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Pauli gradings on Lie superalgebras and graded codimension growth

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ABSTRACT

We introduce grading on certain finite dimensional simple Lie superalgebras of type $P(t)$ by elementary abelian 2-group. This grading gives rise to Pauli matrices and is a far generalization of $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-grading on Lie algebra of (2×2)-traceless matrices. We use this grading for studying numerical invariants of polynomial identities of Lie superalgebras. In particular, we compute graded PI-exponent corresponding to Pauli grading.

1. Introduction

In this paper we study algebras over a field F of characteristic zero. Group graded algebras have been intensively studied in the last two decades (see, for example, [3,5,6,10,11,18,19,26]). All possible gradings on matrix algebras over an algebraically closed

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field were described in [3,6]. Recently, all gradings by a finite abelian groups on finite dimensional simple real algebras have also been classified in [7,23]. Many authors have also paid attention to gradings on Lie algebras [5,8,11,19]. Both, in associative and Lie case, an exceptional role is played by gradings which cannot be "refined" – in particular, gradings whose homogeneous components are one-dimensional [3,6,8,19]. Classification of group gradings on Lie superalgebras is only in its initial stages (see, e.g., [4]). Therefore an important role is played by new examples of gradings on Lie superalgebras.

It is well known that abelian gradings are closely connected to automorphism and involution actions on algebra (see, for example, [3]), hence the knowledge of gradings gives us an important information about the group of automorphisms and antiautomorphisms of an algebra. Another application of gradings is the study of graded and non-graded identities and their numerical invariants.

Given an algebra A, one can associate to it an infinite sequence of non-negative integers

$$\{c_n(A)\}, \quad n = 1, 2, \ldots,$$

called *codimensions* of A. The study of asymptotic behavior of $\{c_n(A)\}$ is one of the most important and current approaches in the modern PI-theory [14]. In many cases codimension growth is exponentially bounded. In particular,

$$\dim A = d < \infty \quad \Rightarrow \quad c_n(A) \le d^{n+1}$$

(see [2] and also [15, Proposition 2]). If, in addition, A is endowed with a grading by a group G then one can also define the graded codimension sequence $c_n^G(A)$. For a finite dimensional algebra A, graded and ordinary codimensions satisfy the following inequalities:

$$c_n(A) \le c_n^G(A) \le (\dim A)^{n+1} \tag{1}$$

(see [2]).

As a rule, an investigation of asymptotics of graded codimensions is much easier than a study of non-graded codimensions. This fact was used in our previous papers for obtaining the results on both graded and non-graded codimension growth [16,20–22].

If A is a finite dimensional graded simple algebra then there exist the limits

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \quad exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)} \tag{2}$$

and according to (1) we have

$$exp(A) \le exp^G(A) \le \dim A. \tag{3}$$

It is well known that in many most important cases of algebras (associative, Lie, Jordan, alternative, etc.)

$$exp(A) = \dim A, \tag{4}$$

provided that A is simple and F is algebraically closed [12,13,25]. In this case $exp^G(A)$ is also equal to $\dim A$ for any grading on A. If A is graded simple but not simple in the usual sense then graded and non-graded exponents can differ. For example, if G is a finite abelian group of order $|G| = m$ and A is its group algebra, $A = FG$, then $exp(A) = 1$ whereas $exp^G(A) = m$. Clearly, if A is simple in non-graded sense then A is also graded simple for any G-grading. Relations (3) and (4) show that the conjecture that $exp(A) = exp^G(A)$ holds for associative, Lie, Jordan and alternative algebras over an algebraically closed field.

Nevertheless, in the Lie superalgebra case there exist simple algebras such that $exp(A)$ and $exp^G(A)$ exist and are strictly less than $\dim A$ (see [16,22]). Here we are talking about canonical $\mathbb{Z}_2$-grading on Lie superalgebras. Therefore the study of relations between graded and non-graded PI-exponents is of interest in the general case. In particular, if the conjecture that $exp(A) = exp^G(A)$ is confirmed then it would give us a powerful tool for computing precise asymptotics of codimension growth. Another consequence would be the independence of $exp^G(A)$ on the particular G-grading.

The goal of the present paper is twofold. In the first part we define the so-called Pauli G-grading on the simple Lie superalgebra of the type $L = P(t)$ (in the notation of [17], for general material on Lie superalgebras see also [24]), where t is the power of 2 and G is an elementary abelian 2-group. This grading possesses many remarkable properties. In fact, it is induced from the grading on simple 3-dimensional Lie algebra $sl_2(F)$ by Pauli matrices and is compatible with the canonical $\mathbb{Z}_2$-grading. All non-zero homogeneous components of L are one-dimensional. Also, any even homogeneous element $0 \neq a \in L_g$ is a non-degenerate matrix and for any homogeneous elements $a \in L_g, b \in L_h$ their Lie supercommutator is either zero or non-degenerate. In the second part of the paper we investigate the graded codimension growth of L. We show that all computations are much easier than in the non-graded case due to the remarkable properties of Pauli grading.

Our main result is Theorem 1 below, stating that $exp^G(P(t)) = t^2 - 1 + t\sqrt{t^2-1}$. Note that Theorem 1 is true for $t = 2$ although $P(2)$ is not simple and $exp^G(P(2)) = 3 + 2\sqrt{3}$ holds for both Pauli grading and the canonical $\mathbb{Z}_2$-grading (see [20]).

Theorem 1. *Let L be a Lie superalgebra of the type $P(t)$, $t = 2^q, q \geq 1$, equipped with G-grading given in Proposition 2. Then G-graded PI-exponent of L exists and*

$$exp^G(L) = t^2 - 1 + t\sqrt{t^2-1}.$$

2. Pauli gradings

Let L be an algebra over a field F and let G be a group. One says that L is G-graded if L has a vector space decomposition

$$L = \bigoplus_{g\in G} L_g$$

such that $L_g L_h \subseteq L_{gh}$ for all $g, h \in G$. Subspaces $L_g, g \in G$, are called homogeneous components of L. Any element $a \in L_g$ is called homogeneous of degree $\deg a = g$. The subset

$$Supp\ L = \{g \in G | L_g \neq 0\}$$

is said to be the support of the grading. A subspace $V \subseteq L$ is called homogeneous if

$$V = \bigoplus_{g\in G} V \cap L_g.$$

Let A and B be two associative algebras and let G and H be two groups. Suppose that A and B are endowed by G- and H-gradings, respectively,

$$A = \bigoplus_{g\in G} A_g, \quad B = \bigoplus_{h\in H} B_h.$$

Then one can introduce $G \times H$-grading on the tensor product $A \otimes B$ by setting

$$(A \otimes B)_{gh} = A_g \otimes B_h.$$

An associative algebra R is said to be a superalgebra if R has some $\mathbb{Z}_2$-grading, that is

$$R = R^{(0)} \oplus R^{(1)},\ R^{(0)}R^{(0)} + R^{(1)}R^{(1)} \subseteq R^{(0)},\ R^{(0)}R^{(1)} + R^{(1)}R^{(0} \subseteq R^{(1)}.$$

A special case of associative superalgebras which we will use later is the $\mathbb{Z}_2$-graded $n \times n$ matrix algebra $R = M_{k,l}(F)$ with

$$R = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\} = R^{(0)} \oplus R^{(1)}, \quad R^{(0)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, R^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where $n = k + l$, A, B, C, D are $k \times k, k \times l, l \times k$ and $l \times l$ matrices, respectively. In particular, when $k = l$ we have $\mathbb{Z}_2$-grading on $M_{2k}(F)$ which will be used for the definition of Lie superalgebra $P(k)$.

Recall now that $\mathbb{Z}_2$-graded non-associative algebra $L = L^{(0)} \oplus L^{(1)}$ is called a *Lie superalgebra* if it satisfies homogeneous relations

$$ab + (-1)^{|a||b|}ba = 0,\ a(bc) = (ab)c + (-1)^{|a||b|}b(ac) = 0$$

for all $a, b, c \in L^{(0)} \cup L^{(1)}$ where $|x| = 0$ if $x \in L^{(0)}$ and $|x| = 1$ if $x \in L^{(1)}$. In particular, any associative superalgebra $R = R^{(0)} \oplus R^{(1)}$ with the new product called supercommutator, defined for homogeneous elements as

$$[a, b] = ab - (-1)^{|a||b|}ba$$

becomes a Lie superalgebra.

Let $L^{(0)} \oplus L^{(1)}$ be a Lie superalgebra and let G be a group. Then a G-grading

$$L = \oplus_{g \in G} L_g$$

is called *compatible* with $\mathbb{Z}_2$-grading of L if $L_g \subseteq L^{(0)}$ or $L_g \subseteq L^{(1)}$ for all $g \in G$.

For defining the Pauli grading on the associative matrix algebra $M_{2^q}(F)$ we start with $q = 1$. Consider 2×2 matrices

$$\sigma_0 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \sigma_1 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \sigma_2 = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \sigma_3 = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}. \quad (5)$$

Matrices (5) are closely related to Pauli matrices.

$$\sigma_x = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \sigma_y = \left\{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\}, \sigma_z = \left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}.$$

It is well-known that the linear span $L =< \sigma_x, \sigma_y, \sigma_z >$ is closed under Lie commutator and $L \simeq su(2)$ as Lie algebra whereas the span $< \sigma_0, \sigma_1, \sigma_2, \sigma_3 >$ as an associative algebra is isomorphic to $M_2(F)$. Denote by $G =< a >_2 \times < b >_2$ the product of two cyclic groups of order 2 with generators a and b, respectively. Clearly, G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the decomposition

$$R = M_2(F) = R_e \oplus R_a \oplus R_b \oplus R_{ab} \quad (6)$$

is a G-grading, where

$$R_e =< \sigma_0 >, R_a =< \sigma_1 >, R_b =< \sigma_2 >, R_{ab} =< \sigma_3 > .$$

We call the grading (6) on $M_2(F)$ *Pauli grading* on $M_2(F)$.

We generalize this construction to matrices of arbitrary size $2^q, q \geq 2$ in the following way. Let $R = R_1 \otimes \cdots \otimes R_q$ where all $R_1, \ldots, R_q$ are isomorphic to the 2×2 matrix algebra $M_2(F)$. Let also

$$G_0 = G_1 \times \ldots \times G_q, G_j =< a_j >_2 \times < b_j >_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2, j = 1, \ldots, q. \quad (7)$$

Then R has a basis consisting of elements

$$c = x_1 \otimes \cdots \otimes x_q \tag{8}$$

where all $x_1, \ldots, x_q$ are of the type (5). Then in the Kronecker realization of tensor product of matrices for transpose involution T we have

$$c^T = (x_1 \otimes \cdots \otimes x_q)^T = x_1^T \otimes \cdots \otimes x_q^T.$$

In particular, the element c of the type (5) is symmetric if and only if the number of matrices σ_3 among $x_1, \ldots, x_q$ is even and $c^T = -c$ if and only if the number of σ_3 is odd.

All $R_1, \ldots, R_q$ have Pauli grading as defined earlier and we can extend these gradings to their tensor product R. Then we obtain G_0-grading on R

$$R = \bigoplus_{g \in G_0} R_g$$

where $R_g =< x_1 \otimes \cdots \otimes x_q >$ and all $x_1, \ldots, x_q$ are of the type (5). Moreover, we have

$$\deg(x_1 \otimes \cdots \otimes x_q) = \deg x_1 \cdots \deg x_q \tag{9}$$

where

$$\deg x_i = \begin{cases} e_i, & \text{if} \quad x_i = \sigma_0 \\ a_i, & \text{if} \quad x_i = \sigma_1, \\ b_i, & \text{if} \quad x_i = \sigma_2, \\ a_i b_i, & \text{if} \quad x_i = \sigma_3 \end{cases} \tag{10}$$

and $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are defined in (5).

Combining all previous arguments we get the following.

Proposition 1. *The following assertions hold:*

1) *Relations (5), (9), (10) define G_0-grading on the matrix algebra $R = M_{2^q}(F)$, where G_0 is the elementary abelian 2-group defined in (7);*
2) $\dim R_g = 1$ *for every $g \in G_0$;*
3) *R has a homogeneous in G_0-grading basis consisting of products (8) and any basis element is either symmetric or skew-symmetric under transpose involution;*
4) *Every non-zero homogeneous element is invertible; and*
5) *Lie subalgebra sl_{2^q} of traceless matrices is homogeneous in this grading.* □

Applying Proposition 1, we construct a grading on some simple Lie superalgebras. Recall that $P(t)$ (in the notation [17]) is a Lie superalgebra $L \subset M_{t,t}(F)$ with

$$L^{(0)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \right\}, \quad L^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A, B and C are $t \times t$ matrices, $trA = 0$, $B^T = B, C^T = -C$ and $X \to X^T$ is the transpose involution on $M_t(F)$. We equip L with an abelian grading in the following way. Let

$$t = 2^q, \quad R = R_1 \otimes \cdots \otimes R_q, \quad R_1 = \cdots = R_q = M_2(F)$$

and let G_0 be as in (7). We extend G_0 to

$$G =< a_0 >_2 \times G_0 \simeq (\mathbb{Z}_2)^{2q+1}$$

and define G-grading on L compatible with canonical $\mathbb{Z}_2$-grading. If $X_g \in R$ is homogeneous, $\deg X_g = g \in G_0$, then

$$Y = \left\{ \begin{pmatrix} X_g & 0 \\ 0 & -X_g^T \end{pmatrix} \right\}, \tag{11}$$

is homogeneous in L, $\deg Y = g$ for all $X_g \in sl_{2^q}(F) \subset R$,

$$\text{if} \quad X_g \quad \text{is symmetric then} \quad Y = \left\{ \begin{pmatrix} 0 & X_g \\ 0 & 0 \end{pmatrix} \right\} \tag{12}$$

is homogeneous, $\deg Y = a_0 g$

$$\text{if} \quad X_g \quad \text{is skew then} \quad Y = \left\{ \begin{pmatrix} 0 & 0 \\ X_g & 0 \end{pmatrix} \right\} \tag{13}$$

is homogeneous, $\deg Y = a_0 g$. The following proposition is an immediate consequence of Proposition 1 and multiplication rule of L.

Proposition 2. *Let*

$$G_0 =< a_1 >_2 \times < b_1 >_2 \times \cdots \times < a_q >_2 \times < b_q >_2$$

and

$$G =< a_0 >_2 \times G_0$$

be elementary abelian 2-groups. Then (11), (12) and (13) define a G-grading on $L = P(2^q)$ compatible with the canonical $\mathbb{Z}_2$-grading. All homogeneous components of L are 1-dimensional. If

$$g = a_0 g_0, h = a_0 h_0, g_0, h_0 \in G_0, \quad 0 \neq X_g \in L_g, X_h \in L_h$$

and both X_g, X_h are either of the type (12) or of the type (13) then $[X_g, X_h] = 0$. In all other cases $[X_g, X_h]$ is an invertible element of $M_{2^q}(F)$. $\square$

3. Graded PI-exponent

We recall some key notions from the theory of identities and their numerical invariants. We refer the reader to [1,9,14] for details. Consider an absolutely free algebra $F\{X\}$ with a free generating set

$$X = \bigcup_{g\in G} X_g, \quad |X_g| = \infty \quad \text{for any} \quad g \in G.$$

One can define a G-grading on $F\{X\}$ by setting $\deg_G x = g$, when $x \in X_g$, and extend this grading to the entire $F\{X\}$ in the natural way. A polynomial $f(x_1, \ldots, x_n)$ in homogeneous variables $x_1 \in X_{g_1}, \ldots, x_n \in X_{g_n}$ is called a *graded identity* of a G-graded algebra A if $f(a_1, \ldots, a_n) = 0$ for any $a_1 \in A_{g_1}, \ldots, a_n \in A_{g_n}$. The set $Id^G(A)$ of all graded identities of A forms an ideal of $F\{X\}$ which is stable under graded homomorphisms $F\{X\} \to F\{X\}$.

First, let G be finite, $G = \{g_1, \ldots, g_k\}$ and

$$X = X_{g_1} \bigcup \cdots \bigcup X_{g_k}.$$

Denote by $P_{n_1,\ldots,n_k}$ the subspace of $F\{X\}$ of multilinear polynomials of total degree $n = n_1 + \cdots + n_k$ in variables

$$x_1^{(1)}, \ldots, x_{n_1}^{(1)} \in X_{g_1}, \ldots, x_1^{(k)}, \ldots, x_{n_k}^{(k)} \in X_{g_k}.$$

Then the value

$$c_{n_1,\ldots,n_k}(A) = \dim \frac{P_{n_1,\ldots,n_k}}{P_{n_1,\ldots,n_k} \cap Id^G(A)}$$

is called a *partial codimension* of A while

$$c_n^G(A) = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, \ldots, n_k} c_{n_1,\ldots,n_k}(A) \tag{14}$$

is called a *graded codimension* of A. Recall that the *support of the grading* is the set

$$Supp\ A = \{g \in G | A_g \neq 0\}.$$

Note that if $Supp\ A \neq G$, say, $Supp\ A = \{g_1, \ldots, g_d\}$, $d < k$, then the value

$$\sum_{n_1+\cdots+n_d=n} \binom{n}{n_1, \ldots, n_d} \dim \frac{P_{n_1,\ldots,n_d}}{P_{n_1,\ldots,n_d} \cap Id^G(A)} \tag{15}$$

coincides with (14).

Denote

$$P_{n_1,\dots,n_k}(A) = \frac{P_{n_1,\dots,n_k}}{P_{n_1,\dots,n_k} \cap Id^G(A)}. \tag{16}$$

For finding a lower bound for PI-exponent we need the following observation.

Lemma 1. *Let A be a G-graded algebra with the support $SuppA = \{g_1, \dots, g_d\} \subseteq G$. Let also $\dim A_g = 1$ for any $g \in SuppA$. Then*

(1) if $P_{n_1,\dots,n_d}(A) \neq 0$ then $\dim P_{n_1,\dots,n_d}(A) = 1$,
(2) $\dim P_{n_1,\dots,n_d}(A) = 1$ if and only if there exist $u_1 \in A_{g_1}, \dots, u_d \in A_{g_d}$ and a monomial $m(u_1, \dots, u_d) = m \neq 0$ on $u_1, \dots, u_d$ such that every u_j appears in m exactly n_j times, $j = 1, \dots, d$.

Proof. First, let $P_{n_1,\dots,n_d}(A) \neq 0$. Then there exists a multilinear homogeneous polynomial

$$f = f(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(d)}, \dots, x_{n_d}^{(d)}) \in P_{n_1,\dots,n_d}$$

which is not an identity of A. That is, one can find $u_1 \in A_{g_1}, \dots, u_d \in A_{g_d}$ such that $f(u_1, \dots, u_d) \neq 0$. If

$$g = g(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(d)}, \dots, x_{n_d}^{(d)}) \in P_{n_1,\dots,n_d} \setminus Id^G(A)$$

then

$$g(u_1, \dots, u_1, \dots, u_d, \dots, u_d) = \lambda f(u_1, \dots, u_1, \dots, u_d, \dots, u_d)$$

for some scalar λ since $\dim A_g = 1$ for $g = g_1^{n_1} \cdots g_d^{n_d}$. Hence $g - \lambda f \equiv 0$ is an identity of A. This proves (1).

Now let $\dim P_{n_1,\dots,n_d}(A) = 1$, that is $P_{n_1,\dots,n_d}(A) \neq 0$. Then there exist

$$f = f(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(d)}, \dots, x_{n_d}^{(d)}) \in P_{n_1,\dots,n_d} \setminus Id^G(A)$$

and $u_1 \in A_{g_1}, \dots, u_d \in A_{g_d}$ such that

$$f(\underbrace{u_1, \dots, u_1}_{n_1}, \dots, \underbrace{u_d, \dots, u_d}_{n_d}) \neq 0$$

in A. Hence, at least one monomial of f has a non-zero value under evaluation $\varphi : F\{X\} \mapsto A$, where

$$\varphi(x_j^{(i)}) = u_i, \quad 1 \leq i \leq d, \quad 1 \leq j \leq n_i.$$

This implies (2), and have we completed the proof. □

Corollary 1.

$$c_n^G = \sum \binom{n}{n_1, \ldots, n_d} \tag{17}$$

where the sum in (17) is taken over all tuples $(n_1, \ldots, n_d)$ *such that*

$$P_{n_1,\ldots,n_d}(A) \neq 0. \tag{18}$$

Moreover, for the inequality (18) it suffices to check the condition (2) of Lemma 1. □

Now we go back to the Lie superalgebra

$$L = L^{(0)} \oplus L^{(1)} = P(t), \quad t = 2^q,$$

with the G-grading presented in Proposition 2. First, we give an upper bound for $exp^G(L)$. Note that Stirling formula for factorials implies the inequalities

$$\frac{1}{n^d}\Phi(n; n_1, \ldots, n_d)^n \leq \binom{n}{n_1, \ldots, n_d} \leq n\Phi(n; n_1, \ldots, n_d)^n \tag{19}$$

where

$$\Phi(n; n_1, \ldots, n_d) = (\frac{n_1}{n})^{-\frac{n_1}{n}} \cdots (\frac{n_d}{n})^{-\frac{n_d}{n}}$$

and $n = n_1 + \cdots + n_d$.

Denote

$$a = \frac{t(t+1)}{2}, b = \frac{t(t-1)}{2}, c = t^2 - 1, d = a + b + c = \dim L.$$

The algebra L has a natural $\mathbb{Z}$-grading

$$L = \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$$

where

$$\mathcal{L}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\}, \mathcal{L}_0 = L^{(0)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \right\}, \mathcal{L}_1 = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}.$$

All remaining components $\mathcal{L}_k$, $k \neq 0, \pm 1$, are zero. Clearly, $P_{n_1,\ldots,n_d}(L) \neq 0$ only if

$$|n_1 + \cdots + n_a - n_{a+1} \cdots - n_{a+b}| \leq 1 \tag{20}$$

where $\{g_1, \ldots, g_d\} \subseteq G$ is the support $SuppL$. It follows from Corollary 1 and (19) that

$$\frac{1}{n^d}\max\{\Phi(n;n_1,\dots,n_d)^n\} \le c_n^G(L) \le n^d \max\{\Phi(n;n_1,\dots,n_d)^n\} \tag{21}$$

where the maximum is taken over all $n_1,\dots,n_d$ satisfying (20).

First, consider the case where the left side of (20) is equal to zero. Then we rewrite

$$\Phi(n;n_1,\dots,n_d) = \Phi(x_1,\dots,x_d)$$

where $x_1+\cdots+x_d = 1$, $x_1,\dots,x_d \ge 0$,

$$\Phi(x_1,\dots,x_d) = x_1^{-x_1}\cdots x_d^{-x_d} \tag{22}$$

and

$$x_1+\cdots+x_a = x_{a+1}+\cdots+x_{a+b}.$$

It is easy to see that the maximal value of the function (22) is achieved when

$$x_1=\cdots=x_a,\ x_{a+1}=\cdots=x_{a+b},\ x_{a+b+1}=\cdots=x_{a+b+c}.$$

Denote $x=x_1, y=x_{a+b}, z=x_{a+b+c}$. Then (22) does not exceed

$$\widetilde{\Phi}=\widetilde{\Phi}(x,y,z)=x^{-ax}y^{-by}z^{-cz}$$

and x,y,z satisfy the relations $ax=by$, $ax+by+cz=1$. These relations imply

$$\widetilde{\Phi}^{-1} = z^{(t^2-1)z}(1-(t^2-1)z)^{(1-(t^2-1)z)}(t^2(t^2-1))^{\frac{(t^2-1)z-1}{2}}$$

as a function of z. Then

$$g(z)=\ln\widetilde{\Phi}^{-1} = cz\ln z+(1-cz)\ln(1-cz)-\frac{1}{2}(1-cz)\ln(ct^2).$$

Direct calculations show that $g'(z)=0$ only if

$$z=z_0=(t^2-1+t\sqrt{t^2-1})^{-1}$$

and $g''(z_0)>0$. Hence, in z_0 the function $g(z)$ has a local minimum. Moreover,

$$g(z_0)=-\ln(t^2-1+t\sqrt{t^2-1}).$$

It follows that

$$\widetilde{\Phi}\le t^2-1+t\sqrt{t^2-1}$$

and

$$\sqrt[n]{c_n^G(L)} \le n^{\frac{d}{n}}(t^2-1+t\sqrt{t^2-1}) \tag{23}$$

as follows from (21) in the case $n_1+\cdots+n_a = n_{a+1}+\cdots+n_{a+b}$.

If $n_1+\cdots+n_a-n_{a+1}-\cdots-n_{a+b} = -1$ then

$$\binom{n}{n_1,\ldots,n_d} \le \binom{n+1}{n_1+1,n_2,\ldots,n_d}$$

and

$$\sqrt[n]{c_n^G(L)} \le (n+1)^{\frac{d}{n+1}}(t^2-1+t\sqrt{t^2-1}). \tag{24}$$

Similarly, if $n_1+\cdots+n_a-n_{a+1}-\cdots-n_{a+b} = 1$ then

$$\sqrt[n]{c_n^G(L)} \le (n-1)^{\frac{d}{n-1}}(t^2-1+t\sqrt{t^2-1}) \tag{25}$$

since

$$\binom{n}{n_1,\ldots,n_d} \le n\binom{n-1}{n_1,\ldots,n_{a+b-1},n_{a+b}-1,n_{a+b+1},\ldots,n_d}.$$

Inequalities (23), (24) and (25) give us the following.

Lemma 2.

$$exp^G(L) \le t^2-1+t\sqrt{t^2-1}.$$

Now we will get the same lower bound.

Lemma 3.

$$exp^G(L) \ge t^2-1+t\sqrt{t^2-1}. \tag{26}$$

Proof. Recall that L is $\mathbb{Z}$-graded algebra, $L = \mathcal{L}_{-1}\oplus\mathcal{L}_0\oplus\mathcal{L}_1$, and $a = \dim\mathcal{L}_1$, $b = \dim\mathcal{L}_{-1}$, $c = \dim\mathcal{L}_0$. Consider a collection

$$X = \{\underbrace{x_1,\ldots,x_1}_{b},\ldots,\underbrace{x_a,\ldots,x_a}_{b}\}$$

where $x_1,\ldots,x_a$ are homogeneous in G-grading elements $\mathcal{L}_1$ with pairwise distinct degree in G-grading. Similarly, we take

$$Y = \{\underbrace{y_1,\ldots,y_1}_{a},\ldots,\underbrace{y_b,\ldots,y_b}_{a}\},$$

with homogeneous $y_1, \dots, y_b \in \mathcal{L}_{-1}$, $deg_G y_i$ are distinct. Renaming elements of X, Y we write

$$X = \{x^{(1)}, \dots, x^{(ab)}\}, \quad Y = \{y^{(1)}, \dots, y^{(ab)}\}.$$

We remark that any $x_i, 1 \le i \le a$ appears among $x^{(1)}, \dots, x^{(ab)}$ exactly b times. Similarly, any $y_j, 1 \le j \le b$, appears among $y^{(1)}, \dots, y^{(ab)}$ exactly a times. Consider supercommutators

$$z_1 = [x^{(1)}, y^{(1)}], \dots, z_{ab} = [x^{(ab)}, y^{(ab)}].$$

By Proposition 2 all z_i are invertible in $M_{2t}(F)$ matrices homogeneous in G-grading of L. Also,

$$z_1, \dots, z_{ab} \in L^{(0)} \simeq sl_{2t}(F).$$

Note that $xy = \pm yx$ for any homogeneous $x, y \in L^{(0)}$. It follows that for any $i = 1, \dots, ab$ there exists $z_i' \in L^{(0)}$ homogeneous in G-grading such that

$$[z_i', z_i] = 2z_i' z_i \neq 0$$

where the product $z_i' z_i$ is taken in the associative algebra $M_{2t}(F)$. Hence, the left-normed Lie commutators

$$z_k^{(i)} = [z_i', \underbrace{z_i, \dots, z_i}_{k}] = 2^k z_i' z_i^k, \ k = 1, 2, \dots,$$

are non-zero homogeneous elements of $L^{(0)}$.

As before, one can find homogeneous $u_1, \dots, u_{ab} \in L^{(0)}$ and linearly independent homogeneous $v_1, \dots, v_c \in L^{(0)}$ such that

$$w_k = [z_k^{(1)}, u_1, z_k^{(2)}, u_2, \dots, z_k^{(ab)}, u_{ab}] \neq 0$$

and

$$w_{k,s} = [w_k, w_1', \underbrace{v_1, \dots, v_1}_{s}, w_2', \underbrace{v_2, \dots, v_2}_{s}, \dots, w_c', \underbrace{v_c, \dots, v_c}_{s}] \neq 0$$

for some homogeneous $w_1', \dots, w_c' \in L^{(0)}$.

If u is a monomial on $x_1, \dots, x_a, y_1, \dots, y_b, v_1, \dots, v_c$ in L then we will denote by $\mathrm{Deg}_{x_i} u, \mathrm{Deg}_{y_i} u, \mathrm{Deg}_{v_i} u$ the total number of factors x_i, y_i and v_i in u, respectively. Then

$$\mathrm{Deg}_{x_i} w_{k,s} \ge kb \quad \text{for all} \quad i = 1, \dots, a,$$
$$\mathrm{Deg}_{y_i} w_{k,s} \ge ka \quad \text{for all} \quad i = 1, \dots, b,$$
$$\mathrm{Deg}_{v_i} w_{k,s} \ge s \quad \text{for all} \quad i = 1, \dots, c.$$

Total degrees Deg on $\{x_\alpha, y_\beta, v_\gamma\}$ are as follows:

$$\mathrm{Deg} z_k^{(i)} = 2k+1, \mathrm{Deg} w_k = 2ab(k+1), \mathrm{Deg} w_{k,s} = 2ab(k+1) + c(s+1) = n.$$

Denote

$$n_i = \mathrm{Deg}_{x_i} w_{k,s},\ i = 1,\dots,a, \tag{27}$$

$$n_{a+i} = \mathrm{Deg}_{y_i} w_{k,s},\ i = 1,\dots,b, \tag{28}$$

$$n_{a+b+i} = \mathrm{Deg}_{v_i} w_{k,s},\ i = 1,\dots,c. \tag{29}$$

If

$$m_1 = \dots = m_a = kb, m_{a+1} = \dots = m_{a+b} = ka, m_{a+b+1} = \dots = m_{a+b+c} = s,$$

and

$$m = m_1 + \dots + m_{a+b+c} = 2abk + cs$$

then $n - m = 2ab + c$ and

$$\binom{n}{n_1,\dots,n_d} \geq \binom{m}{m_1,\dots,m_d} \geq$$
$$\geq \frac{1}{m^d}\Phi(m; m_1,\dots,m_d)^m = \frac{1}{m^d}\widetilde{\Phi}(\frac{kb}{m}, \frac{ka}{m}, \frac{s}{m})^m. \tag{30}$$

Denote $\frac{k}{s} = \alpha$. Then

$$\frac{s}{m} = \frac{s}{2abk + cs} = \frac{1}{\frac{k}{s}\cdot\frac{t^2(t^2-1)}{2} + t^2 - 1} = \frac{1}{t^2 - 1 + \alpha\frac{t^2(t^2-1)}{2}}.$$

Note that if

$$\beta = \frac{2}{t\sqrt{t^2-1}}$$

then

$$t^2 - 1 + \beta\frac{t^2(t^2-1)}{2}) = t^2 - 1 + t\sqrt{t^2-1}$$

and

$$\widetilde{\Phi}(\bar{x}, \bar{y}, \bar{z}) = \Phi_{max} = t^2 - 1 + t\sqrt{t^2-1}$$

provided that

$$\bar{z} = (t^2 - 1 + t\sqrt{t^2-1})^{-1},\ a\bar{x} = b\bar{y},\ a\bar{x} + d\bar{y} + c\bar{z} = 1.$$

In particular, if

$$\alpha = \frac{k}{s} \to \beta$$

then

$$\widetilde{\Phi}(\frac{kb}{m}, \frac{ka}{m}, \frac{s}{m}) \to \Phi_{max}.$$

More precisely, for any $\varepsilon > 0$ there exists real δ such that the inequality

$$|\frac{k}{s} - \frac{2}{t\sqrt{t^2-1}}| < \delta \tag{31}$$

implies

$$\Phi(m; m_1, \ldots, m_d) \geq t^2 - 1 + t\sqrt{t^2-1} - \varepsilon. \tag{32}$$

Fix one pair (k, s) with the relation (31) and take

$$m = 2abk + cs, \quad \bar{n}_1 = m + 2ab + c, \quad n_i$$

as in (27), (28), (29). Then we have for any $r = 1, 2, \ldots,$

$$\binom{r\bar{n}_1}{rn_1, \ldots, rn_d} \geq \frac{1}{(r\bar{n}_1)^d}\Phi(rm; rm_1, \ldots, rm_d)^{rm} = \frac{1}{(r\bar{n}_1)^d}\Phi(m; m_1, \ldots, m_d)^{rm}$$
$$\geq \frac{1}{(r\bar{n}_1)^d}(t^2 - 1 + t\sqrt{t^2-1} - \varepsilon)^{rm}$$

as follows from (30), (32).

Denote $\bar{n}_r = r\bar{n}_1$. For any given $\rho > 0$ we can choose $\bar{n}_1$ large enough and suppose that

$$\frac{rm}{\bar{n}_r} = \frac{\bar{n}_r - (2ab + c)r}{\bar{n}_r} = 1 - \frac{2ab + c}{\bar{n}_1} > 1 - \rho$$

from which it follows that

$$c^G_{\bar{n}_r} \geq \frac{1}{\bar{n}_r^d}(t^2 - 1 + t\sqrt{t^2-1} - \varepsilon)^{1-\rho}. \tag{33}$$

Since $\bar{n}_{r+1} - \bar{n}_r = 2ab + c = const$ and

$$\binom{n'}{n'_1, \ldots, n'_d} \geq \binom{n}{n_1, \ldots, n_d}$$

as soon as

$$n' = n + 1, n'_1 \geq n_1, \ldots, n'_d \geq n_d,$$

(33) implies the inequality

$$exp^G(L) \geq t^2 - 1 + t\sqrt{t^2 - 1} - \varepsilon.$$

Recall that $\varepsilon > 0$ is arbitrary, hence (26) follows and we are done. □

Proof of Theorem 1. The assertions of Theorem 1 now follow from Lemmas 2 and 3. □

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