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ON EXISTENCE OF PI-EXPONENTS OF UNITAL ALGEBRAS

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Abstract. We construct a family of unital non-associative algebras $\{T_\alpha |\ 2 < \alpha \in \mathbb{R}\}$ such that $\underline{exp}(T_\alpha) = 2$, whereas $\alpha \le \overline{exp}(T_\alpha) \le \alpha + 1$. In particular, it follows that ordinary PI-exponent of codimension growth of algebra T_α does not exist for any $\alpha > 2$. This is the first example of a unital algebra whose PI-exponent does not exist.

1. **Introduction.** We consider numerical invariants associated with polynomial identities of algebras over a field of characteristic zero. Given an algebra A, one can construct a sequence of non-negative integers $\{c_n(A)\}, n = 1, 2, \ldots$, called the *codimensions* of A, which is an important numerical characteristic of identical relations of A. In general, the sequence $\{c_n(A)\}$ grows faster than $n!$. However, there is a wide class of algebras with exponentially bounded codimension growth. This class includes all associative PI-algebras [2], all finite-dimensional algebras [2], Kac-Moody algebras [12], infinite-dimensional simple Lie algebras of Cartan type [9], and many others. If the sequence $\{c_n(A)\}$ is exponentially bounded then the following natural question arises: does the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)} \tag{1.1}$$

exist and what are its possible values? In case of existence, the limit (1.1) is called the PI-*exponent of* A, denoted as $exp(A)$. At the end of 1980's, Amitsur conjectured that for any associative PI-algebra, the limit (1.1) exists and is a non-negative integer. Amitsur's conjecture was confirmed in [5, 6]. Later, Amitsur's conjecture was also confirmed for finite-dimensional Lie and Jordan algebras [4, 15]. Existence of $exp(A)$ was also proved for all finite-dimensional simple algebras [8] and many others.

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Nevertheless, the answer to Amitsur's question in the general case is negative: a counterexample was presented in [14]. Namely, for any real $\alpha > 1$, an algebra R_α was constructed such that the lower limit of $\sqrt[n]{c_n(A)}$ is equal to 1, whereas the upper limit is equal to α. It now looks natural to describe classes of algebras in which for any algebra A, its PI-exponent $exp(A)$ exists. One of the candidates is the class of all finite-dimensional algebras. Another one is the class of so-called special Lie algebras. The next interesting class consists of unital algebras, it contains in particular, all algebras with an external unit. Given an algebra A, we denote by $A^\sharp$ the algebra obtained from A by adjoining the external unit. There is a number of papers where the existence of $exp(A^\sharp)$ has been proved, provided that $exp(A)$ exists [11, 16, 17]. Moreover, in all these cases, $exp(A^\sharp) = exp(A) + 1$.

The main goal of the present paper is to construct a series of unital algebras such that $exp(A)$ does not exist, although the sequence $\{c_n(A)$ is exponentially bounded (see Theorem 3.1 and Corollary 3.1 below). All details about polynomial identities and their numerical characteristics can be found in [1, 3, 7].

2. **Definitions and preliminary structures.** Let A be an algebra over a field F and let $F\{X\}$ be a free F-algebra with an infinite set X of free generators. The set $Id(A) \subset F\{X\}$ of all identities of A forms an ideal of $F\{X\}$. Denote by $P_n = P_n(x_1, \ldots, x_n)$ the subspace of $F\{X\}$ of all multilinear polynomials on $x_1, \ldots, x_n \in X$. Then $P_n \cap Id(A)$ is actually the set of all multilinear identities of A of degree n. An important numerical characteristic of $Id(A)$ is the sequence of non-negative integers $\{c_n(A)\}, n = 1, 2, \ldots$, where

$$c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}.$$

If the sequence $\{c_n(A)\}$ is exponentially bounded, then the lower and the upper PI-exponents of A, defined as follows

$$\underline{exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)},$$

are well-defined. An existence of ordinary PI-exponent (1.1) is equivalent to the equality $\underline{exp}(A) = \overline{exp}(A)$.

In [14], an algebra $R = R(\alpha)$ such that $\underline{exp}(R) = 1, \overline{exp}(R) = \alpha$, was constructed for any real $\alpha > 0$. Slightly modifying the construction from [14], we want to get for any real $\alpha > 2$, an algebra R_α with $\underline{exp}(R_\alpha)^\sharp = 2$ and $\alpha \le \overline{exp}(R^\sharp) \le \alpha + 1$.

Clearly, polynomial identities of $A^\sharp$ strongly depend on the identities of A. In particular, we make the following observation. Note that if $f = f(x_1, \ldots, x_n)$ is a multilinear polynomial from $F\{X\}$ then $f(1 + x_1, \ldots, 1 + x_n) \in F\{X\}^\sharp$ is the sum

$$f = \sum f_{i_1,\ldots,i_k}, \quad \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\},\ 0 \le k \le n, \tag{2.1}$$

where $f_{i_1,\ldots,i_k}$ is a multilinear polynomial on $x_{i_1}, \ldots, x_{i_k}$ obtained from f by replacing all $x_j, j \ne i_1, \ldots, i_k$ with 1.

Remark 2.1. A multilinear polynomial $f = f(x_1, \ldots, x_n)$ is an identity of $A^\sharp$ if and only if all of its components $f_{i_1,\ldots,i_k}$ on the left hand side of (2.1) are identities of A.

The next statement easily follows from Remark 2.1.

Remark 2.2. Suppose that an algebra A satisfies all multilinear identities of an algebra B of degree $\deg f = k \le n$ for some fixed n. Then $A^\sharp$ satisfies all identities of $B^\sharp$ of degree $k \le n$.

Using results of [13], we obtain the following inequalities.

Lemma 2.1. ([13, Theorem 2]) *Let A be an algebra with an exponentally bounded codimension growth. Then $\overline{exp}(A^\sharp) \le \overline{exp}(A) + 1$.* □

Lemma 2.2. ([13, Theorem 3]) *Let A be an algebra with an exponentally bounded codimension growth satisfying the identity (2.2). Then $\underline{exp}(A^\sharp) \ge \underline{exp}(A) + 1$.* □

Given an integer $T \ge 2$, we define an infinite-dimensional algebra B_T by its basis

$$\{a, b, z_1^i, \dots, z_T^i |\ i = 1, 2, \dots\}$$

and by the multiplication table

$$z_j^i a = \begin{cases} z_{j+1}^i & \text{if} \quad \mathrm{j} \le \mathrm{T} - 1, \\ 0 & \text{if} \quad \mathrm{j} = \mathrm{T} \end{cases}$$

for all $i \ge 1$ and

$$z_T^i b = z_1^{i+1}, \quad i \ge 1.$$

All other products of basis elements are equal to zero. Clearly, algebra B_T is right nilpotent of class 3, that is

$$x_1(x_2x_3) \equiv 0 \tag{2.2}$$

is an identity of B_T. Due to (2.2), any nonzero product of elements of B_T must be left-normed. Therefore we omit brackets in the left-normed products and write $(y_1y_2)y_3 = y_1y_2y_3$ and $(y_1 \cdots y_k)y_{k+1} = y_1 \cdots y_{k+1}$ if $k \ge 3$.

We will use the following properties of algebra B_T.

Lemma 2.3. ([14, Lema 2.1]) *Let $n \le T$. Then $c_n(B_T) \le 2n^3$.* □

Lemma 2.4. ([14, Lema 2.2]) *Let $n = kT + 1$. Then*

$$c_n(B_T) \ge k! = \left(\frac{N-1}{T}\right)!.$$

□

Lemma 2.5. ([14, Lema 2.3]) *Any multilinear identity $f = f(x_1, \dots, x_n)$ of degree $n \le T$ of algebra B_T is an identity of B_{T+1}.* □

Let $F[\theta]$ be a polynomial ring over F on one indeterminate θ and let $F[\theta]_0$ be its subring of all polynomials without free term. Denote by Q_N the quotient algebra

$$Q_N = \frac{F[\theta]_0}{(Q^{N+1})},$$

where (Q^{N+1}) is an ideal of $F[\theta]$ generated by Q^{N+1}. Fix an infinite sequence of integers $T_1 < N_1 < T_2 < N_2 \dots$ and consider the algebra

$$R = B(T_1, N_1) \oplus B(T_2, N_2) \oplus \cdots, \tag{2.3}$$

where $B(T, N) = B_T \otimes Q_N$.

Let R be an algebra of the type (2.3). Then the following lemma holds.

Lemma 2.6. *For any $i \ge 1$, the following equalities hold:*

(a) if $T_i \le n \le N_i$ then

$$P_n \cap Id(R) = P_n \cap Id(B(T_i, N_i) \oplus B(T_{i+1}, N_{i+1})) = P_n \cap Id(B_{T_i} \oplus B_{T_{i+1}});$$

(b) if $N_i < n \le T_{i+1}$ then

$$P_n \cap Id(R) = P_n \cap Id(B(T_{i+1}, N_{i+1})) = P_n \cap (Id(B_{T_{i+1}})).$$

Proof. This follows immediately from the equality $B(T_i, N_i)^{N_i+1} = 0$ and from Lemma 2.5. □

The folowing remark is obvious.

Remark 2.3. Ler R be an algebra of type (2.3). Then

$$Id(R^\sharp) = Id(B(T_1, N_1)^\sharp \oplus B(T_2, N_2)^\sharp \oplus \cdots).$$

□

3. **The main result.**

Theorem 3.1. *For any real $\alpha > 1$, there exists an algebra R_α with $\underline{exp}(R_\alpha) = 1, \overline{exp}(R_\alpha) = \alpha$ such that $\underline{exp}(R_\alpha^\sharp) = 2$ and $\alpha \le \overline{exp}(R_\alpha^\sharp) \le \alpha + 1$.*

Proof. Note that

$$c_n(A) \le n c_{n-1}(A) \tag{3.1}$$

for any algebra A satisfying (2.2). We will construct R_α of type (2.3) by a special choice of the sequence $T_1, N_1, T_2, N_2, \ldots$ depending on α. First, choose T_1 such that

$$2m^3 < \alpha^m \tag{3.2}$$

for all $m \ge T_1$. By Lemma 2.4, algebra B_{T_1} has an overexponential codimenson growth. Hence there exists $N_1 > T_1$ such that

$$c_n(B_{T_1}) < \alpha^n \quad \text{for all} \quad n \le N_1 - 1 \quad \text{and} \quad c_{N_1}(B_{T_1}) \ge \alpha^{N_1}.$$

Consider an arbitrary $n > N_1$. By Remark 2.1, we have

$$c_n(R^\sharp) \le \sum_{k=0}^{n} \binom{n}{k} c_k(R) = \Sigma_1' + \Sigma_2',$$

where

$$\Sigma_1' = \sum_{k=0}^{N_1} \binom{n}{k} c_k(R), \quad \Sigma_2' = \sum_{k=N_1+1}^{n} \binom{n}{k} c_k(R).$$

By Lemma 2.6, we have $\Sigma_1' + \Sigma_2' \le \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{k=0}^{N_1} \binom{n}{k} c_k(B_{T_1}), \quad \Sigma_2 = \sum_{k=0}^{n} \binom{n}{k} c_k(B_{T_2}).$$

Then for any $T_2 > N_1$, an upper bound for Σ_2 is

$$\Sigma_2 \le \sum_{k=0}^{n} \binom{n}{k} 2k^3 \le 2n^3 \sum_{k=0}^{n} \binom{n}{k} = 2n^3 2^n, \tag{3.3}$$

which follows from (3.2), provided that $n \le T_2$.

Let us find an upper bound for Σ_1 assuming that n is sufficiently large. Clearly,

$$\Sigma_1 \le N_1 \alpha^{N_1} \sum_{k=0}^{N_1} \binom{n}{k} \tag{3.4}$$

which follows from the choice of N_1, relation (3.1), and the equality $B(T_1, N_1)^n = 0$ for all $n \ge N_1 + 1$. Since $N_1 \alpha^{N_1}$ is a constant for fixed N_1, we only need to estimate the sum of binomial coefficients.

From the Stirling formula

$$m! = \sqrt{2\pi m}(\frac{m}{e})^m e^{\frac{1}{12m+\theta_m}}, \quad 0 < \theta_m < 1,$$

it follows that

$$\binom{n}{k} \leq \sqrt{\frac{n}{k(n-k)}} \cdot \frac{n^n}{k^k(n-k)^{n-k}}. \tag{3.5}$$

Now we define the function $\Phi : [0;1] \to \mathbb{R}$ by setting

$$\Phi(x) = \frac{1}{x^x(1-x)^{1-x}}.$$

It is not difficult to show that Φ increases on $[0;1/2]$, $\Phi(0) = 1$, and $\Phi(x) \leq 2$ on $[0;1]$. In terms of the function Φ we rewrite (3.5) as

$$\binom{n}{k} \leq \sqrt{\Phi\left(\frac{k}{n}\right)} \cdot \Phi\left(\frac{k}{n}\right)^n < 2\Phi\left(\frac{k}{n}\right)^n \leq 2\Phi\left(\frac{N_1}{n}\right)^n \tag{3.6}$$

provided that $n > 2N_1$. Now (3.4) and (3.6) together with (3.3) imply

$$\Sigma_1 \leq 2N_1\alpha^{N_1}(N_1+1)\Phi\left(\frac{N_1}{n}\right)^n, \quad \Sigma_2 \leq 2n^3 2^n.$$

Since

$$\lim_{n\to\infty} \Phi\left(\frac{N_1}{n}\right)^n = 1$$

and $\Phi(x)$ increases on $(0;1/2]$, there exists $n > 2N_1$ such that

$$2N_1(N_1+1)\alpha^{N_1}\Phi\left(\frac{N_1}{n}\right)^n + 2n^3 2^n < (2+\frac{1}{2})^n. \tag{3.7}$$

Now we take T_2 to be equal to the minimal $n > 2N_1$ satisfying (3.7). Note that for such T_2 we have

$$c_n(R^\sharp) < (2+\frac{1}{2})^n$$

for $n = T_2$.

As soon as T_2 is choosen, we can take N_2 as the minimal n such that $c_n(B_{T_2}) \geq \alpha^n$. Then again, $c_m(R) < m\alpha^m$ if $m < N_2$. Repeating this procedure, we can construct an infinite chain $T_1 < N_1 < T_2 < N_2 \cdots$ such that

$$c_n(R) < \alpha^n + 2n^3 \tag{3.8}$$

for all $n \neq N_1, N_2, \ldots$,

$$\alpha^n \leq c_n(R) < \alpha^n + n(\alpha^{n-1} + 2n^3) \tag{3.9}$$

for all $n = N_1, N_2, \ldots$ and

$$2N_j(N_j+1)\alpha^{N_j}\Phi\left(\frac{N_j}{T_{j+1}}\right)^{T_{j+1}} + 2T_{j+1}^3 \cdot 2^{T_{j+1}} < (2+\frac{1}{2^j})^{T_{j+1}} \tag{3.10}$$

for all $j = 1, 2, \ldots$.

Let us denote by R_α the just constructed algebra R of type (2.3). Then (3.10) means that

$$c_n(R_\alpha^\sharp) < (2+\frac{1}{2^j})^n \tag{3.11}$$

if $n = T_{j+1}, j = 1, 2, \ldots$. It follows from inequality (3.11) that

$$\underline{exp}(R_\alpha^\sharp) \leq 2. \tag{3.12}$$

On the other hand, since R_α is not nilpotent, it follows that

$$\underline{exp}(R^\sharp_\alpha) \geq 1. \tag{3.13}$$

Since the PI-exponent of non-nilpotent algebra cannot be strictly less than 1, relations (3.12), (3.13) and Lemma 2.2 imply

$$\underline{exp}(R_\alpha) = 1, \ \underline{exp}(R^\sharp_\alpha) = 2.$$

Finally, relations (3.8), (3.9) imply the equality $\overline{exp}(R_\alpha) = \alpha$. Applying Lemma 2.1, we see that $\overline{exp}(R^\sharp_\alpha) \leq \alpha + 1$. The inequality $\alpha = \overline{exp}(R_\alpha) \leq \overline{exp}(R^\sharp_\alpha)$ is obvious, since R_α is a subalgebra of $R^\sharp_\alpha$, Thus we have completed the proof of Theorem 3.1. □

As a consequence of Theorem 3.1 we get an infinite family of unital algebras of exponential codimension growth without ordinary PI-exponent.

Corollary 3.1. *Let $\beta > 2$ be an arbitrary real number. Then the ordinary PI-exponent of unital algebra $R^\sharp_\beta$ from Theorem 3.1 does not exist. Moreover, $\underline{exp}(R^\sharp_\beta) = 2$, whereas $\beta \leq \overline{exp}(R^\sharp_\beta) \leq \beta + 1$.* □

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