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Codimension Growth of Solvable Lie Superalgebras

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Abstract. We study numerical invariants of identities of finite-dimensional solvable Lie superalgebras. We define new series of finite-dimensional solvable Lie superalgebras L with non-nilpotent derived subalgebra L' and discuss their codimension growth. For the first algebra of this series we prove the existence and integrality of $\exp(L)$.

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1. Introduction

Let A be an algebra over a field F of characteristic zero. One can define an infinite sequence $\{c_n(A)\}, n = 1, 2, \dots,$ of non-negative integers associated with A called *codimension sequence*. It measures the quantity of polynomial identities of A. For many classes of algebras the sequence $\{c_n(A)\}$ is exponentially bounded. In particular, this holds for associative PI-algebras [14], [15], for finite-dimensional algebras [1], [11], for Kac-Moody Lie algebras [20], [21], and many others. In this case the sequence $(c_n(A))^{1/n}$ has the lower and upper limits $\underline{\exp}(A)$ and the $\overline{\exp}(A)$ called the *lower* and *upper* PI-*exponents* of A, respectively. If $\underline{\exp}(A) = \overline{\exp}(A)$ then there exists an ordinary limit called the PI-*exponent* $\exp(A)$ of A. At the end of 1980's Amitsur conjectured that $\exp(A)$ exists and is an integer for every associative PI-algebra A. Amitsur's conjecture was proved in [7], [8]. Later the existence and integrality of PI-exponent was proved for finite-dimensional Lie and Jordan algebras [4], [5], [6], [10], [11], [22]. On the other hand, there are infinite-dimensional solvable Lie algebras with fractional PI-exponents [2], [18], [24].

None of these results can be generalized to Lie superalgebras. There is an infinite series of finite-dimensional superalgebras $P(t)$, $t \geq 2$, where all $P(3), P(4), \dots$ are simple whereas $P(2)$ is not. For $L = P(2)$ it was proved in [12] that $\exp(L)$ exists and is not an integer. Due to [12], there is a serious reason to expect that PI-exponent is fractional for any simple superalgebra $P(t)$, $t \geq 3$.

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For infinite-dimensional Lie superalgebras only some partial results are known [16], [25]. In particular, in [25] it was shown that PI-exponent of a Lie superalgebra L exists and is an integer, provided that its commutator subalgebra L^2 is nilpotent. Note that by the Lie Theorem, the subalgebra L^2 is nilpotent for any finite-dimensional solvable Lie algebra L. Unfortunately, finite-dimensional Lie superalgebras in general do not satisfy this condition. Hence the result of [25] cannot be applied to finite-dimensional solvable Lie superalgebras. Although there are examples of finite-dimensional Lie superalgebras with the fractional PI-exponent, the following conjecture looks natural: Is it true that any finite-dimensional solvable Lie superalgebra has an integer exponent?

In this paper we construct new series of finite-dimensional solvable Lie superalgebras $S(t)$, $t = 2, 3, \ldots$, with non-nilpotent derived subalgebras. For $S(2)$ we prove the existence and integrality of PI-exponent (Theorem 4.3). We also discuss the following related question concerning graded identities. Every Lie superalgebra $L = L_0 \oplus L_1$ is endowed by the natural $\mathbb{Z}_2$-grading. Hence one can also study asymptotic behavior of graded codimension sequence $\{c_n^{\mathrm{gr}}(L)\}$. It was mentioned in [1] that $c_n(A) \le c_n^{\mathrm{gr}}(A)$ for any algebra $A = \oplus_{g\in G} A_g$ graded by a finite group G. Hence $\exp(A) \le \exp^{\mathrm{gr}}(A)$. In the associative case there are examples where this inequality is strong. For instance, if $A = F[G]$ is the group algebra of a finite abelian group G then $\exp(A) = 1$ whereas $\exp^{\mathrm{gr}}(A) = |G|$. For Lie superalgebras similar examples are unknown. On the other hand, there are many examples of simple (associative and nonassociative) algebras with $\exp^{\mathrm{gr}}(A) = \exp(A)$. In the present paper we give the first example in the class of solvable Lie superalgebras, namely, we prove that $\exp^{\mathrm{gr}}(S(2)) = \exp(S(2))$ (Theorem 5.3).

2. Generalities

Let A be an algebra over F and let $F\{X\}$ be the absolutely free algebra over F with an infinite set of generators X. A non-associative polynomial $f = f(x_1, \ldots, x_n) \in F\{X\}$ is said to be an *identity* of A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. All identities of A form an ideal $\mathrm{Id}(A)$ of $F\{X\}$.

Denote by P_n the subspace in $F\{X\}$ of all multilinear polynomials on $x_1, \ldots, x_n \in X$. Then $P_n \cap \mathrm{Id}(A)$ is the set of all multilinear identities of A of degree n. Since $\mathrm{char}\, F = 0$, the sequence of subspaces

$$\{P_n \cap \mathrm{Id}(A)\}, \qquad n = 1, 2, \ldots,$$

completely defines the ideal $\mathrm{Id}(A)$. Denote

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)} \quad \text{and} \quad c_n(A) = \dim P_n(A).$$

The sequence of integers $\{c_n(A)\}$, $n = 1, 2, \ldots$, called the *codimension sequence* of A, is an important numerical characteristic of $\mathrm{Id}(A)$. The analysis of the asymptotic behavior of $\{c_n(A)\}$ is one of the main approaches of the study of identities of algebras.

As it was mentioned in the introduction, there is a wide class of algebras A such that $c_n(A) \le a^n$ for some constant a. In this case one can define the *lower* and the *upper* PI-exponents of A as follows:

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \qquad \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)},$$

respectively. If the ordinary limit exists we can define the (*ordinary*) PI-*exponent*

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}.$$

A powerful tool for computing codimensions is the representation theory of the symmetric group S_n. One can define an S_n-action on the subspace P_n of multilinear polynomials by setting

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for $\sigma \in S_n$. Then P_n becomes an FS_n-module. Since $P_n \cap \mathrm{Id}(A)$ is stable under S_n-action, then $P_n(A)$ is also an FS_n-module and its S_n-character

$$\chi_n(A) = \chi(P_n(A))$$

is called the n-th *cocharacter* of A. By Maschke's Theorem, $P_n(A)$ is completely reducible, so

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{1}$$

where χ_λ is the irreducible S_n-character corresponding to the partition λ of n. All details concerning S_n-representations can be found in [13]. The total sum of multiplicities in (1) is called the n-th *colength* of A,

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda.$$

Clearly,

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda \tag{2}$$

where $d_\lambda = \deg \chi_\lambda$ is the dimension of the corresponding irreducible representation and the multiplicities m_λ are taken from (1). It is well-known that the colength sequence $\{l_n(A)\}$ is polynomially bounded for any finite-dimensional algebra A.

Proposition 2.1 ([3, Theorem 1]). *Let* $\dim A = d$. *Then, for all* $n \ge 1$,

$$l_n(A) \le d(n+1)^{d^2+d}.$$

Throughout the paper we will omit brackets in left-normed products in non-associative algebras, i.e., $abc = (ab)c$, $abcd = (abc)d$, etc.

3. Lie superalgebras $S(t)$

In this section we introduce an infinite series of finite-dimensional solvable Lie superalgebras with non-nilpotent commutator subalgebra.

First, let R be an arbitrary associative algebra with involution $*\colon R \to R$. Consider an associative algebra Q consisting of (2×2)-matrices over R

$$Q = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \Bigg| \, A, B, C, D \in R \right\}.$$

The algebra Q can be naturally endowed by $\mathbb{Z}_2$-grading $Q = Q_0 \oplus Q_1$, where

$$Q_0 = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & D \end{array} \right) \right\}, \quad Q_1 = \left\{ \left(\begin{array}{cc} 0 & B \\ C & 0 \end{array} \right) \right\}.$$

It is well-known that if we define a (super) commutator brackets by setting

$$[x, y] = xy - (-1)^{|x||y|} yx$$

for homogeneous $x, y \in Q_0 \cup Q_1$, where $|x| = 0$ if $x \in Q_0$ and $|x| = 1$ if $x \in Q_1$, then Q becomes a Lie superalgebra. For basic notions of super Lie theory we refer to [17]. Denote by

$$R^+ = \{x \in R |\ x^* = x\}\ , \quad R^- = \{y \in R |\ y^* = -y\},$$

the subspaces of symmetric and skew elements of R, respectively. Then the subspace

$$L = \left\{ \left(\begin{array}{cc} x & y \\ z & -x^* \end{array} \right) \Bigg| \, x \in R, y \in R^+, z \in R^- \right\} = L_0 \oplus L_1 \tag{3}$$

of Q is a Lie superalgebra under the supercommutator product defined above, where even and odd components are

$$L_0 = \left\{ \left(\begin{array}{cc} x & 0 \\ 0 & -x* \end{array} \right) \right\}, \quad L_1 = \left\{ \left(\begin{array}{cc} 0 & y \\ z & 0 \end{array} \right) \right\}.$$

Note that if $R = M_t(F)$ is a $(t \times t)$-matrix algebra, $t \geq 3$, then its subalgebra $\widetilde{L} \subset L$ consisting of the matrix

$$\left\{ \left(\begin{array}{cc} x & y \\ z & -x^* \end{array} \right) \right\}$$

with traceless matrices x where $x \to x^*$ is the transpose involution is a well-known simple Lie superalgebra $P(t)$ (or $b(t)$ in the notations of [17]).

Now we clarify the structure of R in our case. Let $R = UT_t(F)$ be an algebra of $(t \times t)$-upper triangular matrices over F. It is well-known (see, for example, [19]) that the reflection across the secondary diagonal is the involution on R, hence L defined in (3) is a finite-dimensional Lie superalgebra. We denote this superalgebra by $S(t)$. Its even component $S_0 \simeq UT_t(F)$ is solvable hence the entire L is also solvable (see, for example, [17]). It is not difficult to check that the derived subalgebra L^2 is not nilpotent and we get the following conclusion.

Proposition 3.1. *Let R be the upper triangular $(t\times t)$-matrix algebra with the involution $*\colon R\to R$, the reflection across the secondary diagonal. Then $S(t)=L=L_0\oplus L_1$ well-defined in* (3) *is a finite-dimensional solvable Lie superalgebra,* $\dim L=t(t+1)$, *with non-nilpotent commutator subalgebra.*

Now we will have to deal with the Lie superalgebra $S(2)$. First, we compute supercommutators in the associative superalgebra $Q\simeq UT_2(F)\otimes M_2(F)$. If A,B,C and D are 2×2-matrices then

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix} 0 & AB+BA^*\\ 0 & 0\end{pmatrix}, \tag{4}$$

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right]=\begin{pmatrix} 0 & 0\\ -A^*C-CA & 0\end{pmatrix}, \tag{5}$$

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} B & 0\\ 0 & -B^*\end{pmatrix}\right]=\begin{pmatrix} AB-BA & 0\\ 0 & -(AB-BA)^*\end{pmatrix}, \tag{6}$$

$$\left[\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right]=\begin{pmatrix} BC & 0\\ 0 & CB\end{pmatrix}. \tag{7}$$

From now on, we will not use associative multiplication and will omit square brackets in the product of elements of Lie superalgebra $S(2)$. That is, $xy=[x,y]$, $xyz=[[x,y],z]$ and so on for $x,y,z\in S(2)$. Let e_{11},e_{12} and e_{22} be (2×2)-matrix units. Then $e_{11}^*=e_{22}$, $e_{22}^*=e_{11}$, $e_{12}^*=e_{12}$ in R and the matrices

$$a=\begin{pmatrix} e_{11}-e_{22} & 0\\ 0 & e_{11}-e_{22}\end{pmatrix},\quad b=\begin{pmatrix} 0 & e_{11}+e_{22}\\ 0 & 0\end{pmatrix},\quad c=\begin{pmatrix} 0 & 0\\ e_{11}-e_{22} & 0\end{pmatrix},$$

$$d=\begin{pmatrix} e_{11}+e_{22} & 0\\ 0 & -e_{11}-e_{22}\end{pmatrix},\quad x=\begin{pmatrix} e_{12} & 0\\ 0 & -e_{12}\end{pmatrix},\quad y=\begin{pmatrix} 0 & e_{12}\\ 0 & 0\end{pmatrix}$$

form a basis of $S(2)$. By definition a,d and x are even whereas b,c and y are odd. Using (4), (5), (6), (7) we can compute all nonzero products of basis elements,

$$bc=cb=a,\quad bd=-db=-2b,\quad cd=-dc=2c,\quad xa=-ax=-2x,$$

$$xb=-bx=2y,\quad ya=-ay=-2y,\quad yc=cy=-x,\quad yd=-dy=-2y.$$

4. PI-exponent of $S(2)$

Since we will have to deal with multialternating sets of arguments in multilinear and multihomogeneous expressions, it is convenient to use the following agreement. If $f=f(x_1,\ldots,x_n,y_1,\ldots,y_k)$ is a non-associative polynomial, multilinear on $x_1,\ldots,x_n$, then we denote the result of alternation of f on $x_1,\ldots,x_n$ by marking all $x_1,\ldots,x_n$ by one and the same symbol over x_i's. For example,

$$\bar x_1 y\bar x_2\bar x_3=\sum_{\sigma\in S_3}(\operatorname{sgn}\sigma)x_{\sigma(1)}yx_{\sigma(2)}x_{\sigma(3)},\quad\text{or}$$

$$(y\bar x_1\widetilde{x}_1)(\bar x_2\widetilde{x}_2)(\bar x_3\widetilde{x}_3)=$$

$$=\sum_{\sigma\in S_3}\sum_{\tau\in S_3}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)(yx_{\sigma(1)}x_{\tau(1)})(x_{\sigma(2)}x_{\tau(2)})(x_{\sigma(3)}x_{\tau(3)}).$$

Our next goal is to prove the relation

$$y(\widetilde{b\bar{c}})(\widetilde{c\bar{d}})(\widetilde{d\,\bar{b}})\widetilde{a\bar{a}} = 384y. \tag{8}$$

Since $aa = ab = ba = ac = ca = ad = da = 0$, the left hand side of (8) is equal to $y(\widetilde{b\bar{c}})(\widetilde{c\bar{d}})(\widetilde{d\,\bar{b}})aa$. Hence it suffices to show that

$$y(\widetilde{b\bar{c}})(\widetilde{c\bar{d}})(\widetilde{d\,\bar{b}}) = 96y. \tag{9}$$

The left hand side of (9) can be written as the sum

$$\begin{aligned} &y(b\bar{c})(c\bar{d})(d\bar{b}) + y(c\bar{c})(d\bar{d})(b\bar{b}) + y(d\bar{c})(b\bar{d})(c\bar{b})- \\ &\qquad - y(c\bar{c})(b\bar{d})(d\bar{b}) - y(d\bar{c})(c\bar{d})(b\bar{b}) - y(b\bar{c})(d\bar{d})(c\bar{b}). \end{aligned}$$

Direct computations show that

$$\begin{aligned} y(b\bar{c})(c\bar{d})(d\bar{b}) &= y(bc)(cd)(db) + y(bd)(cb)(dc) = 4yacb + 4ybac, \\ y(c\bar{c})(d\bar{d})(b\bar{b}) &= y(cb)(dc)(bd) + y(cd)(db)(bc) = 4yacb + 4ycba, \\ y(d\bar{c})(b\bar{d})(c\bar{b}) &= y(dc)(bd)(cb) + y(db)(bc)(cd) = 4ycba + 4ybac, \\ -y(c\bar{c})(b\bar{d})(d\bar{b}) &= y(cd)(bc)(db) + y(cb)(bd)(dc) = 4ycab + 4yabc, \\ -y(d\bar{c})(c\bar{d})(b\bar{b}) &= y(db)(cd)(bc) + y(dc)(cb)(bd) = 4ybca + 4ycab, \\ -y(b\bar{c})(d\bar{d})(c\bar{b}) &= y(bd)(dc)(cb) + y(bc)(db)(cd) = 4ybca + 4yabc. \end{aligned}$$

Since $yb = 0$ and $yab = -2yb = 0$, we obtain

$$\begin{aligned} y(\widetilde{b\bar{c}})(\widetilde{c\bar{d}})(\widetilde{d\bar{b}}) &= 8yacb + 8ycab + 8ycba = -16ycb - 8xab - 8xba \\ &= 16xb + 16xb - 16ya = 96y \end{aligned}$$

and therefore (9), (8) hold. Equality (8) implies the relation

$$y\,\underbrace{(\widetilde{b\bar{c}})(\widetilde{c\bar{d}})(\widetilde{d\,\bar{b}})\widetilde{a\bar{a}}\cdots(\widetilde{\widetilde{b}\bar{\bar{c}}})(\widetilde{\widetilde{c}\bar{\bar{d}}})(\widetilde{\widetilde{d}\,\bar{\bar{b}}})\widetilde{\widetilde{a}\bar{\bar{a}}}}_{m} \neq 0 \tag{10}$$

for any $m \geq 1$. Consider the multilinear polynomial

$$\begin{aligned} f_m = w(\widetilde{x}_1^{(1)}\bar{z}_1^{(1)})(\widetilde{x}_2^{(1)}\bar{z}_2^{(1)})(\widetilde{x}_3^{(1)}\bar{z}_3^{(1)})\widetilde{x}_4^{(1)}\bar{z}_4^{(1)}\cdots \\ \cdots(\widetilde{\widetilde{x}}_1^{(m)}\bar{\bar{z}}_1^{(m)})(\widetilde{\widetilde{x}}_2^{(m)}\bar{\bar{z}}_2^{(m)})(\widetilde{\widetilde{x}}_3^{(m)}\bar{\bar{z}}_3^{(m)})\widetilde{\widetilde{x}}_4^{(m)}\bar{\bar{z}}_4^{(m)} \end{aligned}$$

of degree $4m+1$. The polynomial f_m depends on $2m$ alternating sets of variables, each of order four. Moreover, f_m assumes a non-zero value under an evaluation $\varphi\colon X \to S(2)$ such that

$$\begin{aligned} &\varphi(w) = y, \quad \varphi(x_1^{(i)}) = b, \quad \varphi(x_2^{(i)}) = c, \quad \varphi(x_3^{(i)}) = d, \quad \varphi(x_4^{(i)}) = a, \\ &\varphi(z_1^{(i)}) = c, \quad \varphi(z_2^{(i)}) = d, \quad \varphi(z_3^{(i)}) = b, \quad \varphi(z_4^{(i)}) = a, \qquad i = 1, \ldots, m. \end{aligned}$$

Denote $n = 8m$ and consider the S_n-action on variables

$$\{x_j^{(i)}, z_j^{(i)} \mid 1 \le j \le 4,\ 1 \le i \le m\}.$$

Under this action the subspace

$$P_{n+1} = P_{n+1}(w, x_j^{(i)}, z_j^{(i)},\ 1 \le j \le 4,\ 1 \le i \le m)$$

becomes an FS_n-module. The structure of the polynomial f_m and the relation $\varphi(f_m) \neq 0$ show that $e_{T_\lambda} f_m$ is not an identity of $S(2)$, where e_{T_λ} is the essential idempotent corresponding to some Young tableau T_λ with the Young diagram D_λ and $\lambda = (2m, 2m, 2m, 2m) \vdash n$. In particular,

$$c_{n+1}(S(2)) \ge \deg \chi_\lambda. \tag{11}$$

From the hook formula for $\deg \chi_\lambda$ and the Stirling formula for factorials we get

$$\deg \chi_\lambda \ge n^{-5} 4^n, \tag{12}$$

provided that $n = 8m$ and $\lambda = (2m)^{(4)}$.

The inequalities (11), (12) give us the lower bound for codimensions $c_n(S(2))$.

Lemma 4.1. *The lower PI-exponent of $S(2)$ satisfies the inequality $\underline{\exp}(S(2)) \ge 4$.*

Proof. Let $n \equiv j \pmod 8$ where $0 \le j \le 7$. If $j = 1$ then $n = 8m + 1$ and

$$c_n(S(2)) \ge \frac{4^{n-1}}{(n-1)^5} \ge \frac{1}{5n^5} 4^n$$

by (11), (12). If $j \neq 1$ then there exist m and $1 \le i \le 8$ such that $n = 8m+1+i$. In this case the polynomial $g = (e_{T_\lambda} f_m) u_1 \cdots u_i$ of degree $8m + 1 + i = n$ is not an identity of $S(2)$ since $\varphi(f_m) = (384)^m y$ for the above mentioned evaluation φ and $ya = -2y$. Hence

$$c_n(S(2)) \ge 4^{-8} n^{-5} 4^n.$$

Therefore $\underline{\exp}(S(2)) \ge 4$ and the proof is complete. ∎

We need another lemma to prove the main result of the paper.

Lemma 4.2. *Let $m_\lambda \neq 0$ in (2) for $A = S(2)$, $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then either $k \le 4$ or $k = 5$ and $\lambda_5 = 1$.*

Proof. Let $m_\lambda \neq 0$ and $k > 4$. Then there exists a Young tableau T_λ such that $e_{T_\lambda} f \notin \mathrm{Id}(A)$ for some multilinear polynomial $f = f(x_1, \ldots, x_n)$. Recall that

$$e_{T_\lambda} = \left(\sum_{\sigma \in R_{T_\lambda}} \sigma \right) \left(\sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\ \tau \right)$$

where R_{T_λ} and C_{T_λ} are the row stabilizer and the column stabilizer of T_λ in S_n, respectively. Note that the polynomial

$$g = g(x_1, \dots, x_n) = \left(\sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\ \tau \right) e_{T_\lambda} f$$

is also non-identity of A. If $k > 5$ then g contains an alternating set of variables $\{x_{i_1}, \dots, x_{i_t}\}$ of order $t \geq 6$. Consider an evaluation $\varphi\colon X \to B = \{a, b, c, d, x, y\}$. The linear subspace $J = \langle x, y \rangle \subset A$ is a nilpotent ideal of A, $J^2 = 0$. If at least two of x_{i_α}, $1 \leq \alpha \leq 6$, lie in J then $\varphi(g) = 0$. But if $\varphi(x_{i_1}), \dots, \varphi(x_{i_t})$ take not more than five distinct values in B then also $\varphi(g) = 0$, due to the skew symmetry of g. This contradiction shows that $k \leq 5$. Similar arguments imply the restriction $\lambda_5 \leq 1$ and hence the proof is complete. ■

Theorem 4.3. *The PI-exponent of the Lie superalgebra $S(2)$ exists and*

$$\exp(S(2)) = 4.$$

Proof. Because of Lemma 4.1 it suffices to prove the inequality

$$\overline{\exp}(S(2)) \leq 4. \tag{13}$$

In light of Lemma 4.2, by Lemma 6.2.4 and Lemma 6.2.5 from [9], we have

$$\deg \chi_\lambda < C n^r 4^n$$

for some constants C, r if $m_\lambda \neq 0$ in (2). Finally, applying Proposition 2.1, we get the inequality (13) and the proof is complete. ■

5. Graded PI-exponent of $S(2)$

Recall the definition of the *graded codimension* of a $\mathbb{Z}_2$-graded algebra. Let $A = A_0 \oplus A_1$ be an F-algebra with $\mathbb{Z}_2$-grading. Denote by $F\{X, Y\}$ the free algebra on two infinite sets of generators X and Y. Let all $x \in X$ be even and all $y \in Y$ odd. Then this parity on $X \cup Y$ induces $\mathbb{Z}_2$-grading on $F\{X, Y\}$. A polynomial $f = f(x_1, \dots, x_m, y_1, \dots, y_n)$ with $x_1, \dots, x_m \in X$, $y_1, \dots, y_n \in Y$ is said to be a *graded identity* of A if $f = f(a_1, \dots, a_m, b_1, \dots, b_n) = 0$ for all $a_1, \dots, a_m \in A_0$, $b_1, \dots, b_n \in A_1$.

Given $0 \leq k \leq n$, denote by $P_{k,n-k}$ the subspace of all multilinear polynomials on $x_1, \dots, x_k \in X, y_1, \dots, y_{n-k} \in Y$ and define the integer

$$c_{k,n-k}(A) = \dim \frac{P_{k,n-k}}{P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(A)}$$

where $\mathrm{Id}^{\mathrm{gr}}(A)$ is the ideal of graded identities of A. Then the value

$$c_n^{\mathrm{gr}}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(A)$$

is called the *graded* n-th *codimension* of A. As in the non-graded case, the limits

$$\underline{\exp}^{\rm gr}(A)=\liminf_{n\to\infty}\sqrt[n]{c_n^{\rm gr}(A)},\qquad \overline{\exp}^{\rm gr}(A)=\limsup_{n\to\infty}\sqrt[n]{c_n^{\rm gr}(A)},$$

$$\exp^{\rm gr}(A)=\lim_{n\to\infty}\sqrt[n]{c_n^{\rm gr}(A)}$$

are called the *lower*, the *upper* and the *ordinary graded* PI-*exponent* of A.

The space $P_{k,n-k}$ has a natural $F[S_k\times S_{n-k}]$-module structure where the symmetric groups S_k and S_{n-k} act on $\{x_1,\ldots,x_k\}$ and on $\{y_1,\ldots,y_{n-k}\}$, respectively. Since $P_{k,n-k}\cap{\rm Id}^{\rm gr}(A)$ is stable under the $(S_k\times S_{n-k})$-action, then the quotient space

$$P_{k,n-k}(A)=\frac{P_{k,n-k}}{P_{k,n-k}\cap{\rm Id}^{\rm gr}(A)}$$

is also an $F[S_k\times S_{n-k}]$-module and its $(S_k\times S_{n-k})$-character has the form

$$\chi_{k,n-k}(A)=\chi(P_{k,n-k}(A))=\sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}}m_{\lambda,\mu}\chi_{\lambda,\mu}. \tag{14}$$

In particular,
$$c_{k,n-k}(A)=\sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}}m_{\lambda,\mu}\deg\chi_\lambda\deg\chi_\mu. \tag{15}$$

The sum of multiplicities

$$l_n^{\rm gr}(A)=\sum_{k=0}^{n}\sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}}m_{\lambda,\mu}$$

is called n-th *graded colength* of A and is polynomially bounded if $\dim A<\infty$ (see [23]) that is, there are constants C,r such that

$$l_n^{\rm gr}(A)\le Cn^r. \tag{16}$$

Recall that $A_0=\langle a,d,x\rangle$, $A_1=\langle b,d,y\rangle$ for our superalgebra $A=S(2)$ and x,y belong to nilpotent ideal J, $J^2=0$.
The same argument as in the proof of Lemma 4.2 gives us the following result.

Lemma 5.1. *Let $A=S(2)$ and let $m_{\lambda,\mu}\neq 0$ in (14). Then $\lambda=(\lambda_1)$ or $\lambda=(\lambda_1,\lambda_2)$ or $\lambda=(\lambda_1,\lambda_2,1)$ and $\mu=(\mu_1)$ or $\mu=(\mu_1,\mu_2)$ or $\mu=(\mu_1,\mu_2,1)$.*

As a consequence of Lemma 5.1 and Lemmas 6.2.4, 6.2.5 from [9] we get the following statement.

Lemma 5.2. *There are constants $c_,r_0,c_1,r_1$ not depending on k such that*

$$\deg\chi_\lambda\le c_0n^{r_0}2^k,\qquad \deg\chi_\mu\le c_1n^{r_1}2^k$$

for all $\lambda\vdash k$, $\mu\vdash(n-k)$ if $m_{\lambda,\mu}\neq 0$ in (14).

Our final result says that $\exp(S(2))$ and $\exp^{\mathrm{gr}}(S(2))$ coincide.

Theorem 5.3. $\exp(S(2)) = \exp^{\mathrm{gr}}(S(2)) = 4$.

Proof. It is well-known (see [1]) that $c_n(A) \le c_n^{\mathrm{gr}}(A)$ for any group graded algebra A. Hence, by Theorem 4.3,

$$\underline{\exp}^{\mathrm{gr}}(S(2)) \ge 4. \tag{17}$$

Let us prove that
$$\overline{\exp}^{\mathrm{gr}}(S(2)) \le 4. \tag{18}$$

By (16), Lemma 5.1 and Lemma 5.2, we have

$$c_{k,n-k}(S(2)) \le c_3 n^{r_3} 2^k 2^{n-k} = c_3 n^{r_3} 2^n$$

for some constants c_3, r_3. Then by definition of graded codimensions,

$$c_n^{\mathrm{gr}}(S(2)) \le c_3 n^{r_3} 2^n \sum_k \binom{n}{k} = c_3 n^{r_3} 4^n.$$

The latter relation proves (18), and the proof is complete. ■

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