

PRODUCTS OF HUREWICZ SPACES IN THE LAVER MODEL

DUŠAN REPOVŠ AND LYUBOMYR ZDOMSKYY

Abstract. This article is devoted to the interplay between forcing with fusion and combinatorial covering properties. We illustrate this interplay by proving that in the Laver model for the consistency of the Borel's conjecture, the product of any two metrizable spaces with the Hurewicz property has the Menger property.

§1. Introduction. A topological space X has the *Menger* property (or, alternatively, is a Menger space) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each \mathcal{V}_n is a finite subfamily of \mathcal{U}_n and the collection $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X . This property was introduced by Hurewicz, and the current name (the Menger property) is used because Hurewicz proved in [12] that for metrizable spaces his property is equivalent to one property of a base considered by Menger in [16]. If in the definition above we additionally require that $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ -cover of X (this means that the set $\{n \in \omega : x \notin \cup \mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the *Hurewicz property* introduced in [13]. Each σ -compact space is obviously a Hurewicz space, and Hurewicz spaces have the Menger property. Contrary to a conjecture of Hurewicz the class of metrizable spaces having the Hurewicz property appeared to be much wider than the class of σ -compact spaces [14, Theorem 5.1]. The properties of Menger and Hurewicz are classical examples of combinatorial covering properties of topological spaces which are nowadays also called selection principles. This is a growing area of general topology, see, e.g., [29]. For instance, Menger and Hurewicz spaces found applications in such areas as forcing [9], Ramsey theory in algebra [30], combinatorics of discrete subspaces [1], and Tukey relations between hyperspaces of compacts [10].

Even before the era of combinatorial covering properties, there was a lot of activity around the study of special sets of reals. These studies resolved many classical questions in general topology and measure theory. As a result, information about special sets of reals is included in standard topology textbooks, such as Kuratowski's *Topology*. The most influential survey on special sets of reals is, probably, Miller's chapter [18] in the *Handbook of Set-Theoretic*

Received April 24, 2016.

2010 *Mathematics Subject Classification*. Primary 03E35, 54D20, Secondary 54C50, 03E05.

Key words and phrases. Menger space, Hurewicz space, semifilter, Laver forcing.

© 2017, Association for Symbolic Logic
1079-8986/17/2303-0003
DOI:10.1017/bsl.2017.24

Topology. The most recent monograph on this topic is written by Bukovsky, see [7]. It complements nicely the classical book [4] of Bartoszyński and Judah. This theory still finds interesting applications in general topology, see, e.g., [11] for the interplay between λ -sets and homogeneity.

The theory of combinatorial covering properties, which originated in [14, 24], can be thought of as a continuation to that of special sets of reals, with emphasis on the behaviour of their open or Borel covers. Some combinatorial covering properties including the Menger and Hurewicz ones are about 15 years older than Gödel's works on L and 40 years older than the method of forcing, and they were introduced in the areas of topology where set-theoretic methods are quite rare even nowadays. E.g., the original idea behind the Menger property, as it is explicitly stated in the first paragraph of [16], was an application in dimension theory. However, since at least [15] it has become clear that the combinatorial covering properties are strongly influenced by axiomatics and hence can be studied with the help of forcing, see, e.g., [3, 8, 9, 19, 25] for the more recent works along these lines. There are equivalences among statements from disciplines with diverse origins (Ramsey theory, game theory, function spaces and convergence, topological groups, dimension theory, covering properties, combinatorial set theory, forcing, hyperspaces, filters, etc.) with combinatorial covering properties. Even though not all of these have found nontrivial applications so far (by translating into the other fields, via an equivalence, the results known for combinatorial covering properties), they are offering an alternative point of view onto the known properties and thus enhance their understanding. E.g., it is shown in [9] that a Mathias forcing associated to a filter \mathcal{F} on ω does not add dominating reals iff \mathcal{F} is Menger as a subspace of 2^ω , thus demonstrating that this property of filters is topological and in this way answering some questions for which it was unclear how the "standard" approaches in this area can be used.

One of the basic questions about a topological property is whether it is preserved by various kinds of products in certain classes of spaces. As usually, the preservation results may be divided into *positive*, asserting that properties under consideration are preserved by products (e.g., the classical Tychonoff theorem) and *negative* which are typically some constructions of spaces possessing certain property whose product fails to have it (e.g., the folklore fact that the Lindelöf property is not preserved even by squares, as witnessed by the Sorgenfrey line). In case of combinatorial covering properties we know that the strongest possible negative result is consistent: Under CH there exist $X, Y \subset \mathbb{R}$ which have the γ -space property with respect to countable Borel covers, whose product $X \times Y$ is not Menger, see [21, Theorem 3.2]. Thus the product of spaces with the strongest combinatorial covering property considered thus far might fail to have even the weakest one. This implies that no positive results for combinatorial covering properties can be obtained outright in ZFC. Unlike the vast majority of topological and combinatorial consequences under CH, the latter one does not follow from any equality among cardinal characteristics of the continuum, see the discussion on [21, p. 2882]. However, there are many other negative

results stating that under certain equality among cardinal characteristics (e.g., $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$, $\mathfrak{b} = \mathfrak{d}$, etc.¹) there are spaces $X, Y \subset \mathbb{R}$ with some combinatorial covering property such that $X \times Y$ is not Menger, see, e.g., [2, 22, 27].

Regarding the positive results, until recently the most unclear situation was with the Hurewicz property and the weaker ones. This was the main motivation for this article. There are two reasons why a product of Hurewicz spaces X, Y can fail to be Hurewicz/Menger. In the first place, $X \times Y$ may simply fail to be a Lindelöf space, i.e., it might have an open cover \mathcal{U} without countable subcover. Then $X \times Y$ is not even a Menger space. This may indeed happen: in ZFC there are two normal spaces X, Y with a covering property much stronger than the Hurewicz one such that $X \times Y$ does not have the Lindelöf property, see [28, Section 3]. However, the above situation becomes impossible if we restrict our attention to metrizable spaces. This second case, on which we concentrate in the sequel, turned out to be sensitive to the ambient set-theoretic universe: under CH there exists a Hurewicz space whose square is not Menger, see [14, Theorem 2.12]. The above result has been achieved by a transfinite construction of length ω_1 , using the combinatorics of the ideal of measure zero subsets of reals. This combinatorics turned out [27, Theorem 43] to require much weaker set-theoretic assumptions than CH. In particular, under the Martin Axiom there are Hurewicz subspaces of the irrationals whose product is not Menger.

The following theorem, which is the main result of this article, shows that an additional assumption in the results from [14, 27] mentioned above is really needed. In addition, it implies that the affirmative answer to [14, Problem 2] is consistent, see [29, Section 2] for the discussion of this problem.

THEOREM 1.1. *In the Laver model for the consistency of the Borel's conjecture, the product of any two Hurewicz spaces has the Menger property provided that it is a Lindelöf space. In particular, the product of any two Hurewicz metrizable spaces has the Menger property.*

This theorem seems to be the first “positive” consistency result related to the preservation by products of combinatorial covering properties weaker than the σ -compactness, in which no further restrictions² on the spaces are assumed. The proof is based on the analysis of continuous maps and names for reals in the model of set theory constructed in [15]. The question whether the product of Hurewicz metrizable spaces is a Hurewicz space in this model remains open. It is worth mentioning here that in the Cohen model there are Hurewicz subsets of \mathbb{R} whose product has the Menger property but fails to have the Hurewicz one, see [21, Theorem 6.6].

As suggested in its formulation, the model we use in Theorem 1.1 was invented by Laver in order to prove that Borel's conjecture is consistent, the

¹We refer the reader to [5] for the definitions and basic properties of cardinal characteristics of the continuum which are mentioned but are not used in the proofs in this article.

²The requirement that the product must be Lindelöf is vacuous for metrizable spaces. Let us note that nowadays the study of combinatorial covering properties concentrates mainly on sets of reals.

latter being the statement that every strong measure zero set is countable. A strong measure zero set is a subset A of the real line with the following property: for every sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive reals there exists a sequence $\langle I_n : n \in \omega \rangle$ of intervals such that $|I_n| < \varepsilon_n$ for all n and A is covered by the I_n 's. Here $|I_n|$ denotes the length of I_n . Obviously, every countable set is a strong measure zero set, and so is every union of countably many strong measure zero sets. Sierpiński proved in [26] that CH implies the existence of uncountable strong measure zero sets, i.e., the negation of Borel's conjecture. Combined with this Laver's result gave the independence of Borel's conjecture. This outstanding result was the first³ instance when a forcing, adding a real, was iterated with countable supports without collapsing cardinals. This work of Laver can be thought of as one of the motivations behind Baumgartner's axiom A and later Shelah's theory of proper forcing.

The conclusion of Theorem 1.1 does not follow from Borel's conjecture: If we add ω_2 many random reals over the Laver model then Borel's conjecture still holds by [4, Section 8.3.B] and we have $\text{cov}(\mathcal{N}) = \text{cof}(\mathcal{N})$, and hence in this model there exists a Hurewicz set of reals whose square is not Menger, see [27]. Thus Borel's conjecture is consistent with the existence of a Hurewicz set of reals with nonMenger square.

Theorem 1.1 seems to be an instance of a more general phenomena, namely that proper posets with fusion affect the behavior of combinatorial covering properties. This happens because sets of reals with certain combinatorial covering properties are forced to have a rather clear structure, which suffices to prove positive preservation results. For instance, the core of the proof of Theorem 1.1 is that Hurewicz subspaces of the real line are concentrated in a sense around their "simpler" subspaces in the Laver model, see Lemma 2.2. As a consequence of corresponding structural results we have proved [33] that the Menger property is preserved by finite products in the Miller model constructed in [17], and there are only \mathfrak{c} many Menger subspaces of \mathbb{R} in the Sacks model constructed in [23], see [10].

We believe that the interplay between forcing with fusion and combinatorial covering properties has many more instances and it is worth considering whether there is some deep reason behind it.

We assume that the reader is familiar with the basics of forcing as well as with standard proper posets used in the set theory of reals.

§2. Proof of Theorem 1.1. We shall first introduce a notion crucial for the proof of Theorem 1.1.

DEFINITION 2.1. A topological space X is called *weakly concentrated* if for every collection $\mathcal{Q} \subset [X]^\omega$ which is cofinal with respect to inclusion, and for every function $R : \mathcal{Q} \rightarrow \mathcal{P}(X)$ assigning to each $Q \in \mathcal{Q}$ a G_δ -set $R(Q)$ containing Q , there exists $Q_1 \in [\mathcal{Q}]^{\omega_1}$ such that $X \subset \bigcup_{Q \in Q_1} R(Q)$.

³According to our colleagues who worked in set theory already in the 70s.

The topology in $\mathcal{P}(\omega)$ is generated by the countable base $\mathcal{B} = \{[s, n] : s \in [\omega]^{<\omega}, n \in \omega\}$, where $[s, n] = \{x \subset \omega : x \cap n = s\}$. Thus any open subset O of $\mathcal{P}(\omega)$ may be identified with $B_O = \{\langle s, n \rangle \in [\omega]^{<\omega} \times \omega : [s, n] \subset O\}$, and vice versa, any $B \subset [\omega]^{<\omega} \times \omega$ gives rise to an open $O_B = \bigcup_{\langle s, n \rangle \in B} [s, n]$. Note that $B \subset B_{O_B}$ for all B . By a *code* for an F_σ subset F of $\mathcal{P}(\omega)$ we mean a sequence $\vec{B} = \langle B_n : n \in \omega \rangle$ of subsets of $[\omega]^{<\omega} \times \omega$ such that $F = \mathcal{P}(\omega) \setminus \bigcap_{n \in \omega} O_{B_n}$. Obviously each F_σ set $F \subset \mathcal{P}(\omega)$ has many codes in the sense of the definition above. For models $V \subsetneq V'$ of ZFC and an F_σ -subset $F \in V'$ of $\mathcal{P}(\omega)$ we say that F is *coded in V* if there exists a code for F which is an element of V . Note that being coded in V doesn't imply being a subset of V : $\mathcal{P}(\omega)$ has codes in V (e.g., $\langle \langle \emptyset, 0 \rangle : n \in \omega \rangle$) but it is not a subset of V as long as there are new reals in V' .

The consideration above also applies to other Polish spaces having a base which can be identified with some "simple" (e.g., constructive would suffice for our purposes) subset of $H(\omega)$, the family of all hereditarily finite sets. Among them are ω^ω , $\mathcal{P}(\omega) \times \omega^\omega$, etc. In particular, since every continuous function from an F_σ -subset F of $\mathcal{P}(\omega)$ to ω^ω is an F_σ -subset of $\mathcal{P}(\omega) \times \omega^\omega$, we may speak about such functions coded in V .

For a subset $X \in V'$ of $\mathcal{P}(\omega)$ and an F_σ -subset Y of X we shall say that Y is *coded in V* if there exists an F_σ -subset F of $\mathcal{P}(\omega)$ coded in V such that $Y = X \cap F$. Similarly, for continuous functions: $f : Y \rightarrow \omega^\omega$ is coded in V if there exists an F_σ -subset F of $\mathcal{P}(\omega)$ such that $Y = X \cap F$ and a continuous $\tilde{f} : F \rightarrow \omega^\omega$ coded in V , such that $f = \tilde{f} \upharpoonright Y$.

The following lemma is the key part of the proof of Theorem 1.1. Its proof is reminiscent of that of [19, Theorem 3.2]. We will use the notation from [15] with only differences being that smaller conditions in a forcing poset are supposed to carry more information about the generic filter, and the ground model is denoted by V .

A subset C of ω_2 is called an ω_1 -club if it is unbounded and for every $\alpha \in \omega_2$ of cofinality ω_1 , if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.

LEMMA 2.2. *In the Laver model every Hurewicz subspace of $\mathcal{P}(\omega)$ is weakly concentrated.*

PROOF. We work in $V[G_{\omega_2}]$, where G_{ω_2} is \mathbb{P}_{ω_2} -generic and \mathbb{P}_{ω_2} is the iteration of length ω_2 with countable supports of the Laver forcing, see [15] for details.

It is well known that a space $X \subset \mathcal{P}(\omega)$ is Hurewicz if and only if $f[X]$ is bounded with respect to \leq^* for every continuous $f : X \rightarrow \omega^\omega$, see [14, Theorem 4.4] or [13]. Let us fix a Hurewicz space $X \subset \mathcal{P}(\omega)$. The Hurewicz property is preserved by F_σ -subspaces because it is obviously preserved by closed subspaces and countable unions. Therefore there exists an ω_1 -club $C \subset \omega_2$ such that for every $\alpha \in C$ and continuous $f : F \rightarrow \omega^\omega$ coded in $V[G_\alpha]$, where F is an F_σ -subspace of X coded in $V[G_\alpha]$, there exists $b \in \omega^\omega \cap V[G_\alpha]$ such that $f(x) \leq^* b$ for all $x \in F$. Indeed, since for every $\alpha < \omega_2$ CH holds in $V[G_\alpha]$, there are at most ω_1 many pairs $\langle F, f \rangle$ such that F is an F_σ -subspace of X coded in $V[G_\alpha]$ and $f : F \rightarrow \omega^\omega$ is a continuous function coded in $V[G_\alpha]$. For every such pair find $\gamma_{F,f} < \omega_2$

and $b_{F,f} \in \omega^\omega \cap V[G_{\gamma_{F,f}}]$ such that $f(x) \leq^* b_{F,f}$ for all $x \in F$. Let $\gamma(\alpha)$ be the supremum of all the $\gamma_{F,f}$ for F, f as above. It is clear that the ω_1 -club $C \in V[G_{\omega_2}]$ of all α such that $\gamma(\beta) < \alpha$ for all $\beta < \alpha$ is as required.

Let $Q \subset [X]^\omega$ be cofinal with respect to the inclusion. Fix a function $R : Q \rightarrow \mathcal{P}(X)$ assigning to each $Q \in Q$ a G_δ -subset $R(Q)$ of $\mathcal{P}(\omega)$ containing Q . By a standard argument (see, e.g., the proof of [6, Lemma 5.10]) there exists an ω_1 -club $D \subset \omega_2$ such that $Q \cap V[G_\alpha] \in V[G_\alpha]$ and $R \upharpoonright (Q \cap V[G_\alpha]) \in V[G_\alpha]$ for⁴ all $\alpha \in D$. Moreover, using CH in the intermediate models as in the previous paragraph, we may also assume that for every $Q_0 \in [X \cap V[G_\alpha]]^\omega \cap V[G_\alpha]$ there exists $Q \in Q \cap V[G_\alpha]$ such that $Q_0 \subset Q$.

Let us fix $\alpha \in C \cap D$. We claim that $X \subset W$, where $W = \bigcup\{R(Q) : Q \in Q \cap V[G_\alpha]\}$. Suppose that, contrary to our claim, there exists $p \in G_{\omega_2}$ and a \mathbb{P}_{ω_2} -name \dot{x} such that $p \Vdash \dot{x} \in \dot{X} \setminus \dot{W}$. By [15, Lemma 11] there is no loss of generality in assuming that $\alpha = 0$. Applying [15, Lemma 14] to a sequence $\langle \dot{a}_i : i \in \omega \rangle$ such that $\dot{a}_i = \dot{x}$ for all $i \in \omega$, we get a condition $p' \leq p$ such that $p'(0) \leq^0 p(0)$, and a finite set U_s of reals for every $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, such that for each $\varepsilon > 0$, $s \in p'(0)$ with $p'(0)\langle 0 \rangle \leq s$, and for all but finitely many immediate successors t of s in $p'(0)$ we have

$$p'(0)_t \hat{\wedge} p' \upharpoonright [1, \omega_2] \Vdash \exists u \in U_s (|\dot{x} - u| < \varepsilon).$$

Fix $Q \in Q \cap V$ containing $X \cap \bigcup\{U_s : s \in p'(0), s \geq p'(0)\langle 0 \rangle\}$ and set $F = X \setminus R(Q)$. Note that F is an F_σ -subset of X coded in V . It follows that $p' \Vdash \dot{x} \in \dot{F}$ because p' is stronger than p that forces $\dot{x} \notin \dot{W} \supset \dot{X} \setminus \dot{F}$. Consider the map $f : F \rightarrow \omega^S$, where $S = \{s \in p'(0) : s \geq p'(0)\langle 0 \rangle\}$, defined as follows:

$$f(y)(s) = [1/\min\{|y - u| : u \in U_s\}] + 1$$

for⁵ all $s \in S$ and $y \in F$. Since F is disjoint from Q which contains all the U_s 's, f is well defined. Since both F and f are coded in V , there exists $b \in \omega^S \cap V$ such that $f(y) \leq^* b$ for all $y \in F$.

It follows from $p' \Vdash \dot{x} \in \dot{F}$ that $p' \Vdash \dot{f}(\dot{x}) \leq^* b$, and hence there exists $p'' \leq p'$ and a finite subset S_0 of S such that $p'' \Vdash \dot{f}(\dot{x})(s) \leq b(s)$ for all $S \setminus S_0$. By replacing p'' with $p''(0)_s \hat{\wedge} p'' \upharpoonright [1, \omega_2]$ for some $s \in p''(0)$, if necessary, we may additionally assume that $p''(0)\langle 0 \rangle \in S \setminus S_0$. Letting $s'' = p''(0)\langle 0 \rangle$, we conclude from the above that $p'' \Vdash \dot{f}(\dot{x})(s'') \leq b(s'')$, which means that

$$p'' \Vdash \min\{|\dot{x} - u| : u \in U_{s''}\} \geq 1/b(s'').$$

On the other hand, by our choice of p' and $p'' \leq p'$ we get that for all but finitely many immediate successors t of s'' in $p''(0)$ we have

$$p''(0)_t \hat{\wedge} p'' \upharpoonright [1, \omega_2] \Vdash \exists u \in U_{s''} |\dot{x} - u| < 1/b(s'')$$

which means $p''(0)_t \hat{\wedge} p'' \upharpoonright [1, \omega_2] \Vdash \min\{|\dot{x} - u| : u \in U_{s''}\} < 1/b(s'')$ and thus leads to a contradiction. ⊥

⁴Here by R we mean the map which assigns to a $Q \in Q$ some code of $\mathcal{P}(\omega) \setminus R(Q)$.

⁵Here $[a]$ is the largest integer not exceeding a .

A subset X of $\mathcal{P}(\omega)$ is called a λ -set if any $A \in [X]^\omega$ is a G_δ -subset of X . Obviously, every weakly concentrated λ -set has size $\leq \omega_1$. Therefore Lemma 2.2 implies [19, Theorem 3.2] because the property of a subset of $\mathcal{P}(\omega)$ considered in the latter theorem easily implies being both Hurewicz and a λ -set, see, e.g., the proof of [31, Theorem 5] for details.

The next lemma can probably be considered as folklore. We present its proof for the sake of completeness.

LEMMA 2.3. *Let $Y \subset \mathcal{P}(\omega)$ be Hurewicz and $Q \subset \mathcal{P}(\omega)$ countable. Then for every G_δ -subset O of $\mathcal{P}(\omega)^2$ containing $Q \times Y$ there exists a G_δ -subset $R \supset Q$ such that $R \times Y \subset O$.*

PROOF. Without loss of generality we shall assume that O is open. Let us write Q in the form $\{q_n : n \in \omega\}$ and set $O_n = \{z \in \mathcal{P}(\omega) : \langle q_n, z \rangle \in O\} \supset Y$. For every n find a cover \mathcal{U}_n of Y consisting of clopen subsets of $\mathcal{P}(\omega)$ contained in O_n . Let $\langle \mathcal{U}'_k : k \in \omega \rangle$ be a sequence of open covers of Y such that each \mathcal{U}_n appears in it infinitely often. Applying the Hurewicz property of Y we can find a sequence $\langle \mathcal{V}_k : k \in \omega \rangle$ such that $\mathcal{V}_k \in [\mathcal{U}'_k]^{<\omega}$ and $Y \subset \bigcup_{k \in \omega} Z_k$, where $Z_k = \bigcap_{m \geq k} \bigcup \mathcal{V}_m$. Note that each Z_k is compact and $Z_k \subset O_n$ for all $n \in \omega$ (because there exists $m \geq k$ such that $\mathcal{U}'_m = \mathcal{U}_n$, and then $Z_k \subset \bigcup \mathcal{V}_m \subset O_n$). Thus $Q \times Y \subset Q \times (\bigcup_{k \in \omega} Z_k) \subset O$. Since Z_k is compact, there exists for every k an open $R_k \supset Q$ such that $R_k \times Z_k \subset O$. Set $R = \bigcap_{k \in \omega} R_k$ and note that $R \supset Q$ and $R \times Y \subset R \times \bigcup_{k \in \omega} Z_k \subset O$. \dashv

Let A be a countable set and $x, y \in \omega^A$. As usually, $x \leq^* y$ means that $\{a \in A : x(a) > y(a)\}$ is finite. The smallest cardinality of an unbounded with respect to \leq^* subset of ω^ω is denoted by \mathfrak{b} . It is well known that $\omega_1 < \mathfrak{b}$ in the Laver model, see [5] for this fact as well as systematic treatment of cardinal characteristics of reals.

The second part of Theorem 1.1 is a direct consequence of Lemma 2.2 and the following.

PROPOSITION 2.4. *Suppose that $\mathfrak{b} > \omega_1$. Let $Y \subset \mathcal{P}(\omega)$ be a Hurewicz space and $X \subset \mathcal{P}(\omega)$ weakly concentrated. Then $X \times Y$ is Menger.*

PROOF. Fix a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers of $X \times Y$ by clopen subsets of $\mathcal{P}(\omega)^2$. For every $Q \in [X]^\omega$ fix a sequence $\langle \mathcal{W}_n^Q : n \in \omega \rangle$ such that $\mathcal{W}_n^Q \in [\mathcal{U}_n]^{<\omega}$ and $Q \times Y \subset \bigcap_{n \in \omega} \bigcup_{m \geq n} \bigcup \mathcal{W}_m^Q$. Letting $O_Q = \bigcap_{n \in \omega} \bigcup_{m \geq n} \bigcup \mathcal{W}_m^Q$ and using Lemma 2.3, we can find a G_δ -subset $R_Q \supset Q$ such that $R_Q \times Y \subset O_Q$. Since X is weakly concentrated, there exists $Q \subset [X]^\omega$ of size $|Q| = \omega_1$ such that $R = \bigcup \{R_Q : Q \in Q\}$ contains X as a subset. Let us fix $x \in X$ and find $Q \in Q$ such that $x \in R_Q$. Then $\{x\} \times Y \subset R_Q \times Y \subset O_Q$. Therefore for every $\langle x, y \rangle \in X \times Y$ there exists $Q \in Q$ such that $\langle x, y \rangle \in O_Q = \bigcap_{n \in \omega} \bigcup_{m \geq n} \bigcup \mathcal{W}_m^Q$. Let us write \mathcal{U}_n in the form $\{U_k^n : k \in \omega\}$ and for every $Q \in Q$ fix a real $b_Q \in \omega^\omega$ with the property $\mathcal{W}_n^Q \subset \{U_k^n : k \leq b_Q(n)\}$. Since $|Q| = \omega_1 < \mathfrak{b}$, there exists $b \in \omega^\omega$ such that $b_Q \leq^* b$ for all $Q \in Q$. It follows from the above that $X \times Y \subset \bigcup_{n \in \omega} \bigcup_{k \leq b(n)} U_k^n$, which completes our proof. \dashv

A family $\mathcal{F} \subset [\omega]^\omega$ is called a *semifilter* if for every $F \in \mathcal{F}$ and $X \subset \omega$, if $|F \setminus X| < \omega$ then $X \in \mathcal{F}$.

The proof of the first part of Theorem 1.1 uses characterizations of the properties of Hurewicz and Menger obtained in [32]. Let $u = \langle U_n : n \in \omega \rangle$ be a sequence of subsets of a set X . For every $x \in X$ let $I_s(x, u, X) = \{n \in \omega : x \in U_n\}$. If every $I_s(x, u, X)$ is infinite (the collection of all such sequences u will be denoted by $\Lambda_s(X)$), then we shall denote by $\mathcal{U}_s(u, X)$ the smallest semifilter on ω containing all $I_s(x, u, X)$. By [32, Theorem 3], a Lindelöf topological space X is Menger (Hurewicz) if and only if for every $u \in \Lambda_s(X)$ consisting of open sets, the semifilter $\mathcal{U}_s(u, X)$ is Menger (Hurewicz). The proof given there also works if we consider only those $\langle U_n : n \in \omega \rangle \in \Lambda_s(X)$ such that all U_n 's belong to a given base of X .

PROOF OF THEOREM 1.1. Suppose that X, Y are Hurewicz spaces such that $X \times Y$ is Lindelöf and fix $w = \langle U_n \times V_n : n \in \omega \rangle \in \Lambda_s(X \times Y)$ consisting of open sets. Set $u = \langle U_n : n \in \omega \rangle, v = \langle V_n : n \in \omega \rangle$, and note that $u \in \Lambda_s(X)$ and $v \in \Lambda_s(Y)$. It is easy to see that

$$\mathcal{U}_s(w, X \times Y) = \{A \cap B : A \in \mathcal{U}_s(u, X), B \in \mathcal{U}_s(v, Y)\},$$

and hence $\mathcal{U}_s(w, X \times Y)$ is a continuous image of $\mathcal{U}_s(u, X) \times \mathcal{U}_s(v, Y)$. By [32, Theorem 3] both of latter ones are Hurewicz, considered as subspaces of $\mathcal{P}(\omega)$, and hence their product is a Menger space by Proposition 2.4 and Lemma 2.2. Thus $\mathcal{U}_s(w, X \times Y)$ is Menger, being a continuous image of a Menger space. It now suffices to use [32, Theorem 3] again. \dashv

§3. Acknowledgments. We would like to thank Marion Scheepers for fruitful discussions of products of Hurewicz spaces in various models of Borels's conjecture. We also thank Boaz Tsaban, as well as the referees, for many comments and suggestions which improved our presentation. The first author was partially supported by the Slovenian Research Agency grant P1-0292. The second author would like to thank the Austrian Science Fund FWF (Grants I 1209-N25 and I 2374-N35) for generous support of this research.

REFERENCES

- [1] L. F. AURICHI, *D-spaces, topological games, and selection principles*. *Topology Proceedings*, vol. 36 (2010), pp. 107–122.
- [2] L. BABINKOSTOVA, *On some questions about selective separability*. *Mathematical Logic Quarterly*, 55 (2009), pp. 539–541.
- [3] D. BARMAN and A. DOW, *Proper forcing axiom and selective separability*. *Topology and its Applications*, vol. 159 (2012), pp. 806–813.
- [4] T. BARTOSZYŃSKI and H. JUDAH, *Set Theory. On the Structure of the Real Line*, A. K. Peters, Ltd., Wellesley, MA, 1995.
- [5] A. BLASS, *Combinatorial cardinal characteristics of the continuum*, *Handbook of Set Theory* (M. Foreman, A. Kanamori, and M. Magidor, editors), Springer, Dordrecht, 2010, pp. 395–491.
- [6] A. BLASS and S. SHELAH, *There may be simple P_{\aleph_1} - and P_{\aleph_2} -points and the Rudin-Keisler ordering may be downward directed*. *Annals of Pure and Applied Logic*, vol. 33 (1987), pp. 213–243.

- [7] L. BUKOVSKY, *The Structure of the Real Line*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series), 71, Birkhäuser/Springer Basel AG, Basel, 2011.
- [8] D. CHODOUNSKY, O. GUZMAN, and M. HRUŠÁK, *Mathias-Prikry and Laver type forcing; Summable ideals, coideals, and $+$ -selective filters*. *Archive for Mathematical Logic*, vol. 55 (2016), pp. 493–504.
- [9] D. CHODOUNSKÝ, D. REPOVŠ, and L. ZDOMSKYY, *Mathias forcing and combinatorial covering properties of filters*. *The Journal of Symbolic Logic*, vol. 80 (2015), pp. 1398–1410.
- [10] P. GARTSIDE, A. MEDINI, and L. ZDOMSKYY, *The Tukey Order, Hyperspaces, and Selection Principles*, work in progress.
- [11] M. HRUŠÁK and J. VAN MILL, *The existence of a meager in itself CDH metric space is independent of ZFC*. *Proceedings of the American Mathematical Society*, doi:10.1090/proc/13434.
- [12] W. HUREWICZ, *Über die Verallgemeinerung des Borelschen Theorems*. *Mathematische Zeitschrift*, vol. 24 (1925), pp. 401–421.
- [13] ———, *Über Folgen stetiger Funktionen*. *Fundamenta Mathematicae*, vol. 9 (1927), pp. 193–204.
- [14] W. JUST, A. W. MILLER, M. SCHEEPERS, and P. J. SZEPTYCKI, *The combinatorics of open covers. II. Topology and its Applications*, vol. 73 (1996), pp. 241–266.
- [15] R. LAVER, *On the consistency of Borel's conjecture*. *Acta Mathematica*, vol. 137 (1976), pp. 151–169.
- [16] K. MENGER, *Einige Überdeckungssätze der Punktmengenlehre*. *Sitzungsberichte. Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie)*, vol. 133 (1924), pp. 421–444.
- [17] A. MILLER, *Rational perfect set forcing*, *Axiomatic Set Theory* (J. Baumgartner, D. A. Martin, and S. Shelah, editors), Contemporary Mathematics, vol. 31, American Mathematical Society, Providence, RI, 1984, pp. 143–159.
- [18] A. W. MILLER, *Special subsets of the real line*, *Handbook of Set-Theoretic Topology* (K. Kunen and J. E. Vaughan, editors), North Holland, Amsterdam, 1984, pp. 201–233.
- [19] A. W. MILLER and B. TSABAN, *Point-cofinite covers in the Laver model*. *Proceedings of the American Mathematical Society*, vol. 138 (2010), pp. 3313–3321.
- [20] ———, *Selective covering properties of product spaces*. *Annals of Pure and Applied Logic*, vol. 165 (2014), pp. 1034–1057.
- [21] A. W. MILLER, B. TSABAN, and L. ZDOMSKYY, *Selective covering properties of product spaces, II: Gamma spaces*. *Transactions of the American Mathematical Society*, vol. 368 (2016), pp. 2865–2889.
- [22] D. REPOVŠ and L. ZDOMSKYY, *On M -separability of countable spaces and function spaces*. *Topology and its Applications*, vol. 157 (2010), pp. 2538–2541.
- [23] G. E. SACKS, *Forcing with perfect closed sets*, *Axiomatic Set Theory* (D. Scott, editor), Proceedings of the Symposium on Pure Mathematics, Vol. XIII, Part I, American Mathematical Society, Providence, RI, 1971, pp. 331–355.
- [24] M. SCHEEPERS, *Combinatorics of open covers. I. Ramsey theory*. *Topology and its Applications*, vol. 69 (1996), pp. 31–62.
- [25] M. SCHEEPERS and F. TALL, *Lindelöf indestructibility, topological games and selection principles*. *Fundamenta Mathematicae*, vol. 210 (2010), pp. 1–46.
- [26] W. SIERPIŃSKI, *Sur un ensemble non dénombrable, dont toute image continue est de mesure nulle*. *Fundamenta Mathematicae*, vol. 11 (1928), pp. 302–303.
- [27] M. SCHEEPERS and B. TSABAN, *The combinatorics of Borel covers*. *Topology and its Applications*, vol. 121 (2002), pp. 357–382.
- [28] S. TODORČEVIĆ, *Aronszajn orderings*. *Djuo Kurepa memorial volume. Publications de l'Institut Mathématique (Beograd)*, vol. 57 (1995), no. 71, pp. 29–46.
- [29] B. TSABAN, *Selection principles and special sets of reals*, *Open Problems in Topology II* (E. Pearl, editor), Elsevier Science Publishing, Amsterdam, 2007, pp. 91–108.
- [30] ———, *Algebra, selections, and additive Ramsey theory*, preprint, 2015, <http://arxiv.org/pdf/1407.7437.pdf>.

- [31] B. TSABAN and L. ZDOMSKYY, *Hereditarily Hurewicz spaces and Arhangel'ski sheaf amalgamations*. *Journal of the European Mathematical Society*, vol. 14 (2012), pp. 353–372.
- [32] L. ZDOMSKYY, *A semifilter approach to selection principles*. *Commentationes Mathematicae Universitatis Carolinae*, vol. 46 (2005), pp. 525–539.
- [33] ———, *Products of Menger spaces in the Miller model*, preprint, <http://www.logic.univie.ac.at/~lzdomsky/>.

FACULTY OF EDUCATION, AND FACULTY OF MATHEMATICS AND PHYSICS
UNIVERSITY OF LJUBLJANA
LJUBLJANA 1000, SLOVENIA

E-mail: dusan.repovs@guest.arnes.si

URL: <http://www.fmf.uni-lj.si/~repovs/index.htm>

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA, WÄHRINGER STRASSE 25
A-1090 WIEN, AUSTRIA

E-mail: lzdomsky@gmail.com

URL: <http://www.logic.univie.ac.at/~lzdomsky/>