

Identities of graded simple algebras

Dušan Repovš^{a,b} and Mikhail Zaicev^c

^aFaculty of Education, University of Ljubljana, Ljubljana, Slovenia; ^bFaculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia; ^cDepartment of Algebra, Faculty of Mathematics and Mechanics, Moscow State University, Moscow, Russia

ABSTRACT

We study identities of finite dimensional algebras over a field of characteristic zero, graded by an arbitrary groupoid Γ . First, we prove that its graded colength has a polynomially bounded growth. For any graded simple algebra A , we prove the existence of the graded PI-exponent, provided that Γ is a commutative semigroup. If A is simple in a non-graded sense, the existence of the graded PI-exponent is proved without any restrictions on Γ .

ARTICLE HISTORY

Received 29 January 2016
Accepted 9 March 2016

COMMUNICATED BY

M. Chebotar

KEYWORDS

Polynomial identities;
graded algebras;
codimensions; exponential
growth

AMS SUBJECT

CLASSIFICATIONS

Primary: 17B01; 16P90;
Secondary: 16R10

1. Introduction

We study numerical characteristics of identities of finite dimensional graded simple algebras over a field of characteristic zero. The main object of our investigations is the asymptotic behaviour of sequences of graded codimensions and graded colengths of such algebras (all necessary definitions and notions will be given in the next section). Given a graded algebra A , one can associate the sequence of so-called *graded codimensions* $\{c_n^{gr}(A), n = 1, 2, \dots\}$. This sequence is an important numerical invariant of graded identities of A . It is known that this sequence is exponentially bounded, that is $c_n^{gr}(A) \leq a^n$ for some real a , provided that $\dim A < \infty$. In this case, the following natural question arises: does the limit

$$\exp^{gr}(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)} \quad (1)$$

exist and what are its possible values? If the limit (1) exists, then it is called the *graded PI-exponent* of A .

In the non-graded case, codimension growth is well understood. Existence and integrality of the (non-graded) PI-exponent were conjectured by Amitsur in 1980s for associative PI-algebras. Amitsur's conjecture was confirmed in [1,2]. Later, the same result was proved for finite dimensional Lie algebras [3–5], Jordan and alternative algebras [6–8] and many other algebraic systems. In the general non-associative case, for any real $\alpha > 1$, examples of

algebras with PI-exponent equal to α were constructed in [9]. Recently, the first example of algebra A such that the PI-exponent of A does not exist was constructed.[10] Nevertheless, for any finite dimensional simple algebra, the PI-exponent does exist.[11]

For graded algebras, there are only partial results of this kind. For example, if A is an associative graded PI-algebra, then its graded PI-exponent always exists and it is an integer.[12] An existence of the \mathbb{Z}_2 -graded PI-exponent for any finite dimensional simple Lie superalgebra has recently been proved in [13]. Note that for finite dimensional Lie superalgebras, both graded and ordinary PI-exponents can be fractional.[11,14,15] The main purpose of this paper is to prove the existence of graded PI-exponents for any finite dimensional graded simple algebra (see Theorem 2).

Another important numerical characteristic of identities of an algebra A is the so-called *colength sequence* $l_n(A)$. Except for its independent interest, asymptotic behaviour of $\{l_n(A)\}$ plays an important role in the studies of asymptotics of $\{c_n(A)\}$. The polynomial type of growth of $\{l_n(A)\}$ is very convenient for investigations of codimension growth.

Polynomial upper bounds of the colength for any associative PI-algebra were established in [16]. For an arbitrary (non-associative) finite dimensional algebra, the same restriction was obtained in [17]. In the case of finite dimensional Lie superalgebras, polynomial growth of \mathbb{Z}_2 -graded colength has recently been confirmed in [18]. In order to get the main result of the paper, we will find the polynomial upper bound for graded colength of a finite dimensional graded algebra (see Theorem 1).

2. Preliminaries

Let Γ be a groupoid. An F -algebra A is said to be Γ -graded if there is a vector space decomposition

$$A = \bigoplus_{g \in \Gamma} A_g$$

and $A_g A_h \subseteq A_{gh}$ for all $g, h \in \Gamma$. An element $a \in A$ is called *homogeneous of degree g* if $a \in A_g$ and in this case, we write $\deg_\Gamma a = g$. A subspace $V \subseteq A$ is homogeneous iff $V = \bigoplus_{g \in \Gamma} (V \cap A_g)$. We call A *graded simple*, if it has no homogeneous ideals. For instance, if Γ is a group and $A = F[\Gamma]$ is its group algebra, then A is Γ -graded simple but is not simple in the usual sense. On the other hand, any simple algebra with an arbitrary grading is graded simple.

We recall some key notions from the theory of graded and ordinary identities and their numerical invariants. We refer the reader to [19,20] for details. Consider an absolutely free algebra $F\{X\}$ with a free generating set

$$X = \bigcup_{g \in \Gamma} X_g, \quad |X_g| = \infty \quad \text{for any } g \in \Gamma.$$

One can define a Γ -grading on $F\{X\}$ by setting $\deg_\Gamma x = g$, when $x \in X_g$, and extend this grading to the entire $F\{X\}$ in the natural way. A polynomial $f(x_1, \dots, x_n)$ in homogeneous variables $x_1 \in X_{g_1}, \dots, x_n \in X_{g_n}$ is called a *graded identity* of a Γ -graded algebra A if $f(a_1, \dots, a_n) = 0$ for any $a_1 \in A_{g_1}, \dots, a_n \in A_{g_n}$. The set $Id^{gr}(A)$ of all graded identities of A forms an ideal of $F\{X\}$ which is stable under graded homomorphisms $F\{X\} \rightarrow F\{X\}$.

First, let Γ be finite, $\Gamma = \{g_1, \dots, g_t\}$ and $X = X_{g_1} \cup \dots \cup X_{g_t}$. Denote by P_{n_1, \dots, n_t} the subspace of $F\{X\}$ of multilinear polynomials of total degree $n = n_1 + \dots + n_t$ in variables $x_1^{(1)}, \dots, x_{n_1}^{(1)} \in X_{g_1}, \dots, x_1^{(t)}, \dots, x_{n_t}^{(t)} \in X_{g_t}$. Then the value

$$c_{n_1, \dots, n_t}(A) = \dim \frac{P_{n_1, \dots, n_t}}{P_{n_1, \dots, n_t} \cap \text{Id}^{gr}(A)}$$

is called a *partial codimension* of A while

$$c_n^{gr}(A) = \sum_{n_1 + \dots + n_t = n} \binom{n}{n_1, \dots, n_t} c_{n_1, \dots, n_t}(A) \tag{2}$$

is called a *graded codimension* of A . Recall that the *support of the grading* is the set

$$\text{Supp } A = \{g \in \Gamma \mid A_g \neq 0\}.$$

Note that if $\text{Supp } A \neq \Gamma$, say, $\text{Supp } A = \{g_1, \dots, g_k\}$, $k < t$, then the value

$$\sum_{n_1 + \dots + n_k = n} \binom{n}{n_1, \dots, n_k} \dim \frac{P_{n_1, \dots, n_k}}{P_{n_1, \dots, n_k} \cap \text{Id}^{gr}(A)} \tag{3}$$

coincides with (2). This allows us to consider (3) as the definition of the graded codimension of A even if Γ is infinite, provided that $\text{Supp } A = \{g_1, \dots, g_k\}$.

For convenience, denote

$$P_{n_1, \dots, n_k}(A) = \frac{P_{n_1, \dots, n_k}}{P_{n_1, \dots, n_k} \cap \text{Id}^{gr}(A)}. \tag{4}$$

Given $1 \leq j \leq k$, consider the action of the symmetric group S_{n_j} on P_{n_1, \dots, n_k} defined by

$$\sigma f \left(\dots, x_1^{(j)}, \dots, x_{n_j}^{(j)}, \dots \right) = f \left(\dots, x_{\sigma(1)}^{(j)}, \dots, x_{\sigma(n_j)}^{(j)}, \dots \right).$$

Then the spaces P_{n_1, \dots, n_k} and $P_{n_1, \dots, n_k}(A)$ become $F[H]$ -modules, where $H = S_{n_1} \times \dots \times S_{n_k}$. Any $F[H]$ -module $P_{n_1, \dots, n_k}(A)$ is decomposed into the sum of irreducible $F[H]$ -submodules and in the languages of group characters, it can be written as

$$\chi_H(P_{n_1, \dots, n_k}(A)) = \sum_{\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k} m_{\lambda^{(1)}, \dots, \lambda^{(k)}} \chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}. \tag{5}$$

Here, $\chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}$ is the character of the irreducible H -representation defined by the k -tuple $(\lambda^{(1)}, \dots, \lambda^{(k)})$ of partitions $\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k$ and $m_{\lambda^{(1)}, \dots, \lambda^{(k)}}$ is the multiplicity of the corresponding $F[H]$ -module in $P_{n_1, \dots, n_k}(A)$. The integer

$$l_{\lambda^{(1)}, \dots, \lambda^{(k)}}(A) = \sum_{\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k} m_{\lambda^{(1)}, \dots, \lambda^{(k)}} \tag{6}$$

is called the *partial colength*, whereas the integer

$$l_n^{gr}(A) = \sum_{n_1 + \dots + n_k = n} l_{n_1, \dots, n_k}(A) \quad (7)$$

is called the *graded colength* of A .

As it was mentioned in the introduction, graded codimensions are exponentially bounded if A is finite dimensional. Namely,

$$c_n^{gr}(A) \leq d^{n+1} \quad (8)$$

where $d = \dim A$ (see [21] and also [7, Proposition 2]). This result was proved in [7,21] under the assumption that Γ is a finite group. The same argument is valid for an arbitrary groupoid. Relation (8) allows us to consider upper and lower limits of $\sqrt[n]{c_n^{gr}(A)}$ and we can define the lower and the upper graded PI-exponents as follows:

$$\underline{\exp}^{gr}(A) = \liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}, \quad \overline{\exp}^{gr}(A) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}.$$

If the lower and the upper limits coincide, then we also define the graded PI-exponent by

$$\exp^{gr}(A) = \underline{\exp}^{gr}(A) = \overline{\exp}^{gr}(A).$$

Representation theory of symmetric groups is a useful tool for studying asymptotics of codimension growth. Basic notions of S_n -representations can be found in [22] and its application to PI-theory in [19,20].

Recall that, given a partition $\lambda \vdash n$, there is exactly one (up to isomorphism) irreducible S_n -representation defined by λ . Its character and dimension are denoted by χ_λ and $\chi_\lambda(1) = d_\lambda$, respectively. For the group $H = S_{n_1} \times \dots \times S_{n_k}$, any irreducible representation is defined by the k -tuple of partitions $\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k$ and its character and dimension are $\chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}$. Moreover,

$$\chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}(1) = d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}}, \quad (9)$$

respectively. In particular, (5) and (9) imply the equality

$$\begin{aligned} c_{n_1, \dots, n_k}(A) &= \chi_H(P_{n_1, \dots, n_k}(A))(1) \\ &= \sum_{\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k} m_{\lambda^{(1)}, \dots, \lambda^{(k)}} d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}}. \end{aligned} \quad (10)$$

Let $d \geq 1$ be a fixed integer and let $\nu = (\nu_1, \dots, \nu_q) \vdash m$ be a partition of m with $q \leq d$. Dimension of an irreducible $F[S_m]$ -module with the character χ_ν is closely connected with the following function:

$$\Phi(\nu) = \frac{1}{\binom{\nu_1}{m}^{\frac{\nu_1}{m}} \cdots \binom{\nu_d}{m}^{\frac{\nu_d}{m}}}.$$

Here, we assume that $\nu_{q+1} = \dots = \nu_d = 0$ in the case $q < d$ and $0^0 = 1$. The values $\Phi(\nu)^m$ and d_ν are close in the following sense.

Lemma 1: (see [11, Lemma 1]) *Let $m \geq 100$. Then*

$$\frac{\Phi(\nu)^m}{m^{d^2+d}} \leq d_\nu \leq m\Phi(\nu)^m.$$

We will use the following property of Φ . Let ν and ρ be any two partitions of m , such that $\nu = (\nu_1, \dots, \nu_p)$, $\rho = (\rho_1, \dots, \rho_q)$, $p, q \leq d$ and $q = p$ or $q = p + 1$, $\rho_{p+1} = 1$. As before, we consider ρ and ν as partitions with d components. We say that the Young diagram D_ρ is obtained from diagram D_ν by pushing down one box if there exists $1 \leq i < j \leq d$ such that $\rho_i = \nu_i - 1$, $\rho_j = \nu_j + 1$ and $\rho_t = \nu_t$ for all remaining $1 \leq t \leq d$.

Lemma 2: (see [11, Lemma 3], [23, Lemma 2]) *Let D_ρ be obtained from D_ν by pushing down one box. Then $\Phi(\rho) \geq \Phi(\nu)$.*

3. Polynomial growth of graded colength

Consider a finite dimensional Γ -graded algebra A with the support $Supp A = \{g_1, \dots, g_k\}$, $A = A_{g_1} \oplus \dots \oplus A_{g_k}$. Let

$$d_1 = \dim A_{g_1}, \dots, d_k = \dim A_{g_k}$$

be dimensions of the homogeneous components. Recall that an irreducible $F[S_t]$ -module corresponding to the partition $\mu \vdash t$ can be realized as a minimal left $F[S_t]$ -ideal generated by an essential idempotent e_{T_λ} where T_λ is some Young tableaux with Young diagram D_λ . For $H = S_{n_1} \times \dots \times S_{n_k}$, any irreducible $F[H]$ -module is isomorphic to the tensor product of $F[S_{n_1}], \dots, F[S_{n_k}]$ -modules with characters $\chi_{\lambda^{(1)}}, \dots, \chi_{\lambda^{(k)}}$, respectively. The following remark easily follows from the construction of essential idempotents and therefore we omit the proof.

Lemma 3: *Let $\lambda^{(1)} = (\lambda_1^{(1)}, \dots, \lambda_{q_1}^{(1)})$, \dots , $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_{q_k}^{(k)})$ be partitions of n_1, \dots, n_k , respectively. Suppose that the multiplicity $m_{\lambda^{(1)}, \dots, \lambda^{(k)}}$ on the right hand side of (5) is non-zero. Then $q_1 \leq d_1, \dots, q_k \leq d_k$.*

For convenience, we shall assume as before that $q_1 = d_1, \dots, q_k = d_k$ even if q_i is strictly less than d_i for some i .

Denote by $R = R\{X_{g_1} \cup \dots \cup X_{g_k}\}$ the relatively free algebra of the variety $var A$ of Γ -graded algebras generated by A . Denote by $R_{d_1, \dots, d_k}^{n_1, \dots, n_k}$ the subspace of polynomials in R of degree n_1 in the set of variables $\{X_1^{(1)}, \dots, X_{d_1}^{(1)}\} \subseteq X_{g_1}$, of degree n_2 in $\{X_1^{(2)}, \dots, X_{d_2}^{(2)}\} \subseteq X_{g_2}$, etc.

Lemma 4: *Multiplicities on the right hand side of (5) satisfy the inequalities*

$$m_{\lambda^{(1)}, \dots, \lambda^{(k)}} \leq \dim R_{d_1, \dots, d_k}^{n_1, \dots, n_k}.$$

Proof: Let $\tilde{P}_{n_1, \dots, n_k}$ be the subspace of multilinear polynomials of degree $n = n_1 + \dots + n_k$ on $x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(k)}, \dots, x_{n_k}^{(k)}$ in R . Here, $x_j^{(i)} \in X_{g_i}$ for all $1 \leq i \leq k, 1 \leq j \leq n_i$. Then $\tilde{P}_{n_1, \dots, n_k}$ is isomorphic to $P_{n_1, \dots, n_k}(A)$ as an $F[H]$ -module. Denote for brevity $q = m_{\lambda^{(1)}, \dots, \lambda^{(k)}}$ and consider the $F[H]$ -submodule

$$M = M_1 \oplus \dots \oplus M_q$$

of $\tilde{P}_{n_1, \dots, n_k}$, where M_1, \dots, M_q are isomorphic irreducible $F[H]$ -modules with H -character $\chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}$. Any M_j is generated as an $F[H]$ -module by a multilinear polynomial of the type

$$\begin{aligned}
 & f_j \left(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(k)}, \dots, x_{n_k}^{(k)} \right) \\
 &= e_{T_{\lambda^{(1)}}} \cdots e_{T_{\lambda^{(k)}}} h_j \left(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(k)}, \dots, x_{n_k}^{(k)} \right)
 \end{aligned}$$

with a multilinear polynomial $h_j \in \tilde{P}_{n_1, \dots, n_k}$.

One can split the set of indeterminates $x_1^{(1)}, \dots, x_{n_k}^{(k)}$ into a disjoint union of subsets

$$\begin{aligned}
 P_1^{(1)} &= \left\{ x_1^{(1)}, \dots, x_{\lambda_1^{(1)}}^{(1)} \right\}, \\
 P_2^{(1)} &= \left\{ x_{\lambda_1^{(1)}+1}^{(1)}, \dots, x_{\lambda_1^{(1)}+\lambda_2^{(1)}}^{(1)} \right\}, \\
 &\dots \\
 P_{d_1}^{(1)} &= \left\{ x_{n_1-\lambda_{d_1}^{(1)}+1}^{(1)}, \dots, x_{n_1}^{(1)} \right\}, \\
 &\dots \\
 P_1^{(k)} &= \left\{ x_1^{(k)}, \dots, x_{\lambda_1^{(k)}}^{(k)} \right\}, \\
 P_2^{(k)} &= \left\{ x_{\lambda_1^{(k)}+1}^{(k)}, \dots, x_{\lambda_1^{(k)}+\lambda_2^{(k)}}^{(k)} \right\}, \\
 &\dots \\
 P_{d_k}^{(k)} &= \left\{ x_{n_k-\lambda_{d_k}^{(k)}+1}^{(k)}, \dots, x_{n_k}^{(k)} \right\}
 \end{aligned}$$

so that all f_1, \dots, f_q are symmetric on any subset $P_1^{(1)}, \dots, P_{d_k}^{(k)}$.

Now, we identify all variables in each symmetric subset, that is, we apply a homomorphism $\varphi : R \rightarrow R$ such that

- $\varphi(x_\alpha^{(j)}) = x_1^{(j)}$ if $1 \leq \alpha \leq \lambda_1^{(j)}$,
- \dots
- $\varphi(x_\alpha^{(j)}) = x_{d_j}^{(j)}$ if $\lambda_1^{(j)} + \dots + \lambda_{d_j-1}^{(j)} < \alpha \leq n_j$

for all $j = 1, \dots, k$. Then all $\varphi(f_1), \dots, \varphi(f_q)$ lie in $R_{d_1, \dots, d_k}^{n_1, \dots, n_k}$. Note that the total linearization of each of $\varphi(f_j)$ is equal to f_j with a non-zero coefficient independent of j . Hence, a non-trivial linear relation $\alpha_1 \varphi(f_1) + \dots + \alpha_q \varphi(f_q) = 0$ implies the same relation $\alpha_1 f_1 + \dots + \alpha_q f_q = 0$. But f_1, \dots, f_q belong to distinct irreducible summands M_1, \dots, M_q , respectively. In particular, they are linearly independent. Hence, q does not exceed $\dim R_{d_1, \dots, d_k}^{n_1, \dots, n_k}$, and the proof is completed. \square

Now, we restrict the dimension of $R_{d_1, \dots, d_k}^{n_1, \dots, n_k}$.

Lemma 5: *Let $A = A_{g_1} \oplus \dots \oplus A_{g_k}$ be a Γ -graded algebra with the support $\{g_1, \dots, g_k\}$ and let $d_1 = \dim A_{g_1}, \dots, d_k = \dim A_{g_k}$. Then*

$$\dim R_{d_1, \dots, d_k}^{n_1, \dots, n_k} \leq (d_1 + \dots + d_k)(n_1 + 1)^{d_1^2} \cdots (n_k + 1)^{d_k^2}. \quad (11)$$

Proof: Let $\{a_1^{(g_1)}, \dots, a_{d_i}^{(g_i)}\}$ be a basis of the subspace A_{g_i} , $1 \leq i \leq k$. Consider a polynomial ring $F[Y]$, where $Y = Y_{g_1} \cup \dots \cup Y_{g_k}$ and

$$Y_{g_i} = \left\{ y_{m,j}^{g_i} \mid 1 \leq m \leq d_i, j = 1, 2, \dots \right\}.$$

Then algebra $F[Y]$ can be naturally endowed by a Γ -grading with $\text{Supp } F[Y] = \{g_1, \dots, g_k\}$ if we set $\deg_{\Gamma} y = g_i$ when $y \in Y_{g_i}$. Denote $\tilde{A} = A \otimes F[Y]$ and fix elements

$$z_j^{g_i} = \sum_{m=1}^{d_i} a_m^{(g_i)} \otimes y_{m,j}^{g_i}, \quad j = 1, 2, \dots,$$

in \tilde{A} . Then $\text{alg}\{z_j^{g_i}\}$ is also a Γ -graded algebra, where $\deg z_j^{g_i} = g_i$. Moreover, $\tilde{A} \simeq R$ and $R_{d_1, \dots, d_k}^{n_1, \dots, n_k}$ is a subspace of $A \otimes T$, where T is the subspace of $F[Y]$ spanned by monomials of degree at most n_t in the set of indeterminates $\{y_{m,j}^{g_t} \mid 1 \leq m, j \leq d_t\}$, $t = 1, \dots, k$. Clearly,

$$\dim T \leq (n_1 + 1)^{d_1^2} \dots (n_k + 1)^{d_k^2} \quad (12)$$

hence (11) follows from (12). □

Now, we are ready to get an upper bound for graded colength of A .

Theorem 1: Let $A = \bigoplus_{g \in \Gamma} A_g$ be a finite dimensional algebra graded by groupoid Γ with $\text{Supp } A = \{g_1, \dots, g_k\}$. Let also $\dim A_{g_i} = d_i$, $1 \leq i \leq k$. Then the n th graded colength of A satisfies the inequality

$$l_n^{gr} \leq d(n+1)^{k+d_1^2+\dots+d_k^2+d_1+\dots+d_k}$$

where $d = \dim A = d_1 + \dots + d_k$.

Proof: By Lemma 3, the total number of partitions $\lambda_i \vdash n_i$ does not exceed $(n_i + 1)^{d_i}$ for any $i = 1, \dots, k$. Hence, by (6) and Lemmas 4 and 5, we have

$$l_{n_1, \dots, n_k}(A) \leq d(n_1 + 1)^{d_1^2+d_1} \dots (n_k + 1)^{d_k^2+d_k}$$

and

$$l_n^{gr} \leq d(n+1)^{k+d_1^2+\dots+d_k^2+d_1+\dots+d_k}$$

as follows from (7). □

4. Existence of graded PI-exponents

We begin this section with a technical result connecting dimensions of irreducible representations of symmetric groups and multinomial coefficients. Given a partition $\mu = (\mu_1, \dots, \mu_t) \vdash m$, we denote by $q\mu(q\mu_1, \dots, q\mu_t)$ the partition of qm , where $q \geq 1$ is an arbitrary integer. We also define the *height* $ht(\mu)$ as $ht(\mu) = t$. Recall that $d_{\mu} = \chi_{\mu}(1)$ is the *dimension* of the corresponding irreducible representation of S_m .

Lemma 6: Let n_1, \dots, n_k be positive integers, $n_1 + \dots + n_k = n \geq 100$. Let also $\lambda^{(1)}, \dots, \lambda^{(k)}$ be partitions of n_1, \dots, n_k , respectively, such that $ht(\lambda^{(1)}), \dots, ht(\lambda^{(k)}) \leq d$. If $q \geq 100$ then

$$\binom{qn}{qn_1, \dots, qn_k} d_{q\lambda^{(1)}} \cdots d_{q\lambda^{(k)}} \geq \left(\frac{1}{qn}\right)^{k(d^2+d+1)} \left[\frac{1}{n^{2k}} \binom{n}{n_1, \dots, n_k} d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} \right]^q.$$

Proof: Given non-negative real $\alpha_1, \dots, \alpha_k$ with $\alpha_1 + \cdots + \alpha_k = 1$, we denote

$$\Phi(\alpha_1, \dots, \alpha_k) = \frac{1}{(\alpha_1)^{\alpha_1} \cdots (\alpha_k)^{\alpha_k}}.$$

From the Stirling formula for factorials, it easily follows that

$$\frac{1}{m^k} \Phi\left(\frac{m_1}{m}, \dots, \frac{m_k}{m}\right)^m \leq \binom{m}{m_1, \dots, m_k} \leq m \Phi\left(\frac{m_1}{m}, \dots, \frac{m_k}{m}\right)^m, \quad (13)$$

where m_1, \dots, m_k are non-negative integers and $m_1 + \cdots + m_k = m$. Applying (13) to

$$P = \binom{qn}{qn_1, \dots, qn_k}$$

we obtain

$$P > \left(\frac{1}{qn}\right)^k \Phi\left(\frac{qn_1}{qn}, \dots, \frac{qn_k}{qn}\right)^{qn} = \left(\frac{1}{qn}\right)^k \left[\Phi\left(\frac{n_1}{n}, \dots, \frac{n_k}{n}\right)^n \right]^q.$$

Applying again (13), we get

$$P > \left(\frac{1}{qn}\right)^k \left[\frac{1}{n} \binom{n}{n_1, \dots, n_k} \right]^q. \quad (14)$$

It follows from Lemma 1 and (13) that

$$\begin{aligned} d_{q\lambda^{(1)}} \cdots d_{q\lambda^{(k)}} &> \left(\frac{1}{qn_1} \cdots \frac{1}{qn_k}\right)^{k(d^2+d)} \left[\Phi(\lambda^{(1)})^{n_1} \cdots \Phi(\lambda^{(k)})^{n_k} \right]^q \\ &> \left(\frac{1}{qn}\right)^{k(d^2+d)} \left[d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} \right]^q. \end{aligned} \quad (15)$$

Now, our lemma is a consequence of (14) and (15). □

Recall that A is a d -dimensional Γ -graded algebra. Now, let

$$a = \overline{\exp}^{gr}(A) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}. \quad (16)$$

The next lemma is the main step of the proof of Theorem 2.

Lemma 7: *Let Γ be a commutative semigroup and let a in (16) be strictly greater than 1. If A is graded simple, then for any $\varepsilon > 0$ and any $\delta > 0$, there exists an increasing sequence of positive integers $n^{(1)}, n^{(2)}, \dots$ such that*

- (i) $\sqrt[n]{c_n^{gr}(A)} > (1 + \delta)(1 - \varepsilon)$ for all $n = n^{(1)}, n^{(2)}, \dots$; and
- (ii) $n^{(q+1)}, n^{(q)} \leq d$, for all $q = 1, 2, \dots$

Proof: Clearly, there exists an integer $n^{(1)}$ such that

$$c_{n^{(1)}}^{gr}(A) > (a - \varepsilon)^{n^{(1)}}$$

and $n^{(1)}$ can be chosen arbitrarily large. There are also $n_1, \dots, n_k \geq 0$ such that $n_1 + \dots + n_k = n^{(1)}$ and

$$\binom{n^{(1)}}{n_1, \dots, n_k} c_{n_1, \dots, n_k}(A) > \frac{1}{(n^{(1)} + 1)^k} (a - \varepsilon)^{n^{(1)}}.$$

(see (2)). Without loss of generality, we can suppose that $k = |\text{Supp } A|$. Consider the $H = S_{n_1} \times \dots \times S_{n_k}$ -action on P_{n_1, \dots, n_k} . It follows from (6), (7), (10) that there exist partitions $\lambda^{(1)} \vdash n_1, \dots, \lambda^{(k)} \vdash n_k$ such that

$$d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} > \frac{1}{l_{n^{(1)}}^{gr}(A)} c_{n_1, \dots, n_k}(A).$$

By Theorem 1, we have

$$l_{n^{(1)}}^{gr}(A) < d(n^{(1)} + 1)^{k(d+1)^2}$$

hence

$$d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} > \frac{1}{d(n^{(1)} + 1)^{k(d+1)^2}} c_{n_1, \dots, n_k}(A).$$

and

$$\binom{n^{(1)}}{n_1, \dots, n_k} d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} > \frac{1}{d(n^{(1)} + 1)^{k(d+1)^2+k}} (a - \varepsilon)^{n^{(1)}}. \tag{17}$$

There exists a multilinear polynomial

$$f = f(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(k)}, \dots, x_{n_k}^{(k)}) \notin Id^{gr}(A)$$

where $x_j^{(i)} \in X_{g_i}$ for all $1 \leq i \leq k, 1 \leq j \leq n_k$ such that f generates an irreducible $F[H]$ -module with the character

$$\chi_H(F[H]f) = \chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}.$$

There are homogeneous in Γ -grading $a_1^1, \dots, a_{n_1}^1, \dots, a_1^k, \dots, a_{n_k}^k$ with $\deg_\Gamma a_j^i = g_i$ such that

$$Q = f(a_1^1, \dots, a_{n_k}^k) \neq 0.$$

Since Γ is associative and commutative, it follows that Q is homogeneous in Γ -grading of A . Therefore, one can find $d' \leq d$ and homogeneous $c_1, \dots, c_{d'} \in A$ satisfying the inequality

$$(Q * c_1 * \dots * c_{d'}) * Q \neq 0 \tag{18}$$

where $a * b$ denotes the right or the left multiplication by b (otherwise $\text{Span} \langle Q \rangle$ is a graded ideal of A). Denote

$$g_1 = f_1 = f(x_{1,1}^{(1)}, \dots, x_{1,n_1}^{(1)}, \dots, x_{1,1}^{(k)}, \dots, x_{1,n_k}^{(k)}),$$

where $x_{\alpha,\beta}^{(i)}$ are new homogeneous variables, $\deg_{\Gamma} x_{\alpha,\beta}^{(i)} = g_i$, and take

$$g_2 = (f_1 * z_1 * \cdots * z_{d'}) * f_2,$$

where $\deg_{\Gamma} z_1 = \deg_{\Gamma} c_1, \dots, \deg_{\Gamma} z_{d'} = \deg_{\Gamma} c_{d'}$,

$$f_2 = f(x_{2,1}^{(1)}, \dots, x_{2,n_1}^{(1)}, \dots, x_{2,1}^{(k)}, \dots, x_{2,n_k}^{(k)}),$$

with $x_{\alpha,\beta}^{(i)} \in X_{g_i}$. Then $g_2 \in P_{2n_1+q_1, \dots, 2n_k+q_k}$ is not an identity of A , as follows from (18), and $q_1, \dots, q_k \geq 0, q_1 + \cdots + q_k = d'$.

The square $H \times H$ of group H acts on $\{x_{\alpha,\beta}^{(i)}\}$ where the first copy of H acts on $\{x_{1,\beta}^{(i)}\}$, while the second copy acts on $\{x_{2,\beta}^{(i)}\}$ and

$$\chi_{H \times H}(F[H \times H]g_2) = \chi_{\lambda^{(1)}, \dots, \lambda^{(k)}} \otimes \chi_{\lambda^{(1)}, \dots, \lambda^{(k)}}.$$

Denote $n^{(2)} = 2n^{(1)} + d'$. Repeating this procedure, we construct for all $q = 3, 4, \dots$ a multilinear polynomial

$$g_q = g_q(x_{1,1}^{(1)}, \dots, x_{1,n_1}^{(1)}, \dots, x_{q,1}^{(k)}, \dots, x_{q,n_k}^{(k)}, z_1, z_2, \dots)$$

of degree $n^{(q)}$ such that:

- (i) all $x_{\alpha,\beta}^{(i)}, z_j$ are homogeneous and $x_{\alpha,\beta}^{(i)} \in X_{g_i}$;
- (ii) g_q is not an identity of A ;
- (iii) $n^{(q)} = qn^{(1)} + d^{(q)}$, $d^{(q)} \leq (q-1)d \leq dq$, $n^{(q)} - n^{(q-1)} \leq d$; and
- (iv) q copies $H^q = H \times \cdots \times H$ of H act on g_q permuting $x_{\alpha,\beta}^{(i)}$ and g_q generates an irreducible $F[H^q]$ -module M with

$$\chi(M) = (\chi_{\lambda^{(1)}, \dots, \lambda^{(k)}})^{\otimes q}.$$

Denote $H(q) = S_{qn_1} \times \cdots \times S_{qn_k}$. Given $1 \leq i \leq k$, group S_{qn_i} acts on $\{x_{1,1}^{(i)}, \dots, x_{q,n_i}^{(i)}\}$. We can induce the H^q -action on M to the $H(q)$ -action. Consider the decomposition of $\tilde{M} = F[H(q)]g_q$ into irreducible components,

$$\chi_{H(q)}(\tilde{M}) = \sum_{\rho^{(1)} \vdash qn_1, \dots, \rho^{(k)} \vdash qn_k} t_{\rho^{(1)}, \dots, \rho^{(k)}} \chi_{\rho^{(1)}, \dots, \rho^{(k)}}.$$

It follows by the Richardson–Littlewood rule that for any $1 \leq i \leq k$, either $\rho^{(i)} = q\lambda^{(i)}$ or $\rho^{(i)}$ is obtained from $\lambda^{(i)}$ by putting down one or more boxes of $D_{q\lambda^{(i)}}$. Then, by Lemma 2, we have $\Phi(\rho^{(i)}) \geq \Phi(q\lambda^{(i)}) = \Phi(\lambda^{(i)})$. Now, Lemma 1 implies the inequality

$$d_{\rho^{(1)}} \cdots d_{\rho^{(k)}} > \left(\frac{1}{qn^{(1)}} \right)^{k(d+1)^2} d_{q\lambda^{(1)}} \cdots d_{q\lambda^{(k)}}. \quad (19)$$

Recall that g_q is not an identity of A . Hence, there exist integers $p_{q,1}, \dots, p_{q,k} \geq 0$ such that $p_{q,1} + \dots + p_{q,k} = d^{(k)}$ and

$$c_{qn_1+p_{q,1}, \dots, qn_k+p_{q,k}}(A) \geq d_{\rho^{(1)}} \cdots d_{\rho^{(k)}}.$$

In particular,

$$c_{n^{(q)}}^{gr}(A) \geq \binom{qn^{(1)}}{qn_1, \dots, qn_k} d_{\rho^{(1)}} \cdots d_{\rho^{(k)}}. \quad (20)$$

Note that for any partition $\mu \vdash m, \nu \vdash m$ with $\Phi(\mu) \geq \Phi(\nu)$, it follows by Lemma 1 that

$$d_\mu \geq \frac{1}{m^{d^2+d}} \Phi(\mu)^m \geq \frac{1}{m^{d^2+d}} \Phi(\nu)^m \geq \frac{1}{m^{d^2+d+1}} d_\nu.$$

Then, by Lemma 6 and (19), the right hand side of (20) is not less than

$$\left(\frac{1}{qn^{(1)}}\right)^{2k(d+1)^2} \left(\frac{1}{n^{(1)}}\right)^{2kq} \left[\binom{n^{(1)}}{n_1, \dots, n_k} d_{\lambda^{(1)}} \cdots d_{\lambda^{(k)}} \right]^q.$$

Now, (17) implies the following inequality

$$c_{n^{(q)}}^{gr}(A) \geq \left(\frac{1}{qn^{(1)}}\right)^{2k(d+1)^2} \left(\frac{1}{n^{(1)}}\right)^{2kq} \left(\frac{1}{d(n^{(1)}+1)}\right)^{2k(d+1)^2q} (a-\varepsilon)^{n^{(q)}}.$$

Since $a > 1$, by the assumptions of the lemma, we then have

$$(a-\varepsilon)^{qn^{(1)}} \geq \frac{(a-\varepsilon)^{n^{(q)}}}{a^{qd}}$$

for all small enough ε . Hence

$$\sqrt[n]{c_n^{gr}(A)} > D(a-\varepsilon),$$

where $D = D_1 D_2$,

$$D_1 = \left(\frac{1}{n^{(q)}}\right)^{\frac{2k(d+1)^2}{n^{(q)}}}, \quad D_2 = \left(\frac{1}{n^{(1)}}\right)^{\frac{2k}{n^{(1)}}} \left(\frac{1}{d(n^{(1)}+1)}\right)^{\frac{2k(d+1)^2}{n^{(1)}}} \left(\frac{1}{a}\right)^{\frac{d}{n^{(1)}+d}}.$$

For small $\delta_1, \delta_2 > 0$, one can choose $n^{(1)}$ such that $D_1 > (1-\delta_1)$ and $D_2 > (1-\delta_2)$ for all $n^{(q)}, q \geq 1$. Finally, we can take δ_1, δ_2 small enough and get the inequality

$$\sqrt[n]{c_n^{gr}(A)} > (1-\delta)(a-\varepsilon),$$

for all $n = n^{(q)}, q = 1, 2, \dots$ □

Remark 1: In the proof of the previous lemma, we used associativity and commutativity of Γ only for getting relation (18). In case of an arbitrary groupoid Γ , the element Q in (18) can be non-homogeneous in Γ -grading and hence an ideal I generated by Q in A can be strictly less than A . But if A is simple in a non-graded sense, then $I = A$ and relation (18) and Lemma 7 hold.

For completing the proof of the main result, we need the following remark. Denote by $\text{Ann } A$ the annihilator of A .

Lemma 8: *Let A be a Γ -graded algebra with a finite support of order k . If $\text{Ann } A = 0$, then*

$$c_{n+1}^{gr}(A) > \frac{1}{8kn^k} c_n^{gr}(A)$$

for all sufficiently large n .

Proof: Denote $\text{Supp } A = \{g_1, \dots, g_k\}$. It follows from (2) that there exists $n_1, \dots, n_k \geq 0$, $n_1 + \dots + n_k = n$ such that

$$\frac{1}{2n^k} c_n^{gr}(A) < \frac{1}{(n+1)^k} c_n^{gr}(A) < c_{n_1, \dots, n_k}(A). \quad (21)$$

Recall that $c_{n_1, \dots, n_k}(A) = \dim P_{n_1, \dots, n_k}(A)$ (see 4). Denote by U_i , $1 \leq i \leq k$, the subspace of polynomials f in $P_{n_1, \dots, n_k}(A)$ such that $\varphi(f)_{A_{g_i}} = 0$ for all graded evaluations $\varphi : F\{X\} \rightarrow A$. Similarly, let W_i , $1 \leq i \leq k$, be the subspace of polynomials $h \in P_{n_1, \dots, n_k}(A)$ satisfying $A_{g_i} \varphi(h) = 0$ for all graded evaluations $\varphi : F\{X\} \rightarrow A$. Denote

$$V = U_1 \cap \dots \cap U_k \cap W_1 \cap \dots \cap W_k.$$

If $f \in V$, then all values of f in A lie in $\text{Ann } A = 0$, that is $V = 0$. Suppose that

$$\dim U_1, \dots, \dim U_k, \dots, \dim W_1, \dots, \dim W_k > \frac{4k-1}{4k} N$$

where $N = c_{n_1, \dots, n_k}(A)$. Then $\dim V > (2k \cdot \frac{4k-1}{4k} - (2k-1))N = \frac{1}{2}N$, that is $V \neq 0$, a contradiction. It follows that $\dim U_i < \frac{4k-1}{4k}N$ or $\dim W_i < \frac{4k-1}{4k}N$ for at least one i . Let, for instance, $\dim U_1 < \frac{4k-1}{4k}N$. Denote by T the codimension of U_1 in $P_{n_1, \dots, n_k}(A)$. Then

$$T > \frac{1}{4k} c_{n_1, \dots, n_k}(A) > \frac{1}{8kn^k} c_n^{gr}(A)$$

as follows from (21). Now, if f_1, \dots, f_T are linearly independent modulo U_1 elements from $P_{n_1, \dots, n_k}(A)$, then $f_1 z, \dots, f_T z$ are linearly independent elements in $P_{n_1+1, n_2, \dots, n_k}(A)$, provided that z is a new homogeneous indeterminate, $\deg_\Gamma z = g_1$. Hence

$$c_{n+1}^{gr}(A) \geq c_{n_1+1, n_2, \dots, n_k}(A) \geq T > \frac{1}{8kn^k} c_n^{gr}(A),$$

and we are done. □

Now, we are ready to prove the main result of this paper.

Theorem 2: *Let Γ be a commutative semigroup and let $A = \bigoplus_{g \in \Gamma} A_g$ be a finite dimensional Γ -graded algebra. If A is graded simple, then there exists the limit*

$$\exp^{gr}(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}.$$

Proof: Denote

$$a = \overline{\exp}^{gr}(A) = \limsup_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}.$$

If $a = 0$, then A is nilpotent and $\overline{\exp}^{gr}(A) = 0$. If A is not nilpotent, then $a \geq 1$. In the case $a = 1$, the lower limit of $\sqrt[n]{c_n^{gr}(A)}$ is also 1 and we are done.

Let now $a > 1$. By Lemma 7, there exists a sequence $n^{(1)} < n^{(2)} < \dots$ such that

$$c_n^{gr} \geq (1 - \delta)^n (a - \varepsilon)^n$$

for all $n = n^{(i)}, i \geq 1$, and $\varepsilon, \delta > 0$ can be chosen arbitrarily small.

Now, let $m = n^{(i)}, m' = n^{(i+1)}$ and let $m < n < m'$. Then $n = m + p, 1 \leq p < d$. By Lemma 8, we have

$$c_n^{gr}(A) = c_{m+p}^{gr}(A) > \left(\frac{1}{(8k(m+p))} \right)^p (1-\delta)^n (a-\varepsilon)^m > \left(\frac{1}{8kn} \right)^d \frac{1}{(a-\varepsilon)^d} (1-\delta)^n (a-\varepsilon)^n. \tag{22}$$

Clearly, inequality (22) also holds for all $n = n^{(1)}, n^{(2)}, \dots$, and for all small $\varepsilon, \delta > 0$. Hence

$$\liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)} \geq (1 - \delta)(a - \varepsilon).$$

Since ε, δ are arbitrary, we can conclude that

$$\underline{\exp}^{gr}(A) = \liminf_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)} = a$$

and the proof of the theorem is completed. □

Finally, note that associativity and commutativity of Γ was used only in the proof of Lemma 7 (see Remark 1). Hence, for an arbitrary groupoid Γ , we have obtained the following result.

Theorem 3: *Let $A = \bigoplus_{g \in \Gamma} A_g$ be a finite dimensional algebra graded by a groupoid Γ . If A is simple, then there exists its graded PI-exponent*

$$\exp^{gr}(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(A)}.$$

Acknowledgements

We thank the referee for comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author was supported by the SRA [grant number P1-0292-0101], [grant number J1-7025-0101] and [grant number J1-6721-0101]. The second author was partially supported by the RFBR [grant number 16-01-00113a].

References

- [1] Giambruno A, Zaicev M. On codimension growth of finitely generated associative algebras. *Adv. Math.* **1998**;140:145–155.
- [2] Giambruno A, Zaicev M. Exponential codimension growth of P.I. algebras: an exact estimate. *Adv. Math.* **1999**;142:221–243.
- [3] Giambruno A, Regev A, Zaicev M. On the codimension growth of finite-dimensional Lie algebras. *J. Algebra.* **1999**;220:466–474.
- [4] Giambruno A, Regev A, Zaicev M. Simple and semisimple Lie algebras and codimension growth. *Trans. Am. Math. Soc.* **2000**;352:1935–1946.
- [5] Zaitsev MV. Integrality of exponents of growth of identities of finite-dimensional Lie algebras. *Izv. Ross. Akad. Nauk Ser. Mat.* **2002**;66:23–48. Russian.
- [6] Giambruno A, Zaicev M. Multialternating Jordan polynomials and codimension growth of matrix algebras. *J. Linear Algebra Appl.* **2007**;422:372–379. English translation in *Izv. Math.* **2002**;66:463–487.
- [7] Giambruno A, Zaicev M. Codimension growth of special simple Jordan algebras. *Trans. Amer. Math. Soc.* **2010**;362:3107–3123.
- [8] Giambruno A, Shestakov I, Zaicev M. Finite-dimensional non-associative algebras and codimension growth. *Adv. Appl. Math.* **2011**;47:125–139.
- [9] Giambruno A, Mishchenko S, Zaicev M. Codimensions of algebras and growth functions. *Adv. Math.* **2008**;217:1027–1052.
- [10] Zaicev M. On existence of PI-exponents of codimension growth. *Electr. Res. Announc. Math. Sci.* **2014**;21:113–119.
- [11] Giambruno A, Zaicev M. On codimension growth of finite dimensional Lie superalgebras. *J. London Math. Soc.* **2012**;95:534–548.
- [12] Giambruno A, La Mattina D. Graded polynomial identities and codimensions: computing the exponential growth. *Adv. Math.* **2010**;225:859–881.
- [13] Repovš D, Zaicev M. Graded PI-exponents of simple Lie superalgebras. *Ark. Mat.* **2016**;54:147–156.
- [14] Repovš D, Zaicev M. Graded identities of some simple Lie superalgebras. *Algebras Represent. Theory.* **2014**;17(5):1401–1412.
- [15] Repovš D, Zaicev M. Graded codimensions of Lie superalgebra $b(2)$. *J. Algebra.* **2015**;422:1–10.
- [16] Berele A, Regev A. Applications of hook Young diagrams to PI algebras. *J. Algebra.* **1983**;82:559–567.
- [17] Giambruno A, Mishchenko S, Zaicev M. Algebras with intermediate growth of the codimensions. *Adv. Appl. Math.* **2006**;37:360–377.
- [18] Zaicev M. Graded identities in finite-dimensional algebras of codimensions of identities in associative algebras. *Mosc. Univ. Math. Bull.* **2015**;70:234–236.
- [19] Giambruno A, Zaicev M. Polynomial identities and asymptotic methods, Vol. 122, Mathematical surveys and monographs. Providence (RI): American Mathematical Society; **2005**.
- [20] Drensky V. Free algebras and PI-algebras. Graduate course in algebra. Singapore: Springer-Verlag Singapore; **2000**.
- [21] Bahturin Yu, Drensky V. Graded polynomial identities of matrices. *Linear Algebra Appl.* **2002**;357:15–34.
- [22] James J, Kerber A. The representation theory of the symmetric group, Vol. 16, Encyclopedia of mathematics and its applications. London: Addison-Wesley; **1981**.
- [23] Zaitsev M, Repovš D. Four dimensional simple algebra with a fractional PI-exponent. *Mat. Zam.* **2014**;95:538–553.