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Exponential growth of codimensions of identities of algebras with unity

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Abstract. The asymptotic behaviour is studied of exponentially bounded sequences of codimensions of identities of algebras with unity. A series of algebras is constructed for which the base of the exponential increases by exactly 1 when an outer unity is adjoined to the original algebra. It is shown that the PI-exponents of unital algebras can take any value greater than 2, and the exponents of finite-dimensional unital algebras form a dense subset in the domain $[2,\infty)$.

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§ 1. Introduction

1.1. In this paper we study functions that characterize the number of identity relations holding in one or another algebra. With every algebra A over a field F of characteristic zero, we can associate an integer sequence $\{c_n(A)\}$, $n=1,2,\dots$, constructed from its multilinear identities. The asymptotic behaviour of this sequence contains certain information on the structure of the algebra A itself. For example, if A is an associative algebra, then $c_n(A)=1$ for all n if and only if A is a commutative non-nilpotent algebra. But if $c_n(A)=0$ for some $n>1$, then A is nilpotent, $A^n=0$ (and conversely). It was recently shown that $\{c_n(A)\}$ is asymptotically increasing, that is, there exists a positive integer t such that $c_{t+j}\leqslant c_{t+j+1}$ for all $j=0,1,\dots$. If $c_{m-1}>c_m$, then this value of t is closely connected with the nilpotency class of the Jacobson radical of the algebra A (the result was announced in [1], a complete proof was published in [2]). If the field F is algebraically closed and A is simple, then $c_n(A)\sim d^n$, where $d=\dim A$ (see [3]). Here the relation $c_n(A)\sim d^n$ means that

$$\lim_{n\to\infty}\sqrt[n]{c_n(A)}=d.$$

The same effect is also observed in the case of Lie algebras (see [4]), Jordan algebras, alternative algebras, and a number of other classes (see [5]). For Lie algebras there is a well-known open problem of classification of infinite-dimensional simple Lie algebras. At present, this problem is apparently far from its solution,

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but certain information on the structure of such an algebra L can be obtained if $\{c_n(L)\}$ has exponential growth (see [6]).

1.2. The presence or absence of unity in an algebra substantially affects the structure of its identities. For example, if A is an associative algebra with unity, then the set of all its identities is completely determined by the system of so-called proper identities (see [7]). If, in addition, A satisfies all identities of the 2×2 matrix algebra, then asymptotically for its T-ideal there exist only countably many variants, which can be described explicitly (see [8]). If $\{c_n(A)\}$ grows polynomially, then $c_n(A) = qn^k + O(n^{k-1})$ for some integer k and positive rational number q (see [9]). Later it was shown that for a fixed k one can find a suitable algebra for any $q \in \mathbb{Q}$, $q > 0$ (see [10]). It was also proved in the same paper that if A is a unital algebra, then

$$\frac{1}{k!} \leqslant q \leqslant \sum_{i=2}^{k} \frac{(-1)^i}{i!} \simeq \frac{1}{e}.$$

Another positive effect of the presence of a unity manifested itself in the proof of the following conjecture. As a refinement of Amitsur's conjecture, Regev conjectured that

$$c_n(A) \simeq Cn^{\frac{t}{2}} d^n$$

for any associative PI-algebra, where t and n are integers, $C = \text{const}$. After a series of partial results, in 2008 Regev's conjecture was confirmed for algebras with unity (see [11], [12]). And only recently the validity of this conjecture was proved in the general case (see [1], [2]).

In [13], for all real $\gamma > 1$, examples were constructed of finite-dimensional algebras with exponential growth of codimensions $c_n \sim \gamma' \approx \gamma$. As shown in [14], for finite-dimensional algebras with unity, exponential growth cannot be slower than 2^n.

In [15], it was pointed out that if A is an associative PI-algebra, and $A^{\#}$ the algebra obtained from A by adjoining an outer unity, then $\exp(A^{\#})$ is equal to $\exp(A)$ or $\exp(A)+1$. This simple assertion follows from the results of [16], [17], where not only was the existence of the limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

proved for any associative PI-algebra A, but also a procedure for the calculation of this quantity was proposed. Nevertheless, this observation made it possible to propose the conjecture that $\exp(A^{\#})$ is always equal to $\exp(A)$ or $\exp(A)+1$. The first nontrivial example confirming this conjecture was constructed in [14], another example was proposed in [18], and in [19] a series of examples was presented, in which for any algebra A from paper [13] with $\exp(A) = \gamma \in \mathbb{R}$, $1 \leqslant \gamma \leqslant 2$, its extension $A^{\#}$ has exponent $\exp(A^{\#}) = \gamma + 1 \in [2,3]$. We also note that in [20] the author proposed a method of construction, from a Lie algebra L over a field F, of a Poisson algebra that is equal to $L \oplus F$ as a vector space and contains L as a Lie subalgebra of codimension 1. The algebra $L \oplus F$ can be regarded as a natural modification of the algebra $L^{\#}$. Somewhat later, the same author proved that $\exp(L \oplus F) = \exp(L) + 1$ (see [21]).

1.3. The main goal of the present paper is the construction of a family of algebras A_γ, $\gamma \in \mathbb{R}$, $\gamma > 1$, for which $\exp(A_\gamma) = \gamma$ (Theorem 1) and $\exp(A_\gamma^\#) = \gamma + 1$ (Theorem 2). Note that in the construction of these examples we use infinite periodic words and Sturmian words, the combinatorial properties of which are used to obtain asymptotic estimates.

Apart from one more confirmation of the aforementioned conjecture, these results show that any real number $\gamma \geqslant 2$ can be realized as the PI-exponent of a unital algebra (see Corollary 1). Furthermore, Theorem 2 and several combinatorial properties of infinite words imply that the PI-exponents of finite-dimensional unital algebras form a dense subset in the domain $[2, \infty)$.

One can get acquainted with the foundations of the theory of identity relations and the quantitative PI-theory from the monographs [22]–[24].

§ 2. Basic notions and constructions

2.1. Let A be an algebra over a field F, and let $F\{X\}$ be an absolutely free F-algebra with an infinite set of generators X. A polynomial $f = f(x_1, \dots, x_n) \in F\{X\}$, $x_1, \dots, x_n \in X$, is called an identity of A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. The set of all identities $\operatorname{Id}(A)$ of the algebra A forms an ideal in $F\{X\}$. Let P_n denote the subspace of all multilinear polynomials in $x_1, \dots, x_n$ in $F\{X\}$. Then $P_n \cap \operatorname{Id}(A)$ is the set of all multilinear identities of degree n of the algebra A. It is well known that in the case of zero characteristic of the ground field, the ideal $\operatorname{Id}(A)$ is completely determined by the set of subspaces $\{P_n \cap \operatorname{Id}(A)\}$, $n = 1, 2, \dots$. Let $P_n(A)$ denote the quotient space

$$P_n(A) = \frac{P_n}{P_n \cap \operatorname{Id}(A)},$$

and let $c_n(A)$ denote its dimension

$$c_n(A) = \dim P_n(A).$$

The quantity $c_n(A)$ is called the nth codimension of identities of the algebra A (or simply the nth codimension of A) and is one of the quantitative characteristics of the set of identity relations of A. Studying the asymptotic behaviour of the sequence $\{c_n(A)\}$ is one of the key problems of quantitative PI-theory.

In the general case, $\{c_n(A)\}$ may have super-exponential growth. For example, if $A = F\{X\}$, then

$$c_n(A) = \frac{1}{n} C_{2n-2}^{n-1} n!;$$

if A is a free associative algebra, then $c_n(A) = n!$, and if A is a free Lie algebra, then $c_n(A) = (n-1)!$. But in many cases the growth of the sequence $\{c_n(A)\}$ is bounded by an exponential function. The class of algebras with exponentially bounded codimension growth contains all associative PI-algebras (see [25]), all finite-dimensional algebras (see [26]) of any signature, the Kac-Moody algebras (see [27]), infinite-dimensional simple Lie algebras of Cartan type (see [28]), and quite a number of others. In this case, the upper and lower limits

$$\overline{\exp}(A) = \varlimsup_{n\to\infty} \sqrt[n]{c_n(A)}, \qquad \underline{\exp}(A) = \varliminf_{n\to\infty} \sqrt[n]{c_n(A)}$$

are defined, and are called the upper and lower PI-exponents of A. If the ordinary limit exists, that is,

$$\overline{\exp}(A) = \underline{\exp}(A),$$

then it is called the (ordinary) PI-exponent.

2.2. In the study of the asymptotics of the growth of $\{c_n(A)\}$, a useful tool is the theory of representations of symmetric groups. The group S_n naturally acts on P_n:

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Furthermore, the subspace $P_n \cap \mathrm{Id}(A)$ is invariant under this action, and therefore $P_n(A)$ is also endowed with the structure of an $F[S_n]$-module. All the requisite information on the theory of representations of symmetric groups and its application in the study of identity relations can be found in [29], [22]–[24]. Since representations of the group S_n are completely reducible, the module $P_n(A)$ decomposes into a direct sum of irreducible $F[S_n]$-modules, which fact it is convenient to write down in the language of character theory. The character $\chi(P_n(A))$ is called the nth cocharacter of A and is denoted by $\chi_n(A)$. The decomposition of $P_n(A)$ into irreducible components is written as the decomposition of $\chi_n(A)$ into a sum of irreducible characters:

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{2.1}$$

where χ_λ is the character of the irreducible representation of S_n corresponding to a partition λ of the number n, while the non-negative integer m_λ is its multiplicity. In particular, relation (2.1) means that

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda, \tag{2.2}$$

where $d_\lambda = \deg \chi_\lambda$ is the dimension of the irreducible representation of S_n corresponding to a partition λ. To obtain estimates of the codimension growth, we need one more quantity, which is called the nth colength of A, defined as

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda,$$

where the m_λ are the coefficients on the right-hand side of (2.2). Obviously,

$$c_n(A) \leqslant l_n(A) \max\{d_\lambda \mid \lambda \vdash n,\, m_\lambda \neq 0\}. \tag{2.3}$$

We need more detailed information on the structure of irreducible $F[S_n]$-modules. Recall that a partition λ of a number n is defined as an ordered tuple of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geqslant \dots \geqslant \lambda_k > 0$ and $\lambda_1 + \dots + \lambda_k = n$. The number $h(\lambda) = k$ is called the height of λ. From a partition λ, we construct a table of n boxes called a Young diagram D_λ. It consists of k rows and contains λ_j boxes in the jth row for every $j = 1, \dots, k$. If the numbers $1, \dots, n$ are written in the boxes of the diagram D_λ, then the construction thus obtained is called a Young tableau T_λ. It is known that any irreducible $F[S_n]$-module is isomorphic to the minimal left ideal $F[S_n]e_{T_\lambda}$ of the group ring of S_n, where the element e_{T_λ} is constructed as follows.

Let R_{T_λ} denote the subgroup of all permutations permuting the numbers $1, \dots, n$ only within the rows of the tableau T_λ. Clearly, $R_{T_\lambda} \simeq S_{\lambda_1} \times \dots \times S_{\lambda_k}$. In similar fashion, the subgroup C_{T_λ} is defined; the elements of this subgroup do not take any number beyond a column of T_λ. We set

$$R(T_\lambda) = \sum_{\sigma \in R_{T_\lambda}} \sigma, \qquad C(T_\lambda) = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau, \qquad e_{T_\lambda} = R(T_\lambda)C(T_\lambda).$$

It is the character of this module that is called the irreducible character χ_λ. The element e_{T_λ} is called a Young symmetrizer and it is a quasi-idempotent of the ring $F[S_n]$, that is, $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$, where γ is a nonzero scalar. In particular, this implies that the element $C(T_\lambda)e_{T_\lambda}$ is not equal to zero and generates the same minimal left ideal $F[S_n]e_{T_\lambda}$. In the context of the action of S_n on the space of multilinear polynomials P_n, this fact makes it possible to draw a simple but important conclusion.

Remark 1. Let M be an irreducible $F[S_n]$-submodule of P_n. Then M is generated as an $F[S_n]$-module by a multilinear polynomial with the following properties:

- the set of variables involved in f splits into a union of disjoint subsets

$$\{x_1, \dots, x_n\} = X_1 \cup \dots \cup X_t,$$

  where $t = \lambda_1$ is the length of the first row of D_λ and $|X_j|$ is the height of the jth column of D_λ, $j = 1, \dots, k$;
- the polynomial f is skew-symmetric with respect to each of the sets $X_1, \dots, X_t$.

2.3. For estimating the dimensions of irreducible representations of S_n, it is convenient to use the function $\Phi(\lambda)$ defined on partitions as follows.

First let $0 \leqslant x_1, \dots, x_d \leqslant 1$ be any real numbers such that $x_1 + \dots + x_d = 1$, $d \geqslant 2$. We set

$$\Phi(x_1, \dots, x_d) = \frac{1}{x_1^{x_1} \cdots x_d^{x_d}}. \tag{2.4}$$

We use the continuity of Φ and the property that if we fix the values of all variables except x_i, x_j, then the maximum of Φ is attained at $x_i = x_j$. Moreover, if $x_i > x_j$, then $\Phi(x_i - \varepsilon, x_j + \varepsilon)$ is increasing as ε increases from 0 to $(x_i - x_j)/2$. But if we fix one of the variables, for example, $x_d = \gamma$, then the maximum is attained at $x_1 = \dots = x_{d-1}$, that is,

$$\max \Phi = \Phi(\theta, \dots, \theta, \gamma), \quad \text{where } (d-1)\theta + \gamma = 1.$$

We use the notation

$$\Phi_{d-1}(\gamma) = \Phi(\underbrace{\theta, \dots, \theta}_{d-1}, \gamma), \qquad (d-1)\theta + \gamma = 1. \tag{2.5}$$

Now let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ and $d \geqslant t$. We write λ in the form $\lambda = (\lambda_1, \dots, \lambda_d)$ even if $d > t$, setting $\lambda_{t+1} = \dots = \lambda_d = 0$. Then

$$\Phi(\lambda) = \Phi\biggl(\frac{\lambda_1}{n}, \dots, \frac{\lambda_d}{n}\biggr).$$

Obviously, the value of $\Phi(\lambda)$ is independent of $d \geqslant t$ if we use the convention $0^0 = 1$.

The value of $\Phi(\lambda)$ and the degree of the character $d_\lambda = \deg \chi_\lambda$ are connected by the following relation.

Lemma 1 (see [30], Lemma 1). *Let $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ be a partition of n into $t \leqslant d$ components and suppose that $n \geqslant 100$. Then*

$$\frac{\Phi(\lambda)^n}{n^{d^2+d}} \leqslant d_\lambda \leqslant n\Phi(\lambda)^n.$$

We need the following property of Φ. Let $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_q)$ be two partitions of a number n, and suppose that $\lambda_q,\ \mu_q > 0$. We say that the Young diagram D_μ is obtained from the diagram D_λ by pushing one box down if there exist $1 \leqslant i < j \leqslant q$ such that $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$, and $\mu_k = \lambda_k$ for all other $1 \leqslant k \leqslant q$. If, however, $\lambda = (\lambda_1, \dots, \lambda_q)$, $\lambda_q > 0$, $\mu = (\mu_1, \dots, \mu_q, 1) \vdash n$, then D_μ is obtained from D_λ by pushing down one of the boxes if one of the rows of D_μ is shorter by one box than the corresponding row of D_λ, while all other rows, except the last one, have the same length.

Lemma 2 (see [30], Lemma 3, [31], Lemma 2). *Suppose that D_μ is obtained from D_λ by pushing down one box. Then $\Phi(\mu) \geqslant \Phi(\lambda)$.*

We also use the following property of the function $\Phi(x_1, \dots, x_d)$.

Lemma 3 (see [19], Lemma 2). *Let $\Phi(x_1, \dots, x_d)$ be defined by formula* (2.4), *and let $\Phi(z_1, \dots, z_d) = a$ for some fixed values $z_1, \dots, z_d$. Then*

$$\max_{0 \leqslant t \leqslant 1} \bigl\{\Phi(y_1, \dots, y_d, 1-t) \mid y_1 = tz_1,\ \dots,\ y_d = tz_d\bigr\} = a + 1,$$

and the maximum is attained at $t = a/(a+1)$.

In fact, Lemma 3 means that when an additional row is added to the diagram D_λ, the value of $\Phi(\lambda)$ increases by at most 1.

2.4. For constructing examples of algebras with a given nature of behaviour of $\{c_n(A)\}$, we use an approach that was proposed for the first time in [13] and which is based on combinatorial properties of infinite binary words. For this, we recall some notions.

Let $w = w_1 w_2 \dots$ be an infinite word in the binary alphabet, that is, every w_i is equal to 0 or 1. The complexity of w is defined as the function of positive integer argument $\operatorname{Comp}_w(n)$ that is equal to the number of different subwords of length n in w. If w is periodic, then $\operatorname{Comp}_w(n) = \text{const} = T$ for all $n \geqslant T$, where T is the period of w. It is also known that if w is not periodic, then $\operatorname{Comp}_w(n) \geqslant n+1$ for all $n \geqslant 1$ (see [32]). The sum $w_{k+1} + \dots + w_{k+m}$ of a finite subword $u = w_{k+1} \dots w_{k+m}$ is customarily denoted by $h(u)$, and the length by $|u|$.

For a given word w, the quantity

$$\pi(w) = \lim_{n\to\infty} \frac{h(w_1 \dots w_n)}{n} \tag{2.6}$$

is called the slope of w if the limit on the right-hand side of (2.6) exists.

If $\mathrm{comp}_w(n) = n+1$ for all $n \geqslant 1$, then w is called a Sturmian word. Sturmian words possess the following properties (see [32]).

Proposition 1. *Let w be a periodic word or a Sturmian word. Then there exists a constant C such that*

(1) $|h(x) - h(y)| \leqslant C$ *for any finite subwords x and y of the same length*;

(2) *the slope $\pi(w)$ always exists*;

(3) *for any finite subword u in w*,

$$\left|\frac{h(u)}{|u|} - \pi(w)\right| \leqslant \frac{C}{|u|};$$

(4) *for any real $\alpha \in (0,1)$ there exists w such that $\pi(w) = \alpha$ and w is a periodic word if α is a rational number*, *or a Sturmian word if α is irrational*; *moreover*, *we can choose $C = 1$ if w is a Sturmian word*, *or $C = T$ if w is a periodic word with period T*, *and then*

$$\pi(w) = \frac{h(w_1 \dots w_T)}{T}.$$

In what follows we also regard words of 0s only, or of 1s only, as being periodic, and then Proposition 1 also extends to the cases $\alpha = 0$, $\alpha = 1$.

§ 3. Sturmian words and non-associative algebras

In the present section we construct a family of non-associative algebras, the PI-exponents of which take any real values in the domain $[2,\infty)$. The idea of construction of algebras with prescribed codimension growth on the basis of Sturmian words was first proposed and realized in [33], [13], where for any real $1 \leqslant \alpha \leqslant 2$, an algebra A_α with $\exp(A_\alpha) = \alpha$ was constructed. In the recent paper [19] it was proved that if we adjoin an outer unity to A_α, then the exponent exists for the resulting algebra $A_\alpha^\#$ and is equal to $\alpha + 1$. The series of algebras constructed below generalizes the construction proposed in [13]. It should be noted that examples of algebras with an arbitrary PI-exponent $\alpha \geqslant 2$ were also presented in [13], but attempts to use them to construct unital algebras with exponents greater than three were unsuccessful. This is what made construction of new examples necessary.

3.1. Let m and d be positive integers such that $m \geqslant 2$, $d \leqslant m-1$, and let $w = w_1 w_2 \dots$ be an infinite word in the binary alphabet $\{0;1\}$. We consider an infinite sequence $(m_1, m_2, \dots)$ in which $m_j = m + w_j$ for all $j \geqslant 1$. An algebra $A(m,d,w)$ is defined by its basis

$$\{a_i, b, z^i_{jk} \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant m_k,\ k = 1,2,\dots\}$$

and multiplication table

$$z_{jk}^i a_i = \begin{cases} z_{j+1,k}^i & \text{if } j < m_k, \\ 0 & \text{if } j = m_k, \end{cases} \qquad z_{m_k,k}^i b = \begin{cases} z_{1k}^{i+1} & \text{if } i < d, \\ z_{1,k+1}^1 & \text{if } i = d. \end{cases}$$

All the other products of basis elements are equal to zero. We point out some properties of $A(m,d,w)$:

- $A(m,d,w)$ satisfies the identity $x_1(x_2x_3) \equiv 0$,
- the linear span $\langle z_{jk}^i \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant m_k,\ k \geqslant 1\rangle$ is an ideal of $A(m,d,w)$ with zero multiplication of codimension $d+1$,
- if $f = f(x_1,\dots,x_n)$ is a multilinear polynomial of degree $n \geqslant d+3$ that is skew-symmetric with respect to $x_1,\dots,x_{d+3}$, then $f \equiv 0$ is an identity in $A(m,d,w)$,
- if $f = f(x_1,\dots,x_n)$ is a multilinear polynomial of degree $n \geqslant 2d+4$ that is skew-symmetric with respect to $x_1,\dots,x_{d+2}$ and with respect to $x_{d+3},\dots,x_{2d+4}$, then $f \equiv 0$ is an identity in $A(m,d,w)$.

Remark 1 in the preceding section immediately yields the following result.

Lemma 4. *Let $A(m,d,w)$ be an algebra defined by an infinite word w and integer parameters $m \geqslant 2$ and $1 \leqslant d \leqslant m-1$. If*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{3.1}$$

is the nth cocharacter of A, then $m_\lambda \neq 0$ in (3.1) only for $h(\lambda) \leqslant d+2$, where $h(\lambda)$ is the height of λ, that is, the number of rows in the diagram D_λ. Furthermore, if $\lambda = (\lambda_1,\dots,\lambda_{d+2})$ and $m_\lambda \neq 0$, then $\lambda_{d+2} \leqslant 1$.

3.2. To obtain an upper estimate for the growth of $\{c_n(A(m,d,w))\}$, we first need to bound the growth of the colength $\{l_n(A(m,d,w))\}$.

First let A be an arbitrary algebra. Let $R = R(y_1,y_2,\dots)$ denote the relatively free algebra of the variety $\mathrm{var}(A)$ generated by A, and let

$$W_n^{(p)}(A) = \mathrm{Span}\{y_{i_1}\cdots y_{i_n} \mid 1 \leqslant i_1,\ \dots,\ i_n \leqslant p\}$$

denote the linear span of all monomials of degree n in $y_1,\dots,y_p$ with all possible arrangements of brackets, that is, of all homogeneous polynomials of degree n in $y_1,\dots,y_p$ contained in R.

Lemma 5 (see [13], Lemma 4.1). *Let A be an algebra with nth cocharacter $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda\chi_\lambda$. Then for any $\lambda \vdash n$ with $h(\lambda) \leqslant p$ we have the inequality*

$$m_\lambda \leqslant \dim W_n^{(p)}(A). \tag{3.2}$$

Throughout what follows we simply omit brackets in a left-normed product, that is, we write $(zt)v$ as ztv. This agreement is especially convenient when working with the algebras $A(m,d,w)$, since all nonzero products in them are left-normed due to the identity $x_1(x_2x_3) \equiv 0$.

Lemma 6. *Let $A = A(m,d,w)$ be defined by m, d, and an infinite word w. Then*

$$\dim W_n^{(p)}(A) \leqslant d(m+1)n^{(d+1)p}\,\mathrm{Comp}_w(n).$$

Proof. Let W denote the linear span of monomials of the form $ty_{i_1}\cdots y_{i_{n-1}}$, where $t=y_{p+1}$, $1\leqslant i_1,\,\dots,\,i_{n-1}\leqslant p$. Then

$$\dim W_n^{(p)}(A)\leqslant p\dim W.$$

Let y be some element of W. Clearly, y is nonzero if and only if there exists a homomorphism $\sigma\colon R\to A$ for which $\sigma(y)\neq 0$.

In order to obtain an estimate for the dimension of W, we consider the following construction. Let $F\langle a_1,\dots,a_d,b\rangle$ be a free associative algebra with generators $a_1,\dots,a_d,b$, and M be a free right $F\langle a_1,\dots,a_d,b\rangle$-module with one generator x. Then any element of M can be written as a linear combination of elements of the form $xf(a_1,\dots,a_d,b)$, where $f(a_1,\dots,a_d,b)$ is a monomial in $a_1,\dots,a_d,b$.

Now suppose that σ is a homomorphism from R into A. Clearly, it is sufficient to verify the condition $\sigma(y)=0$, $y\in W$, only for all homomorphisms of the form

$$\sigma(t)=z^i_{jk},\qquad \sigma(y_s)=\alpha^s_1a_1+\dots+\alpha^s_da_d+\beta^sb,\quad 1\leqslant s\leqslant p,$$

where α^s_r, β^s are any scalars in F.

We consider the polynomial ring $F[\alpha^s_r,\beta^s]$, $1\leqslant s\leqslant p$, $1\leqslant r\leqslant d$, in which α^s_r, β^s are now regarded as variables. For brevity, we denote this ring by $F[\alpha,\beta]$. Let $\psi\colon W\to M\otimes F[\alpha,\beta]$ be the linear map defined by

$$\psi(ty_{i_1}\cdots y_{i_{n-1}})=x(\alpha^{i_1}_1a_1+\dots+\alpha^{i_1}_da_d+\beta^{i_1}b)\cdots(\alpha^{i_{n-1}}_1a_1+\dots+\alpha^{i_{n-1}}_da_d+\beta^{i_{n-1}}b). \tag{3.3}$$

We observe that if

$$h=\sum\lambda_{i_1\dots i_{n-1}}ty_{i_1}\cdots y_{i_{n-1}},$$

then $\psi(h)=0$ only if $h\equiv 0$ is an identity in A, that is, h is a zero element with respect to the free algebra $R(y_1,y_2,\dots)$. This means that (3.3) correctly defines ψ and that ψ is an embedding of W into $M\otimes F[\alpha,\beta]$.

Let φ^i_{jk} denote the linear map from M into A such that

$$\varphi^i_{jk}(xf(a_1,\dots,a_d,b))=z^i_{jk}f(a_1,\dots,a_d,b), \tag{3.4}$$

where the polynomial on the right-hand side of (3.4) is interpreted as a polynomial in right multiplications by $a_1,\dots,a_d,b$ in A. We set

$$I=\bigcap_{i,j,k}\ker\varphi^i_{jk}.$$

If $y\in M/I\otimes F[\alpha,\beta]$, then for any specification of the variables $\{\alpha^s_r,\beta^s\}$ in F and for any substitution $\varphi^i_{jk}\colon M\to A$, the element y goes to zero. This means that W is embedded in $M/I\otimes F[\alpha,\beta]$. Moreover, W is embedded in $M/I\otimes F[\alpha,\beta]^{(n-1)}$, where $F[\alpha,\beta]^{(n-1)}$ is the subspace of homogeneous polynomials of degree $n-1$ in $F[\alpha,\beta]$. In particular,

$$\dim W\leqslant\dim F[\alpha,\beta]^{(n-1)}\cdot\dim\frac{M}{I}.$$

Obviously,

$$\dim F[\alpha,\beta]^{(n-1)}\leqslant(n-1)^{dp+p}\leqslant n^{(d+1)p}.$$

We now estimate the dimension of M/I from above. We fix the indices i, j, k. First, we observe that the rules of multiplication of basis elements in A imply that there exists exactly one monomial $f^i_{j,k}$ that is not contained in the kernel of $\varphi^i_{j,k}$:

$$f^i_{j,k} = x\,\underbrace{a_i\cdots a_i}_{m_k-j}\,b\,\underbrace{a_{i+1}\cdots a_{i+1}}_{p_1}\,b\cdots b\,\underbrace{a_{i+r}\cdots a_{i+r}}_{p_r}\,b\,\underbrace{a_{i+r+1}\cdots a_{i+r+1}}_{s},$$

where the indices of $a_{i+1},\dots,a_{i+r+1}$ are calculated modulo d, $m_k-j+p_1+\dots+p_r+s+r+1=n-1$, $s\leqslant d$, and all $p_1,\dots,p_r$ are equal to one of $m_k,m_{k+1},\dots$ and are uniquely determined by the subword $w(k,k+n-1)=(w_k,w_{k+1},\dots,w_{k+n-1})$ of length n of the word w. In particular, $f^i_{j,k}=f^i_{j,l}$ and $\ker\varphi^i_{j,k}=\ker\varphi^i_{j,l}$ if $w(k,k+n-1)=w(l,l+n-1)$ in w. Since $1\leqslant i\leqslant d$ and $1\leqslant j\leqslant m+1$, the number of different kernels $\ker\varphi^i_{j,k}$ is at most $d(m+1)\operatorname{Comp}_w(n)$. Consequently,

$$\dim\frac{M}{I}\leqslant d(m+1)\operatorname{Comp}_w(n),\qquad \dim W_n^{(p)}(A)\leqslant d(m+1)n^{(d+1)p}\operatorname{Comp}_w(n),$$

and Lemma 6 is proved.

As a corollary, we obtain an estimate of the colength growth for an algebra defined by a Sturmian word or an infinite periodic word.

Proposition 2. *Let $A=A(m,d,w)$, where w is a Sturmian word or an infinite periodic word. Then*

$$l_n(A)\leqslant 2d^2(m+1)n^{(d+1)(d+3)}(n+1).$$

Proof. By Lemma 4, we have $h(\lambda)\leqslant d+2$ and $\lambda_{d+2}\leqslant 1$ for any partition $\lambda\vdash n$ with nonzero multiplicity m_λ. The number of such partitions does not exceed $2dn^{d+1}$. Therefore Lemmas 5 and 6 yield the required estimate.

3.3. We can now set about obtaining upper estimates of PI-exponents.

Let $A=A(m,d,w)$ be an algebra constructed from an infinite word w, where w is a periodic word or a Sturmian word. If $f=f(z^i_{jk},a_1,\dots,a_d,b)$ is an associative word in the alphabet $\{z^i_{jk},a_1,\dots,a_d,b\}$, then we can speak about its degrees $\deg_b f$, $\deg_{a_i} f$, $\deg_{z^i_{jk}} f$ in the variables, about the total degree $\deg f$, as well as about the value of f in A, if we consider it as a left-normed product of basis elements.

We need one sufficient condition for the fact that $f\neq 0$.

Lemma 7. *For given m, d, w, there exists a sequence $\{\varepsilon_n>0\}$, $n=1,2,\dots$, such that if $f=f(z^i_{jk},a_1,\dots,a_d,b)$ is a monomial of degree n that is not equal to zero in $A(m,d,w)$, then*

$$\frac{\deg_b f}{n}\leqslant\frac{1}{m+\alpha}+\varepsilon_n,$$

where $\alpha=\pi(w)$ is the slope of w. Furthermore, $\varepsilon_n\to 0$ as $n\to\infty$.

Proof. The word f can be written in the form $f=ZPQ$, where Z is a product of basis elements $\{z^i_{jk},a_\alpha,b\}$ of degree $\deg Z\leqslant(m+1)d$, while $Q=Q(a_1,\dots,a_d,b)$, $\deg Q\leqslant(m+1)d$, and

$$P=a_1^{m_k-1}b\cdots a_d^{m_k-1}b\cdots a_1^{m_{k+t-1}-1}b\cdots a_d^{m_{k+t-1}-1}b.$$

Then $\deg_b P = td$ and

$$\begin{aligned}\deg_{a_i} P &= (m_k - 1) + \cdots + (m_{k+t-1} - 1) = m_k + \cdots + m_{k+t-1} - t \\ &= (m-1)t + w_k + \cdots + w_{k+t-1}\end{aligned}$$

for any $i = 1, \ldots, d$. As noted in Proposition 1, there exists for w a constant C such that $|w_k + \cdots + w_{k+t-1} - \alpha t| \leqslant C$. Therefore,

$$\deg P = dmt + d(w_k + \cdots + w_{k+t-1}) \geqslant dt\left(m + \alpha - \frac{C}{t}\right)$$

and $n = \deg f \geqslant \deg P$, $\deg_b f \leqslant td + 2d = (t+2)d$. Consequently,

$$\frac{\deg_b f}{n} \leqslant \frac{1 + 2d/t}{m + \alpha - C/t}.$$

Since $n \leqslant d(m_k + \cdots + m_{k+t-1}) + 2(m+1)d \leqslant d(m+1)t + 2(m+1)d$, it follows that

$$t \geqslant \frac{n}{d(m+1)} - 2$$

and t grows linearly as n grows. Consequently,

$$\lim_{n\to\infty} \frac{\deg_b f}{n} = \frac{1}{m+\alpha},$$

whence the assertion of the lemma follows.

We now obtain an upper estimate for the codimension growth of the algebra $A(m, d, w)$.

Lemma 8. *Let $A = A(m, d, w)$, where w is an infinite periodic word or a Sturmian word with slope $\alpha = \pi(w)$. Then*

$$\overline{\exp}(A) \leqslant \Phi_d\left(\frac{1}{m+\alpha}\right),$$

where the function Φ_d is defined by formula (2.5).

Proof. We fix an arbitrarily small $\varepsilon > 0$ and claim that for it there exists N such that if $n \geqslant N$, $\lambda \vdash n$, and $m_\lambda \neq 0$ in (3.1), then

$$\Phi(\lambda) \leqslant \Phi_d\left(\frac{1}{m+\alpha} + \varepsilon\right).$$

First suppose that $\lambda_{d+1} = 0$, that is, $\lambda = (\lambda_1, \ldots, \lambda_d, 0, 0)$. Then

$$\Phi(\lambda) \leqslant \Phi\left(\frac{1}{d}, \ldots, \frac{1}{d}, 0, 0\right) \leqslant \Phi\left(\underbrace{\theta, \ldots, \theta}_{d}, \frac{1}{m+\alpha}\right) = \Phi_d\left(\frac{1}{m+\alpha}\right).$$

Now suppose that $\lambda_{d+1} \neq 0$. Then by Remark 1, there exists a multilinear polynomial $h = h(x_1, \ldots, x_n)$ that is not an identity of A, but is skew-symmetric with respect to λ_1 sets of variables $X_1, \dots, X_{\lambda_1}$, where $|X_1| = d+1$ or $d+2$ depending on the value of λ_{d+2} (being 0 or 1), while $|X_2| = \dots = |X_{\lambda_{d+1}}| = d+1$. Consequently, there exists a substitution $\varphi\colon X \to \{a_r, b, z^i_{jk}\}$ such that $f = \varphi(h) = f(z^i_{jk}, a_1, \dots, a_d, b)$ is a nonzero monomial in A. Then $\deg_b f \geqslant \lambda_{d+1}$, and by Lemma 7,

$$\frac{\lambda_{d+1}}{n} \leqslant \frac{\deg_b f}{n} \leqslant \frac{1}{m+\alpha} + \varepsilon_n.$$

If $\lambda_{d+2} = 0$, then

$$\Phi(\lambda) \leqslant \Phi\biggl(\underbrace{\theta, \dots, \theta}_{d}, \frac{1}{m+\alpha} + \varepsilon_n, 0\biggr) = \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon_n\biggr) \leqslant \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon\biggr)$$

for all big n, since $\varepsilon_n \to 0$ as n grows, while $\Phi_d(1/(m+\alpha) + x)$ is increasing as x increases. If, however, $\lambda_{d+2} = 1$, then

$$\Phi(\lambda) \leqslant \Phi\biggl(\underbrace{\theta, \dots, \theta}_{d}, \frac{1}{m+\alpha} + \varepsilon_n, \frac{1}{n}\biggr).$$

Since $\varepsilon_n \to 0$ and $1/n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \Phi\biggl(\theta, \dots, \theta, \frac{1}{m+\alpha} + \varepsilon_n, \frac{1}{n}\biggr) = \Phi\biggl(\bar\theta, \dots, \bar\theta, \frac{1}{m+\alpha}, 0\biggr),$$

where $\bar\theta d + 1/(m+\alpha) = 1$. Consequently, there exists n such that

$$\Phi\biggl(\theta, \dots, \theta, \frac{1}{m+\alpha} + \varepsilon_n, \frac{1}{n}\biggr) \leqslant \Phi\biggl(\theta', \dots, \theta', \frac{1}{m+\alpha} + \varepsilon, 0\biggr).$$

Consequently,

$$\begin{aligned}\Phi(\lambda) &\leqslant \Phi\biggl(\theta, \dots, \theta, \frac{1}{m+\alpha} + \varepsilon_n, \frac{1}{n}\biggr) \leqslant \Phi\biggl(\theta', \dots, \theta', \frac{1}{m+\alpha} + \varepsilon, 0\biggr) \\ &= \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon\biggr),\end{aligned}$$

where $\theta' d + 1/(m+\alpha) + \varepsilon = 1$ and $\theta' \geqslant \theta$. Since

$$c_n(A) = \sum m_\lambda d_\lambda \leqslant l_n(A) \max\{d_\lambda \mid \lambda \vdash n,\, m_\lambda \neq 0\},$$

it follows from Lemma 1 and Proposition 2 that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n(A)} \leqslant \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon\biggr)$$

for any fixed $\varepsilon > 0$. Consequently,

$$\overline{\exp}(A) \leqslant \Phi_d\biggl(\frac{1}{m+\alpha}\biggr),$$

and Lemma 8 is proved.

We now pass to a lower estimate for the codimension growth of the algebra $A(m,d,w)$.

Lemma 9. *Let $A(m,d,w)$ be the algebra from Lemma 8. Then*

$$\underline{\exp}(A) \geqslant \Phi_d\left(\frac{1}{m+\alpha}\right),$$

where $\alpha = \pi(w)$ is the slope of w.

Proof. In a free algebra $F\{X\}$, consider the monomial

$$h_1 = zx_1^1x_2^1\cdots x_p^1y_1^1\cdots x_1^dx_2^d\cdots x_p^dy_d^1$$

of degree $(p+1)d+1$, where $p = m_1-1 \geqslant m-1 \geqslant d$. Let $\mathrm{Alt}_1^1\colon P_{(p+1)d+1} \to P_{(p+1)d+1}$ be the alternation operator with respect to $z, x_1^1, x_1^2, \ldots, x_1^d, y_1^1$, and Alt_i^1 be the alternation operator with respect to $x_i^1, x_i^2, \ldots, x_i^d, y_i^1$ for every $2 \leqslant i \leqslant d$. If $p > d$, then we also denote by Alt_{d+j}^1 the alternation with respect to $x_{d+j}^1, x_{d+j}^2, \ldots, x_{d+j}^d$ for every $1 \leqslant j \leqslant p-d$. We set $f_1 = \mathrm{Alt}_1^1 \ldots \mathrm{Alt}_p^1(h_1)$.

Consider the substitution $\varphi\colon X \to A$ under which

$$\varphi(z) = z_{11}^1,\ \varphi(x_1^1) = \cdots = \varphi(x_p^1) = a_1,\ \ldots,\ \varphi(x_1^d) = \cdots = \varphi(x_p^d) = a_d,$$
$$\varphi(y_1^1) = \cdots = \varphi(y_d^1) = b.$$

Then

$$\varphi(f_1) = z_{11}^1 \underbrace{a_1\cdots a_1}_{m_1-1} b\cdots \underbrace{a_d\cdots a_d}_{m_1-1} b = z_{12}^1.$$

Note that the result of the substitution φ does not change (up to a nonzero factor) if it is applied not to the element f_1 itself but to its symmetrization $\mathrm{Sym}\, f_1$, where Sym means symmetrization with respect to the sets $\{x_1^1, \ldots, x_p^1\}$, $\ldots$, $\{x_1^d, \ldots, x_p^d\}$, $\{y_1^1, \ldots, y_d^1\}$. Then the polynomial $\mathrm{Sym}\, f_1$ generates in $P_{(p+1)d+1}$ an irreducible $F[S_{(p+1)d+1}]$-module corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_{d+2})$, where $\lambda_1 = \cdots = \lambda_d = p = m_1-1$, $\lambda_{d+1} = d$, $\lambda_{d+2} = 1$, and the condition $\varphi(\mathrm{Sym}\, f_1) \neq 0$ means that the multiplicity m_λ in the decomposition (2.1) is not equal to zero.

We set $p_1 = p$. Next, for all $j = 2, 3, \ldots$ we construct polynomials $f_2, f_3, \ldots$ as follows. If $f_1, \ldots, f_{j-1}$ are already constructed, then we take

$$h_j = f_{j-1}x_{q+1}^1\cdots x_{q+p_j}^1y_1^j\cdots x_{q+1}^d\cdots x_{q+p_j}^dy_d^j,$$

where $q = p_1 + \cdots + p_{j-1}$, $p_j = m_j - 1$, and define f_j as

$$f_j = \mathrm{Alt}_1^j \ldots \mathrm{Alt}_{p_j}^j(h_j),$$

where $\mathrm{Alt}_1^j, \ldots, \mathrm{Alt}_d^j$ are the alternations with respect to the sets

$$\{x_{q+1}^1, \ldots, x_{q+1}^d, y_1^j\},\ \ldots,\ \{x_{q+d}^1, \ldots, x_{q+d}^d, y_d^j\},$$

respectively. If, however, $p_j > d$, then Alt_{d+i}^j is the alternation with respect to $\{x_{q+d+i}^1, \dots, x_{q+d+i}^d\}$, $1 \leqslant i \leqslant p_j - d$. We extend the action of the substitution $\varphi\colon X \to A$ constructed at the $(j-1)$st step by setting

$$\varphi(x_{q+1}^1) = \dots = \varphi(x_{q+p_j}^1) = a_1, \ \dots, \ \varphi(x_{q+1}^d) = \dots = \varphi(x_{q+p_j}^d) = a_d,$$
$$\varphi(y_1^j) = \dots = \varphi(y_d^j) = b.$$

Then, as before,

$$\varphi(\mathrm{Sym}\, f_j) = \gamma z_{1,j+1}^1 \neq 0,$$

where the symmetrization Sym is carried out over the sets

$$\{x_1^1, x_2^1, \dots, x_{q+p_j}^1\}, \ \dots, \ \{x_1^d, x_2^d, \dots, x_{q+p_j}^d\}, \ \{y_1^1, \dots, y_d^1, \dots, y_1^j, \dots, y_d^j\}.$$

Then, as for $j = 1$, $\mathrm{Sym}\, f_j$ generates an irreducible module with character χ_λ, where $\lambda = (\lambda_1, \dots, \lambda_{d+2})$, $\lambda_1 = \dots = \lambda_d = m_1 + \dots + m_j - j$, $\lambda_{d+1} = jd$, $\lambda_{d+2} = 1$, and $m_\lambda \neq 0$ in (2.1).

Thus, for every positive integer t we have constructed a polynomial f_t of degree

$$n = n(t) = (m_1 + \dots + m_t)d + 1 = tmd + d(w_1 + \dots + w_t) + 1,$$

which is not an identity. Furthermore, f_t takes a nonzero value under the substitution $\varphi\colon X \to A$ when the element b is substituted td times. Then by Lemma 7,

$$\frac{td}{n} \leqslant \frac{1}{m+\alpha} + \varepsilon_n,$$

where $\alpha = \pi(w)$ is the slope of w, and $\varepsilon_n \to 0$ as $n \to \infty$. Furthermore, the symmetrization $\mathrm{Sym}\, f_t$ also is not an identity in A, $\varphi(\mathrm{Sym}\, f_t) = K \cdot \varphi(f_t)$, $K \neq 0$, and generates in P_n an irreducible $F[S_n]$-module with character $\chi_{\lambda^{(n)}}$, where

$$\lambda^{(n)} = (\lambda_1, \dots, \lambda_{d+2}), \qquad \lambda_1 = \dots = \lambda_d = m_1 + \dots + m_t - 1,$$
$$\lambda_{d+1} = td, \qquad \lambda_{d+2} = 1.$$

Consequently,

$$\frac{\lambda_{d+1}}{n} = \frac{1}{m + (w_1 + \dots + w_t)/t + 1/(td)} = \beta, \qquad \Phi(\lambda^{(n)}) = \Phi\biggl(\underbrace{\frac{\lambda_1}{n}, \dots, \frac{\lambda_1}{n}}_{d}, \beta, \frac{1}{n}\biggr).$$

In order to obtain a lower estimate for $\Phi(\lambda^{(n)})$, we use properties of periodic words and Sturmian words. By Proposition 1,

$$\lim_{t\to\infty} \frac{w_1 + \dots + w_t}{t} = \alpha,$$

and since $mtd \leqslant n \leqslant (m+1)td$, the quantity $(w_1 + \dots + w_t)/t$ can be made arbitrarily close to α for all sufficiently large n. Consequently, for any $\varepsilon > 0$, there exists N such that

$$\beta = \frac{1}{m + (w_1 + \dots + w_t)/t + 1/(td)} \geqslant \frac{1}{m+\alpha} - \varepsilon$$

for all $n \geqslant N$. Then from properties of the function Φ, we obtain

$$\Phi(\lambda^{(n)}) \geqslant \Phi\biggl(\underbrace{\theta,\dots,\theta}_{d},\frac{1}{m+\alpha}-\varepsilon,0\biggr)=\Phi_d\biggl(\frac{1}{m+\alpha}-\varepsilon\biggr),$$

where $\theta d+1/(m+\alpha)-\varepsilon=1$.

Since

$$c_n(A)\geqslant d_{\lambda^{(n)}}\geqslant \frac{1}{n^{(d+2)^2+d+2}}\Phi(\lambda^{(n)})^n$$

by Lemma 1, and $\varepsilon>0$ is chosen arbitrarily, it follows that

$$\varliminf_{n(t)\to\infty}\sqrt[n(t)]{c_{n(t)}(A)}\geqslant \Phi_d\biggl(\frac{1}{m+\alpha}\biggr).$$

It remains to observe that $c_n(A)$ is a non-decreasing sequence and that

$$n(t+1)-n(t)\leqslant (m+1)d,$$

whence the equation

$$\underline{\exp}(A)=\varliminf_{n\to\infty}\sqrt[n]{c_{n(t)}(A)}\geqslant \Phi_d\biggl(\frac{1}{m+\alpha}\biggr).$$

Lemma 9 is proved.

Lemmas 8 and 9 immediately give us the main result of this section.

Theorem 1. *Let m and d be integers such that $m\geqslant 2$, $1\leqslant d\leqslant m-1$, and let w be an infinite periodic word or a Sturmian word with slope α. Then the PI-exponent of the algebra $A(m,d,w)$ exists and is equal to*

$$\exp(A)=\Phi_d\biggl(\frac{1}{m+\alpha}\biggr)=\Phi\biggl(\underbrace{\frac{m+\alpha-1}{d(m+\alpha)},\dots,\frac{m+\alpha-1}{d(m+\alpha)}}_{d},\frac{1}{m+\alpha}\biggr).$$

§ 4. Exponents of algebras with adjoined unity

4.1. Recall that if an outer unity is adjoined to an algebra A, then the resulting algebra is denoted by $A^{\#}$. We will adjoin unities to the algebras $A(m,d,w)$ considered in § 3.

We shall need a technical result from [19].

Recall that for a given algebra B we denote by $W_n^{(p)}(B)$ the subspace of all homogeneous polynomials of degree n in $y_1,\dots,y_p$ in the relatively free algebra $R(y_1,y_2,\dots)$ of the variety $\operatorname{var}(B)$ with free generators $y_1,y_2,\dots$.

Lemma 10 (see [19], Lemma 6). *Let B be an arbitrary algebra and suppose that* $\dim W_n^{(p)}(B)\leqslant \alpha n^T$ *for some* $\alpha\in\mathbb{R}$ *and* $T\in\mathbb{N}$. *Then* $\dim W_n^{(p)}(B^{\#})\leqslant \alpha(n+1)^{T+p+1}$.

First we estimate the growth of colength from above.

Lemma 11. *Let $A = A(m, d, w)$ be the algebra from the preceding section, where $m \geqslant 2$, $d \leqslant m - 1$, and w is a Sturmian word or an infinite periodic word. Then*

$$l_n(A^{\#}) \leqslant (n+1)^{3(d+3)^2}$$

for all sufficiently large n.

Proof. By Lemma 6,

$$\dim W_n^{(d+3)}(A) \leqslant d(m+1)n^{(d+1)(d+3)} \operatorname{Comp}_w(n).$$

Since the complexity of a periodic word is a constant, and for a Sturmian word it is equal to $n+1$, it follows that

$$\dim W_n^{(d+3)}(A) \leqslant n^{(d+3)^2}$$

for all sufficiently large n. Therefore

$$\dim W_n^{(d+3)}(A^{\#}) \leqslant (n+1)^{2(d+3)^2},$$

by Lemma 10. It follows from Remark 1 that

$$\chi_n(A^{\#}) = \sum_{\substack{\lambda \vdash n \\ h(\lambda) \leqslant d+3}} m_\lambda \chi_\lambda,$$

and $m_\lambda \leqslant \dim W_n^{(d+3)}(A^{\#}) \leqslant (n+1)^{2(d+3)^2}$. And since the number of partitions $\lambda \vdash n$ with $h(\lambda) \leqslant d+3$ does not exceed $(n+1)^{d+3}$, we have

$$l_n(A^{\#}) \leqslant (n+1)^{3(d+3)^2}.$$

Lemma 11 will be required for an upper estimate of the PI-exponent of the algebra $A(m, d, w)^{\#}$. But first we estimate its codimension growth from below.

Lemma 12. *Let $A = A(m, d, w)$ be defined by parameters $m \geqslant 2$, $d \leqslant m - 1$, and w. Then*

$$\underline{\exp}(A^{\#}) \geqslant \exp(A) + 1.$$

Proof. In the proof of Lemma 9, for any $\delta > 0$, we chose an increasing sequence $n = n(t)$, $t = t_0, t_0 + 1, \dots$, a family of partitions $\lambda^{(n)} \vdash n(t)$, and a set of polynomials f_t, $t \geqslant t_0$, with the following properties:

- the partition λ has the form $\lambda = (\lambda_1, \dots, \lambda_{d+2})$, $\lambda_1 = \dots = \lambda_d = m_1 + \dots + m_t - t$, $\lambda_{d+1} = td$, $\lambda_{d+2} = 1$,
- $\Phi(\lambda^{(n)}) \geqslant \Phi_d(1/(m+\alpha) - \delta)$, where α is the slope of w,
- $n(t+1) - n(t) \leqslant d(m+1)$ for all $t \geqslant t_0$,
- the symmetrization of f_t is not an identity of A and generates an irreducible $F[S_n]$-module with character χ_λ,
- f_t is skew-symmetric with respect to λ_1 sets of variables: one of size $d+2$, $td - 1$ of size $d+1$, and $\lambda_1 - \lambda_{d+1}$ of size d.

Furthermore, $\exp(A) = \Phi_d(1/(m+\alpha))$.

Let $\widetilde{h}_{t,k}$ denote the product

$$\widetilde{h}_{t,k} = f_t z_1 \cdots z_k, \qquad k \geqslant 1.$$

We consider the same substitution φ that produced a nonzero value for f_t and $\operatorname{Sym} f_t$; we extend its action to $\widetilde{h}_{t,k}$ by setting $\varphi(z_1) = \cdots = \varphi(z_k) = 1$. Then, obviously,

$$\varphi(\widetilde{h}_{t,k}) = \varphi(f_t) \neq 0.$$

Moreover, if $k \leqslant td$, then we can include $z_1, \dots, z_k$ in the first k skew-symmetric sets for f_t and carry out additional alternation over extended sets. Furthermore, the rules of multiplication of basis elements of A imply that

$$\varphi(\operatorname{Alt}(\widetilde{h}_{t,k})) = \gamma\varphi(\widetilde{h}_{t,k}),$$

where γ is a nonzero integer coefficient. For the polynomial $f_{t,k} = \operatorname{Alt}(\widetilde{h}_{t,k})$, the variables are also distributed over λ_1 skew-symmetric sets: one of size $d+3$, $k-1$ of size $d+2$, $td-k$ of size $d+1$, and $\lambda_1 - td$ of size d. Moreover, if we carry out symmetrization of this polynomial over the same variables as for f_t, plus symmetrization with respect to $z_1, \dots, z_k$, then the value $\varphi(\operatorname{Sym}(f_{t,k}))$ is also proportional to $\varphi(f_t)$ with a nonzero coefficient. That is, the polynomial $\operatorname{Sym}(f_{t,k})$ generates an irreducible $F[S_{n+k}]$-module with character χ_μ, where

$$\mu = (\mu_1, \dots, \mu_{d+3}), \qquad \mu_1 = \lambda_1, \ \dots, \ \mu_d = \lambda_d, \ \mu_{d+1} = \lambda_{d+1}, \ \mu_{d+2} = k, \ \mu_{d+3} = 1.$$

The fact that all partitions of the form

$$\mu = (\lambda_1, \dots, \lambda_d, k, \lambda_{d+1}, 1), \qquad \mu = (k, \lambda_1, \dots, \lambda_{d+2})$$

have nonzero multiplicities in the character $\chi_{n+k}(A^{\#})$ is proved in similar fashion. In other words, we can add any row (the 1st, $(d+1)$st, or $(d+2)$nd) to the diagram D_λ and obtain the diagram D_μ corresponding to the partition $\mu \vdash n+k$ with nonzero multiplicity.

We estimate from below the maximum value of $\Phi(\mu)$ and k corresponding to this maximum value. We set $\lambda_1/n = u_1, \ \dots, \ \lambda_{d+2}/n = u_{d+2}$, $\beta = \Phi(\lambda)$. Then by Lemma 3,

$$\Phi(\theta u_1, \dots, \theta u_{d+2}, 1-\theta) = 1 + \Phi(\lambda) \tag{4.1}$$

is the maximum value that $\Phi(\mu)$ can take, where $\theta = \beta/(\beta+1)$. This means that if k satisfies the two inequalities

$$\frac{k}{n+k} \leqslant 1 - \theta = \frac{1}{\beta+1} \leqslant \frac{k+1}{n+k+1}, \tag{4.2}$$

then the maximum of $\Phi(\mu)$ is attained either at this k, or at $k+1$. Relation (4.2) is equivalent to the double inequality

$$\frac{n}{\beta} - 1 \leqslant k \leqslant \frac{n}{\beta}. \tag{4.3}$$

Recall that n and k depend on t: $n = n(t)$, $k = k(t)$. Taking into account (4.3) and the choice of $n(t)$, we obtain

$$n(t+1) + k(t+1) - n(t) - k(t) \leqslant \frac{\beta+1}{\beta} d(m+1). \tag{4.4}$$

We set $r = r(t) = n(t) + k(t)$, and denote by $\mu^{(r)}$ a partition of $r(t)$ with maximum value $\Phi(\mu^{(r)})$. Since, as n grows, the quantity $1/(\beta+1)$ is ever more precisely approximated by a fraction of the form $k/(n+k)$, we can assume in view of (4.2) that

$$\Phi(\mu^{(r)}) \geqslant \Phi(\lambda^{(n)}) + 1 - \delta'$$

for all sufficiently large n, where $\delta' > 0$ is any quantity given beforehand, $n = n(t)$, $r = r(t)$. Then in view of Lemma 1, we have

$$c_{r(t)}(A^{\#}) \geqslant \frac{\Phi(\mu^{(r(t))})^n}{n^{(d+2)^2+d+3}} \geqslant \frac{(\Phi(\lambda^{(n)}) + 1 - \delta')^n}{n^{(d+2)^2+d+3}} \geqslant \frac{(\Phi_d(1/(m+\alpha) - \delta) + 1 - \delta')^n}{n^{(d+2)^2+d+3}}. \tag{4.5}$$

Since all the differences $r(t+1) - r(t)$ are bounded by the same constant (see (4.4)), and the sequence $\{c_n(A^{\#})\}$ is nondecreasing, it follows from (4.5) that

$$\varliminf_{n\to\infty} \sqrt[n]{c_n(A^{\#})} \geqslant \Phi_d\left(\frac{1}{m+\alpha} - \delta\right) + 1 - \delta'.$$

Finally, since δ and δ' are any arbitrarily small quantities, we obtain

$$\underline{\exp}(A^{\#}) \geqslant \exp(A) + 1,$$

and Lemma 12 is proved.

4.2. We now obtain an upper estimate for $\overline{\exp}(A)^{\#}$.

Lemma 13. *We have the inequality*

$$\overline{\exp}(A^{\#}) \leqslant \exp(A) + 1.$$

Proof. Since the colength $l_n(A^{\#})$ is polynomially bounded by Lemma 11, it is sufficient to prove that

$$\Phi(\lambda) \leqslant \Phi_d\left(\frac{1}{m+\alpha}\right) + 1 = \exp(A) + 1$$

for any $\lambda \vdash n$ with $m_\lambda \neq 0$ in $\chi_n(A)$, as shown by relation (2.3).

Let $h = h(x_1, \dots, x_n)$ be a multilinear polynomial that is not an identity of $A^{\#}$ and that generates in P_n an irreducible $F[S_n]$-module with character χ_λ. As pointed out earlier, we can assume that h is skew-symmetric with respect to λ_1 sets of variables, and λ_{d+2} of them have size at least λ_{d+2}. If $\lambda_{d+2} = 0$, then

$$\Phi(\lambda) \leqslant \Phi\Bigl(\underbrace{\frac{1}{d+1}, \dots, \frac{1}{d+1}}_{d+1}, 0, 0\Bigr) = d + 1 < 1 + \Phi_d\left(\frac{1}{m+\alpha}\right).$$

Suppose that $\lambda_{d+2} \neq 0$. We fix an arbitrary $\varepsilon > 0$. Since $h \notin \mathrm{Id}(A^{\#})$, there exists a substitution φ of the basis elements of A and 1 in place of the variables $x_1, \ldots, x_n$ for which

$$\varphi(h) = f(z^i_{jk}, a_1, \ldots, a_d, b) = f$$

is a nonzero monomial of degree n' in $\{z^i_{jk}, a_1, \ldots, a_d, b\}$, where $n' = n - n_1$ and n_1 is the number of 1s from $A^{\#}$ substituted in place of $x_1, \ldots, x_n$.

We observe that λ_{d+3} can take only two values, 0 or 1. First suppose that $\lambda_{d+3} = 1$. In this case, h has $r = \lambda_{d+2}$ skew-symmetric sets of variables of size at least $d+2$, and $\lambda_{d+1} - \lambda_{d+2}$ skew-symmetric sets of variables of size $d+1$. One of the elements $\{1, b\}$ is substituted into each of these latter sets. Suppose that b is substituted into exactly k sets, and $\lambda_{d+1} - \lambda_{d+2} = k + t$. Then $n_1 \geqslant r + t$ and

$$n'' = n - r - t \geqslant n - n_1 = n' = \deg_b f \geqslant r + k.$$

If $\lambda_{d+1} > \lambda_{d+2}$, then by transferring boxes from the $(d+1)$st row of the diagram D_λ into the $(d+2)$nd row, we can obtain a partition $\lambda' \vdash n$ for which either the $(d+2)$nd row of $D_{\lambda'}$ has length $r+k$ (if $k \leqslant t$), or the $(d+1)$st row has length $r+k$ (if $k > t$). By deleting this row, we obtain a partition $\mu \vdash n''$ for which $\mu_{d+1} = r + k$. Then

$$\frac{\mu_{d+1}}{n''} \leqslant \frac{\deg_b f}{n'} \leqslant \frac{1}{m+\alpha} + \varepsilon_{n'} \tag{4.6}$$

by Lemma 7, where $n' = n - n_1$ and n_1 is the number of 1s substituted into h in place of $x_1, \ldots, x_n$.

We now obtain an inequality similar to (4.6) for $\lambda_{d+3} = 0$. In this case, $h(x_1, \ldots, x_n)$ depends on $r = \lambda_{d+2}$ skew-symmetric sets of size $d+2$. If both elements $1 \in A^{\#}$, $b \in A$ were substituted into each of them, then the same arguments as above would give us relation (4.6). In the opposite case, we substitute either 1 into $r-1$ of these sets and the element b into all r sets, or, on the contrary, unity into the r sets and the basis element b into $r-1$ sets. By transferring, if necessary, boxes from the $(d+1)$st row of D_λ into the $(d+2)$nd (as for $\lambda_{d+3} = 1$) and deleting a row of length $r+t$ (where k and t are defined in the same way as in the case $\lambda_{d+3} = 1$), we obtain a partition $\mu \vdash n'' = n - r - t$ with $\mu_{d+1} = r + k$. Here, in the first case we obtain the inequalities

$$\mu_{d+1} \leqslant \deg_b f, \qquad n'' \geqslant n - n_1 + 1 \geqslant n - n_1 = n',$$

and in the second case, the inequalities

$$\mu_{d+1} \leqslant \deg_b f + 1, \qquad n'' \geqslant n - n_1 = n'.$$

Obviously, in the first case the partition μ satisfies condition (4.6), and in the second, the condition

$$\frac{\mu_{d+1}}{n''} \leqslant \frac{\deg_b f}{n'} \leqslant \frac{1}{m+\alpha} + \varepsilon_{n'} + \frac{1}{n'}. \tag{4.7}$$

Since (4.6) is a stronger restriction than (4.7), we can assume that $\mu = (\mu_1, \ldots, \mu_{d+1}) \vdash n''$ always satisfies inequality (4.7) in which $n'' \leqslant n$ and $n'' \to \infty$

as $n \to \infty$, while $n' = n - n_1$, where n_1 is the number of 1s substituted into $h(x_1, \dots, x_n)$ in order to obtain a nonzero value.

First, we observe that $\lambda_1 \geqslant n_1$, since h is skew-symmetric with respect to λ_1 sets of variables. We set $x = \lambda_1/n$. Then

$$\Phi(\lambda) \leqslant \Phi\biggl(x, \underbrace{\frac{1-x}{d+2}, \dots, \frac{1-x}{d+2}}_{d+2}\biggr) = H(x).$$

The limit of the function $H(x)$ as $x \to 1$ is equal to 1. In particular, this means that there exists an integer q such that if $\lambda_1 \geqslant n(q-1)/q$, then $\Phi(\lambda) < d$ for all sufficiently large $n \geqslant N$.

We now divide all partitions of $\lambda \vdash n \geqslant N$ into two groups: where $\lambda_1 > n(q-1)/q$, and where $\lambda_1 \leqslant n(q-1)/q$. For all partitions in the first group, the inequality

$$\Phi(\lambda) < d < \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon\biggr)$$

holds due to the choice of q and n. For partitions in the second group, we use relation (4.7). The diagram D_μ is obtained from $D_{\lambda'}$ by deleting a row, and $D_{\lambda'}$ is obtained from D_λ by transferring several boxes down. Therefore $\Phi(\lambda) \leqslant \Phi(\lambda') \leqslant \Phi(\mu) + 1$, by Lemmas 2 and 3. Then it follows from (4.7) that

$$\Phi(\lambda) \leqslant \Phi(\mu) + 1 \leqslant \Phi\biggl(\theta, \dots, \theta, \frac{1}{m+\alpha} + \varepsilon_{n'} + \frac{1}{n'}, \frac{1}{n''}\biggr),$$

due to the properties of Φ, where

$$d\theta + \frac{1}{m+\alpha} + \varepsilon_{n'} + \frac{1}{n'} + \frac{1}{n''} = 1.$$

Since $\lambda_1 \leqslant n(q-1)/q$, it follows that $n' = n - n_1 \geqslant n - \lambda_1 \geqslant n/q$. Therefore $n' \to \infty$, as well as n'', as n grows, and $\varepsilon_{n'} \to 0$. As in the proof of Lemma 7, we obtain that

$$\Phi(\lambda) \leqslant 1 + \Phi\biggl(\theta', \dots, \theta', \frac{1}{m+\alpha} + \varepsilon, 0\biggr) = 1 + \Phi_d\biggl(\frac{1}{m+\alpha} + \varepsilon\biggr)$$

for all sufficiently large n. Since $\varepsilon > 0$ was chosen arbitrarily, we obtain

$$\overline{\exp}(A^{\#}) \leqslant 1 + \Phi_d\biggl(\frac{1}{m+\alpha}\biggr) = \exp(A) + 1.$$

Lemma 13 is proved.

Combination of Lemmas 12 and 13 immediately yields the following result.

Theorem 2. *Let m and d be integers such that $m \geqslant 2$, $m-1 \geqslant d$, and let w be an infinite periodic word or a Sturmian word. If $A = A(m,d,w)$ and $A^{\#}$ is obtained from A by adjoining a unity, then $\exp(A^{\#})$ exists, and $\exp(A^{\#}) = \exp(A) + 1$.*

Corollary 1. *For any real number* $\gamma \geqslant 2$, *there exists* (*in general, a non-associative*) *algebra* A_γ *with unity and with PI-exponent* $\exp(A_\gamma) = \gamma$.

Proof. For given d, the set of values

$$\left\{\Phi_d\left(\frac{1}{m+\alpha}\right) = \exp(A(m,d,w)) \mid 0 \leqslant \alpha \leqslant 1,\ m = d+1, d+2, \dots\right\}$$

covers the entire interval $(d, d+1]$. Consequently, any real number $\gamma > 2$ is realized as an exponent $\exp(A^{\#})$, where $A = A(m,d,w)$ for suitable m, d, and w. For $\gamma = 2$, there are many realizations even in the associative case. For example, for an infinite-dimensional Grassmann algebra G with unity, we have $c_n(G) = 2^{n-1}$ (see [34] or [24], Theorem 4.1.8). Therefore, $\exp(G) = 2$.

The question about the set of values of PI-exponents of finite-dimensional algebras is of independent interest. Clearly, if the field F is countable, then this set is also countable. It was shown in [13] that the set $\{\exp(A) | \dim A < \infty\}$ is dense in $[1, \infty)$, and it was proved in [4] that for a finite-dimensional unital algebra A, the growth of $\{c_n(A)\}$ either is polynomial or is bounded below by the exponential function 2^n.

Another consequence of Theorems 1 and 2 is the fact that the set of PI-exponents of finite-dimensional algebras with unity is a dense subset in the domain $[2, \infty) \subset \mathbb{R}$.

Corollary 2. *For any real* $2 \leqslant \alpha < \beta$, *there exists a finite-dimensional* (*in general, non-associative*) *algebra* B *with unity such that*

$$\alpha \leqslant \exp(B) \leqslant \beta.$$

Proof. Consider the algebra $A(m,d,w)$, where w is an infinite periodic word with period T, and, together with it, a finite-dimensional algebra $B = B(m,d,w)$ with basis

$$\left\{a_1, \dots, a_d, b, z^i_{jk} \mid 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant m + w_j,\ 1 \leqslant k \leqslant T\right\}$$

and multiplication table

$$z^i_{jk} a_i = \begin{cases} z^i_{j+1,k} & \text{if } j < m + w_k, \\ 0 & \text{if } j = m + w_k, \end{cases}$$

$$z^i_{m+w_k,k} b = \begin{cases} z^{i+1}_{1k} & \text{if } i < d, \\ z^1_{1,k+1} & \text{if } i = d,\ k < T, \\ z^1_{11} & \text{if } i = d,\ k = T. \end{cases}$$

It is easy to observe that the algebras $A(m,d,w)$ and $B(m,d,w)$ are PI-equivalent, that is, they have the same identities. But then the algebras $A(m,d,w)^{\#}$ and $B(m,d,w)^{\#}$ are also PI-equivalent. Therefore, $\exp(A(m,d,w)^{\#}) = \exp(B(m,d,w)^{\#})$. In particular, $\exp(B(m,d,w)^{\#}) = \exp(A(m,d,w)) + 1$.

By Proposition 1, for any rational $q \in (0,1)$, there exists a periodic word w with slope $\pi(w) = q$. But then

$$\exp(B(m,d,w)^{\#}) = \Phi_d\left(\frac{1}{m+q}\right) + 1,$$

by Theorem 2. Therefore we can find a rational positive number $q < 1$ such that $\alpha \leqslant \exp(B(m,d,w)^{\#}) \leqslant \beta$.

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