, and hence $D' \cap M$ is predense below q_n, which yields $D' \cap G \cap M \ne \emptyset$. Moreover, since $p_n \restriction \delta_n \in G$, any $p' \in D' \cap G$ is compatible with $p_n \restriction \delta_n$. It follows that for any $p' \in G \cap D' \cap M$, any $p_n' \in M$ witnessing for $p' \in D'$ is as required.

Without loss of generality we may assume that each condition in D_{n+1} decides whether $\dot{\mathcal{U}}_{n+1}$ is an ω-cover of X by elements of $\mathcal{B}$. If p_n' decides that it is not, then we set $p_{n+1} = p_n'$ and take q_{n+1} to be any $(M, \mathbb{P}_{\delta_{n+1}})$-generic satisfying (1), (2) and forcing over $\mathbb{P}_{\delta_{n+1}}$ that $\mathcal{W}$ is $\langle X, M[\dot G_{\delta_{n+1}}], \omega\rangle$-hitting, its existence following by our inductive assumption. Otherwise fix a $\mathbb{P}_{\delta_n}$-name $\dot p_n' \in M$ for a condition in $\mathbb{P}_\delta$ such that q_n forces that $\dot p_n'$ have all the properties of p_n' stated above and an $(M, \mathbb{P}_{\delta_{i+1}})$-generic q_{n+1} such that $q_{n+1} \restriction \delta_n = q_n$, $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}} \dot p_n' \restriction \delta_{n+1} \in \dot G_{\delta_{n+1}}$, and $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}}$ “$\mathcal{W}$ is $\langle X, M[\dot G_{\delta_{n+1}}], \omega\rangle$-hitting”. Consider the $\mathbb{P}_{\delta_{n+1}}$-name $\dot{\mathcal{W}}_{n+1}$ which equals

$$\big\{\langle r, \check B\rangle : B \in \mathcal{B} \ \ \& \ \ \mathbb{P}_{\delta_{n+1}} \ni r \text{ decides } \dot p_n' \text{ as } p_n' \ \ \&$$
$$\text{exists } p \le p_n' \text{ such that } p \restriction \delta_{n+1} = r \quad \text{and} \quad p \Vdash_{\mathbb{P}_\delta} \check B \in \dot{\mathcal{U}}_{n+1}\big\}.$$

It follows that $\dot{\mathcal{W}}_{n+1} \in M$ is a $\mathbb{P}_{\delta_{n+1}}$-name which is forced by q_{n+1} to be an ω-cover of X by elements of $\mathcal{B}$, and hence $q_{n+1} \Vdash_{\mathbb{P}_{\delta_{n+1}}} |\mathcal{W} \cap \dot{\mathcal{W}}_{n+1}| = \omega$. Let $H \ni q_{n+1}$ be $\mathbb{P}_{\delta_{n+1}}$-generic over V and let p_n' be the interpretation $(\dot p_n')^H$. Now we shall work in $V[H]$ for a while. It follows from the above that there exists $m > n$ such that $B_m \in \mathcal{W} \cap \dot{\mathcal{W}}_{n+1}^H$. Consequently, there exist $r \in H$ and $p \le p_n'$ such that $p \restriction \delta_{n+1} = r$ and $p \Vdash_{\mathbb{P}_\delta} \check B_m \in \dot{\mathcal{U}}_{n+1}$. By elementarity we can find such r in M (note that $M[H] \cap \mathbb{P}_{\delta_{n+1}} = M \cap \mathbb{P}_{\delta_{n+1}}$), and hence we can also find $p \in M$ as above. Now let $\dot p_{n+1} \in M$ be a $\mathbb{P}_{\delta_{n+1}}$-name such that q_{n+1} forces that $\dot p_{n+1}$ have all the properties of p stated above. Its existence follows by the maximality principle. This completes our inductive construction.

Exactly as in the proof of [1, Lemma 2.8] one can verify that $q = \bigcup_{n\in\omega} q_n$ is $(M, \mathbb{P}_\delta)$-generic. More precisely, it is easy to see by induction on n that q forces over $\mathbb{P}_\delta$ that $\dot p_{n+1} \le \dot p_n \in \dot G_\delta \cap M$ for all $n \in \omega$. Using this we are going to prove that each $D_n \cap M$ is predense below q. Suppose not. Then we can find $q' \le q$ which is incompatible with all elements of $D_n \cap M$ for some $n \in \omega$. Let $H \ni q'$ be $\mathbb{P}_\delta$-generic. Then $p_n := \dot p_n^H \in H \cap M \cap D_n$ by (2), and hence p_n is a condition in $D_n \cap M$ compatible with q (because $q \in H$), a contradiction.

It suffices to note that $(2)(d)$ clearly ensures that q forces $\mathcal{W}$ to be $\langle X, M[\dot{G}_\delta], \omega\rangle$-hitting. This completes our proof. □

Lemma 2.5. *The Miller, Sacks, and Cohen posets satisfy* (†).

Proof. We shall present the proof only for Miller forcing because it is exactly what is needed for the proof of Theorem 1.1 and because the Sacks case is completely analogous, whereas the Cohen one is trivial.

Before we pass to the proof, let us recall the definition of Miller forcing and fix our notation. By a Miller tree we understand a subtree T of $\omega^{<\omega}$ consisting of increasing finite sequences such that the following conditions are satisfied:

- Every $t \in T$ has an extension $s \in T$ which is splitting in T; i.e., there is more than one immediate successor of s in T.
- If s is splitting in T, then it has infinitely many immediate successors in T.

Miller forcing is the collection $\mathbb{M}$ of all Miller trees ordered by inclusion; i.e., smaller trees carry more information about the generic. This poset was introduced in [11].

For a Miller tree T we denote by $\mathrm{Split}(T)$ the set of all splitting nodes of T, and for some $t \in \mathrm{Split}(T)$ we denote the size of $\{s \in \mathrm{Split}(T) : s \subsetneq t\}$ by $Lev(t, T)$. For a node t in a Miller tree T we denote by T_t the set $\{s \in T : s$ is compatible with $t\}$. It is clear that T_t is also a Miller tree. If $T_1 \leq T_0$ and each $t \in \mathrm{Split}(T_0)$ with $Lev(t, T_0) \leq k$ belongs to $\mathrm{Split}(T_1)$, where $k \in \omega$, then we write $T_1 \leq_k T_0$. It is easy to check (and is well-known) that if $T_{n+1} \leq_n T_n$ for all $n \in \omega$, then $\bigcap_{n\in\omega} T_n \in \mathbb{M}$.

We are now in a position to start the proof. Let M and $\{\phi_i : i \in \omega\}$ be as in the formulation of (†). We can additionally assume that for each $\phi \in \{\phi_i : i \in \omega\}$ there are infinitely many i such that $\phi = \phi_i$. Let $\{D_n : n \in \omega\}$ be the set of all open dense subsets of $\mathbb{M}$ which belong to M. Given $T_0 \in M \cap \mathbb{M}$, construct a sequence $\langle T_n : n \in \omega\rangle \in \mathbb{M}^\omega$ as follows: Assume that T_n has been constructed such that $(T_n)_t \in M$ for every $t \in T_n$ with $Lev(t, T_n) = n$. Given such a $t \in T_n$ and $k \in \omega$ such that $t^\frown k \in T_n$, find $R_{t,k} \leq \phi_n((T_n)_{t^\frown k})$ such that $R_{t,k} \in D_n \cap M$. Now set $T_{n+1} = \bigcup\{R_{t,k} : t \in T_k, Lev(t, T_n) = n, t^\frown k \in T_n\}$ and note that $T_{n+1} \leq_n T_n$ and $(T_{n+1})_r \in M$ for all $r \in T_{n+1}$ with $Lev(r, T_{n+1}) = n + 1$. This completes our construction. It is straightforward to check that $T = \bigcap_{n\in\omega} T_n$ is an $(M, \mathbb{M})$-generic condition forcing $\dot{G} \cap \phi_n[M \cap \mathbb{M}]$ to be infinite for all n. □

Finally we have all the necessary ingredients to complete the proof of Theorem 1.1. Let V be a model of GCH. By [13, Theorem 3.2] there exist γ-subspaces X, Y of 2^ω and a continuous map $\phi : X \times Y \to \omega^\omega$ such that $\phi[X \times Y]$ is dominating; i.e., for every $f \in \omega^\omega$ there exists $\langle x, y\rangle \in X \times Y$ such that $f \leq^* \phi\langle x, y\rangle$. (As usual, $f \leq^* g$ for $f, g \in \omega^\omega$ means that the set $\{n \in \omega : f(n) > g(n)\}$ is finite. Whenever we speak about unbounded or dominating subsets of ω^ω, we always mean with respect to $\leq^*$.) Let $\mathbb{P}$ be the iteration of $\mathbb{M}$ of length ω_2 with countable supports, and let G be $\mathbb{P}$-generic. It is well-known that $V \cap \omega^\omega$ is unbounded[1] in $V[G]$, and hence so is $\phi[X \times Y]$. By a result of Hurewicz [6] (see also [7, Theorem 4.3]) this implies that $X \times Y$ is not Hurewicz in $V[G]$. On the other hand, X and Y remain γ-spaces in $V[G]$ by a combination of Lemmata 2.2, 2.3, 2.4, and 2.5. This completes our proof.

[1]Even more is true: there exists an ultrafilter $\mathcal{U} \in V$ which remains a base for an ultrafilter in $V[G]$; namely, all P-points are like that (see [2]). It is easy to see that the set of enumerating functions of a base of an ultrafilter cannot be bounded.

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