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Productively Lindelöf spaces and the covering property of Hurewicz ☆

Dušan Repovš [a,*], Lyubomyr Zdomskyy [b]

[a] *Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana, PO Box 2964, Ljubljana, 1001, Slovenia*
[b] *Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria*

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ABSTRACT

We prove that under certain set-theoretic assumptions every productively Lindelöf space has the Hurewicz covering property, thus improving upon some earlier results of Aurichi and Tall.

1. Introduction

A topological space X is called *productively Lindelöf* if $X \times Y$ is Lindelöf for every Lindelöf space Y. This terminology was introduced in [5], but the concept itself goes back at least to the classical work of Michael [14] who proved that under CH the space of irrational numbers is not productively Lindelöf. The natural question of whether an additional set-theoretic hypothesis is needed here has become known as *Michael's problem* and is still open. Thus we are at the moment far from a satisfactory understanding of productive Lindelöfness, even for subspaces of the Baire space ω^ω.

In a stream of recent papers of Tall and collaborators it was proven that under certain equalities between cardinal characteristics all metrizable productively Lindelöf spaces have strong covering properties close to the σ-compactness. In modern terminology such covering properties are called *selection principles* and constitute a rapidly growing area of general topology (see e.g., [21]).

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* Corresponding author.
E-mail addresses: dusan.repovs@guest.arnes.si (D. Repovš), lzdomsky@gmail.com (L. Zdomskyy).
URLs: http://www.fmf.uni-lj.si/~repovs/index.htm (D. Repovš), http://www.logic.univie.ac.at/~lzdomsky/ (L. Zdomskyy).

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Trying to describe σ-compactness in terms of open covers, Hurewicz [10] introduced the following property, nowadays called *the Menger property*, which was historically the first selection principle: a topological space X is said to have this property if for every sequence $\langle\mathcal{U}_n\colon n\in\omega\rangle$ of open covers of X there exists a sequence $\langle\mathcal{V}_n\colon n\in\omega\rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\bigcup\mathcal{V}_n\colon n\in\omega\}$ is a cover of X. The current name (the Menger property) is used because Hurewicz proved in [10] that for metrizable spaces his property is equivalent to one basis property considered by Menger in [13], see [4] for more information on combinatorial properties of bases. If in the definition above we additionally require that $\{\bigcup\mathcal{V}_n\colon n\in\omega\}$ is a γ-*cover* of X (this means that the set $\{n\in\omega\colon x\notin\bigcup\mathcal{V}_n\}$ is finite for each $x\in X$), then we obtain the definition of the Hurewicz covering property introduced in [11]. Contrary to a conjecture of Hurewicz the class of metrizable spaces having the Hurewicz property appeared to be much wider than the class of σ-compact spaces [12, Theorem 5.1] (see also [6,21]).

By [3, Theorem 23] and [19, Theorem 18], every productively Lindelöf space has the Hurewicz property if $\mathfrak{d}=\omega_1$ or $\mathrm{add}(\mathcal{M})=\mathfrak{c}$. The following theorem implies both of these results.

Theorem 1.1. *If* $\mathrm{add}(\mathcal{M})=\mathfrak{d}$*, then every productively Lindelöf space has the Hurewicz property.*

If $\mathfrak{b}=\omega_1$, then by [1, Corollary 4.5] every productively Lindelöf space has the Menger property. The following result shows that with a help of an additional combinatorial assumption about filters on ω we can actually infer the Hurewicz property.

Theorem 1.2. *If* $\mathfrak{b}=\omega_1$ *and the Filter Dichotomy holds, then every productively Lindelöf space has the Hurewicz property.*

The *Filter Dichotomy* is the statement that for any non-meager filters $\mathcal{F},\mathcal{G}$ on ω there exists a monotone surjection $\phi:\omega\to\omega$ such that $\phi(\mathcal{F})=\phi(\mathcal{G})$. Here we consider filters on ω with the topology inherited from $\mathcal{P}(\omega)$, the latter being identified with the Cantor space 2^ω via characteristic functions.

By [8, Theorems 1,2], the Filter Dichotomy (abbreviated as FD in the sequel) holds in the Miller model, and hence the premises of Theorem 1.2 do not imply those of Theorem 1.1, for the values of cardinal characteristics in some standard iteration models see [7, p. 480].

It has been noted in [20] that the three progressively weaker hypotheses: *CH*, $\mathfrak{d}=\omega_1$, and ω^ω *is not productively Lindelöf*, imply the respectively weaker conclusions about metrizable productively Lindelöf spaces: σ-*compact, the Hurewicz property*, and *the Menger property*. In [20, Problem 3.13] it is asked whether the stronger hypotheses are necessary in order to obtain the stronger conclusions. Theorems 1.1 and 1.2 may be thought of as a tiny step towards the solution of this problem.

In these results we do not assume that X satisfies any separation axioms. This generality is achieved with the help of set-valued maps, which by the methods developed in [23] lead to a reduction to subspaces of the Baire space. For the definitions of cardinal characteristics used in this paper we refer the reader to [7] or [22].

2. Proofs

Theorem 1.1 will be proved by adding "an ε" to the following deep result:

Theorem 2.1. *Let X be a topological space which admits a compactification whose remainder is Lindelöf. If there exists a cardinal κ of uncountable cofinality and an increasing sequence $\langle X_\alpha\colon\alpha<\kappa\rangle$ of principal subsets of X such that $\bigcup_{\alpha<\kappa}X_\alpha=X$, for every compact $K\subset X$ there exists an ordinal α such that $K\subset X_\alpha$, and the minimal ordinal with this property has countable cofinality, then X is not productively Lindelöf.*

Theorem 2.1 can be proved by almost verbatim repetition of a part of the proof of [15, Theorem 1.2].

Proof of Theorem 1.1. Given a productively Lindelöf space X, we shall show that it has the Hurewicz property.

Combining [23, Lemma 1 and Theorem 2] we conclude that X has the Hurewicz property if and only if all images of X under compact-valued upper semicontinuous maps $\Phi : X \Rightarrow \omega^\omega$ have it. Since any such image of X is productively Lindelöf, it is enough to show that productively Lindelöf subspaces of ω^ω have the Hurewicz property. Therefore we shall assume that $X \subset \omega^\omega$. Suppose to the contrary that X does not have the Hurewicz property. Using [12, Theorem 4.3] and passing to a homeomorphic copy of X, if necessary, we may additionally assume that X is unbounded with respect to $\leqslant^*$. The proof will be completed as soon as we derive a contradiction with $\mathrm{add}(\mathcal{M}) = \mathfrak{d}$.

Let $D = \{d_\alpha\colon \alpha < \mathfrak{d}\}$ be a dominating family. Since $\mathrm{add}(\mathcal{M}) \leqslant \mathfrak{b} \leqslant \mathfrak{d}$ [22], we conclude that $\mathfrak{b} = \mathfrak{d}$. For every $\alpha < \mathfrak{b}$ set $X_\alpha = \{x \in X\colon x \leqslant^* d_\xi \text{ for some } \xi < \alpha\}$. Since X is unbounded and D is dominating, $X_\alpha \neq X$ for all α and $\bigcup_{\alpha<\mathfrak{d}} X_\alpha = X$. Given a compact $K \subset X$ we can find $\alpha < \mathfrak{d}$ such that all elements of K are bounded by f_α, and hence $K \subset X_\alpha$. Observe that each X_α is a union of a family $\mathcal{V}_\alpha$ of fewer than $\mathrm{cov}(\mathcal{M})$ many closed subsets of X, where

$$\mathcal{V}_\alpha = \bigl\{\bigl\{x \in X\colon x(k) \leqslant d_\xi(k) \text{ for all } k \geqslant n\bigr\}\colon \xi < \alpha,\ n \in \omega\bigr\}.$$

By the same argument as in the proof of [2, Lemma 2] we can show that there exists a countable subfamily $\mathcal{V}'$ of $\mathcal{V}_\alpha$ covering K. It follows from the above that the minimal ordinal β such that $K \subset X_\beta$ has countably cofinality. Therefore the sequence $\langle X_\alpha\colon \alpha < \mathfrak{d}\rangle$ satisfies the premises of Theorem 2.1, and hence X is not productively Lindelöf. □

Theorem 1.2 is a direct consequence of Theorem 2.6 below, where the FD is weakened to the following assumption:

(∗) For every non-meager filter $\mathcal{G}$ there exists an unbounded tower $\mathcal{T}$ of cardinality $\mathfrak{b}$ and a monotone surjection $\phi : \omega \to \omega$ such that $\phi(\mathcal{T}) \subset \phi(\mathcal{G})$.

We recall that a *tower of cardinality* κ is a set $\mathcal{T} \subset [\omega]^\omega$ which can be enumerated as $\{T_\alpha\colon \alpha < \kappa\}$, such that for all $\alpha < \beta < \kappa$, $T_\beta \subset^* T_\alpha$ and $T_\alpha \not\subset^* T_\beta$, where $A \subset^* B$ means $|A \setminus B| < \omega$. An *unbounded tower of cardinality* κ is an unbounded with respect to $\leqslant^*$ set $\mathcal{T} \subset [\omega]^\omega$ which is a tower of cardinality κ (here we identify each element of $[\omega]^\omega$ with its increasing enumeration). It is an easy exercise to show that $\mathfrak{t} = \mathfrak{b}$ if and only if there is an unbounded tower of cardinality $\mathfrak{t}$.

We shall use the following fundamental result of Talagrand [18].

Theorem 2.2. *A filter $\mathcal{F}$ is meager if and only if there exists an increasing sequence $\langle n_k\colon k \in \omega\rangle$ of natural numbers such that each $F \in \mathcal{F}$ meets all but finitely many intervals $[n_k, n_{k+1})$.*

The following lemma implies that Theorem 2.6 is indeed an improvement of Theorem 1.2.

Lemma 2.3. *If $\mathfrak{b} = \mathfrak{t}$ and the FD holds, then (∗) holds as well.*

Proof. Let $\mathcal{T}$ be an unbounded tower of cardinality $\mathfrak{b}$ and $\mathcal{F}$ be a non-meager filter. The FD yields a monotone surjection $\phi : \omega \to \omega$ such that $\phi(\langle\mathcal{T}\rangle) = \phi(\mathcal{F})$. Then $\phi(\mathcal{T}) \subset \phi(\mathcal{F})$ and $\phi(\mathcal{T})$ is an unbounded tower of cardinality $\mathfrak{b}$. □

A set $S = \{f_\alpha\colon \alpha < \mathfrak{b}\}$ is called a $\mathfrak{b}$-*scale* if $S \subset \omega^\omega$, all elements of S are increasing, S is unbounded with respect to $\leqslant^*$, and $f_\alpha \leqslant^* f_\beta$ for each $\alpha < \beta < \mathfrak{b}$. It is easy to see that a $\mathfrak{b}$-scale always exists.

Let $\chi : [\omega]^\omega \to \omega^\omega$ be the map assigning to each infinite subset A of ω its enumeration $e_A \in \omega^\omega$ (i.e., $e_A(n)$ is the nth element of A). Then χ is an embedding of $[\omega]^\omega$ into ω^ω which maps every unbounded tower of cardinality $\mathfrak{b}$ onto a $\mathfrak{b}$-scale. It is a direct consequence of [1, Corollary 2.5] that if $\mathfrak{b} = \omega_1$ and X is a productively Lindelöf subspace of ω^ω, then $B \not\subset X$ for any $\mathfrak{b}$-scale B. As a corollary we get the following

Lemma 2.4. *If $\mathfrak{b} = \omega_1$ and X is a productively Lindelöf subspace of $[\omega]^\omega$, then $\mathcal{T} \not\subset X$ for any unbounded tower $\mathcal{T}$.*

In the proof of Theorem 2.6 we shall use set-valued maps, see [16]. By a *set-valued map Φ* from a set X into a set Y we understand a map from X into the power-set $\mathcal{P}(Y)$ of Y and write $\Phi : X \Rightarrow Y$. For a subset A of X we define $\Phi(A) = \bigcup_{x \in A} \Phi(x) \subset Y$. A set-valued map Φ from a topological spaces X to a topological space Y is said to be

- *compact-valued*, if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous*, if for every open subset V of Y the set $\Phi_\subset^{-1}(V) = \{x \in X: \Phi(x) \subset V\}$ is open in X.

A family $\mathcal{F} \subset [\omega]^\omega$ is called *centered*, if $\bigcap \mathcal{F}_1 \in [\omega]^\omega$ for every $\mathcal{F}_1 \in [\mathcal{F}]^{<\omega}$. For a centered family $\mathcal{F}$ we shall denote by $\langle \mathcal{F} \rangle$ the smallest non-principal filter on ω containing $\mathcal{F}$. In other words, $\langle \mathcal{F} \rangle = \{A \subset \omega: \bigcap \mathcal{F}_1 \subset^* A$ for some $\mathcal{F}_1 \in [\mathcal{F}]^{<\omega}\}$.

The following easy lemma is a direct consequence of [17, Claim 3.2(2)].

Lemma 2.5. *If a centered family $\mathcal{F}$ is an image of a topological space X under a compact-valued upper semicontinuous map, then $\langle \mathcal{F} \rangle$ is a countable union of images of finite powers of X under compact-valued upper semicontinuous maps.*

Let $\mathcal{U}$ be a family of subsets of a set X. A subset A of X is called *$\mathcal{U}$-bounded* if $A \subset \bigcup \mathcal{V}$ for some finite $\mathcal{V} \subset \mathcal{U}$. $\mathcal{U}$ is called an *ω-cover* of X if $X \notin \mathcal{U}$ and for every finite $F \subset X$ there exists $U \in \mathcal{U}$ such that $F \subset U$.

Theorem 2.6. *If $\mathfrak{b} = \omega_1$ and the assumption $(*)$ from above holds, then every productively Lindelöf space has the Hurewicz property.*

Proof. Let X be a productively Lindelöf space and let $\langle \mathcal{U}_n: n \in \omega \rangle$ be a sequence of open covers of X. Let us write $\mathcal{U}_n$ in the form $\{U_k^n: k \in \omega\}$. Observe that there is no loss of generality in assuming $U_k^{n+1} \subset U_k^n \subset U_{k+1}^n$ for all $n, k \in \omega$.

The equality $\mathfrak{b} = \omega_1$ implies that ω^ω is not productively Lindelöf [9, Remark 10.5], and hence by [17, Proposition 3.1] all productively Lindelöf spaces have the Menger property. Since the class of productively Lindelöf spaces is closed under finite products, we conclude that all finite powers of X have the Menger property. By [12, Theorem 3.9] there exists a sequence $\langle \mathcal{V}_n: n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for all n and $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$ is an ω-cover of X. It follows from our assumptions on $\mathcal{U}_n$'s that we may assume $|\mathcal{V}_n| = 1$ for all $n \in \omega$, i.e., $\mathcal{V}_n = \{U_{k_n}^n\}$ for some $k_n \in \omega$.

For every $x \in X$ we shall denote by $I_\mathcal{V}(x)$ the set $\{n \in \omega: x \in U_{k_n}^n\}$. By [23, Lemma 2] the set

$$\mathcal{F} = \{A \subset \omega: I_\mathcal{V}(x) \subset^* A \text{ for some } x \in X\}$$

is a countable union of images of X under compact-valued upper semicontinuous maps. Therefore by Lemma 2.5 $\mathcal{G} = \langle \mathcal{F} \rangle$ is a countable union of images of finite powers of X under compact-valued upper semicontinuous maps. Since $\mathcal{V}$ is an ω-cover of X, we conclude that $\mathcal{G} \subset [\omega]^\omega$. By the methods of [17, § 3] it also follows that $\mathcal{G}$ is productively Lindelöf. Two cases are possible.

1. $\mathcal{G}$ is meager. Then by Theorem 2.2 there exists an increasing sequence $\langle m_n\colon n \in \omega\rangle$ of integers such that every element of $\mathcal{U}$ meets all but finitely many intervals $[m_n, m_{n+1})$. It suffices to observe that $U_n = \bigcup\{U^m_{k_m}\colon m \in [m_n, m_{n+1})\}$ is $\mathcal{U}_n$-bounded for every $n \in \omega$ and the sequence $\langle U_n\colon n \in \omega\rangle$ is a γ-cover of X.

2. $\mathcal{G}$ is non-meager. By the assumption $(*)$ from above we can find a monotone surjection $\phi : \omega \to \omega$ and an unbounded tower $\mathcal{T} \subset [\omega]^\omega$ of cardinality $\mathfrak{b}$ such that $\phi(\mathcal{U}) \supset \mathcal{T}$, which contradicts Lemma 2.4 and thus completes our proof. □

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