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NUMERICAL INVARIANTS OF IDENTITIES OF UNITAL ALGEBRAS

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We study polynomial identities of algebras with adjoined external unit. For a wide class of algebras we prove that adjoining external unit element leads to increasing of PI-exponent precisely to 1. We also show that any real number from the interval [2,3] can be realized as PI-exponent of some unital algebra.

Key Words: Codimension; Exponential growth; Fractional PI-exponent; Non-associative unital algebra; Polynomial identity.

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1. INTRODUCTION

We study numerical characteristics of polynomial identities of algebras over a field F of characteristic zero. Given an algebra A over F, one can associate to it the sequence $\{c_n(A)\}$ of non-negative integers called the *sequence of codimensions*. If the growth of $\{c_n(A)\}$ is exponential, then the limiting ratio of consecutive terms is called PI-*exponent* of A and written $exp(A)$. In the present paper, we are mostly interested what happens with PI-exponent if we adjoin to A an external unit element.

The first results in this area were proved for associative algebras. It is known that $exp(A)$ is an integer in the associative case [6], [7]. It was shown in [9] that it follows from the proofs in [6], [7] that either $exp(A^\sharp) = exp(A)$ or $exp(A^\sharp) = exp(A) + 1$ and both options can be realized. Here $A^\sharp$ is the algebra A with adjoined external unit.

The next result was published in [15], following an example of 5-dimensional algebra A with $exp(A) < 2$ constructed in [4]. The point is that in the associative or Lie case PI-exponent cannot be less than 2 ([11], [13]). For a finite dimensional Lie superalgebra, Jordan and alternative algebra PI-exponent is also at least 2. Starting from the example A from [4] it was shown in [15] that $exp(A^\sharp) = exp(A) + 1$. In [15]

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also the following problem was stated: Is it true that always either $exp(A^\sharp) = exp(A)$ or $exp(A^\sharp) = exp(A) + 1$?

An example of 4-dimensional simple algebra A with a fractional PI-exponent was constructed in [2]. It was also shown that $exp(A^\sharp) = exp(A) + 1$. This result was announced in [1]. It was also shown in [1] that if A is itself a unital algebra then $exp(A^\sharp) = exp(A)$.

In the present paper (see Theorem 1) we shall prove that for a previously known series of algebras A_α with $exp(A_\alpha) = \alpha$, $\alpha \in \mathbb{R}, 1 < \alpha < 2$ (see [3]), any extended algebra $A_\alpha^\sharp$ has exponent $\alpha + 1$. That is, we shall show that there exist infinitely many algebras A such that $exp(A^\sharp) = exp(A) + 1$.

Another important question is the following: which real numbers can be realized as PI-exponents of some algebra? For example, if A is any associative PI-algebra or a finite dimensional Lie or Jordan algebra, then $exp(A)$ is an integer (see [5], [6], [7], [14]).

For unital algebras it is only known that if $\dim A < \infty$ then $exp(A)$ cannot be less than 2. As a consequence of the main result of our paper (see Corollary 1) we shall obtain that for any real $\alpha \in [2, 3]$ there exists a unital algebra B_α such that $exp(B_\alpha) = \alpha$.

2. PRELIMINARIES

Let A be an algebra over a field F of characteristic zero, and let $F\{X\}$ be absolutely free algebra over F with a countable set of generators $X = \{x_1, x_2, \dots\}$. Recall that a polynomial $f = f(x_1, \dots, x_n)$ is said to be an *identity* of A if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. The set $Id(A)$ of all polynomial identities of A forms an ideal of $F\{X\}$.

Denote by P_n the subspace of all multilinear polynomials in $F\{X\}$ on $x_1, \dots, x_n$. Then the intersection $Id(A) \cap P_n$ is the space of all multilinear identities of A of degree n.

Denote

$$P_n(A) = \frac{P_n}{Id(A) \cap P_n}.$$

A non-negative integer

$$c_n(A) = \dim P_n(A)$$

is called the nth *codimension* of A. Asymptotic behavior of the sequence $\{c_n(A)\}$, $n = 1, 2, \dots,$ is an important numerical invariant of identities of A. We refer to [8] for an account of basic notions of the theory of codimensions of PI-algebras.

If the sequence $\{c_n(A)\}$ is exponentially bounded, i.e., $c_n(A) \le a^n$ for all n and for some number a (for example in the case when $\dim A < \infty$ and in many other cases), we can define the lower and the upper PI-exponents of A by

$$\underline{exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \qquad \overline{exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)},$$

and (the ordinary) PI-exponent

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

provided that $\underline{exp}(A) = \overline{exp}(A)$.

In order to compute the values of codimensions we can consider symmetric group action on P_n defined by

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)} \qquad \forall \sigma \in S_n.$$

The subspace $P_n \cap Id(A)$ is invariant under this action and we can study the structure of $P_n(A)$ as an S_n-module. Denote by $\chi_n(A)$ the S_n-character of $P_n(A)$, called the nth *cocharacter* of A. Since char $F = 0$ and any S_n-representation is completely reducible, the nth cocharacter has the decomposition

$$\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda, \tag{1}$$

where χ_λ is the irreducible S_n-character corresponding to the partition $\lambda \vdash n$ and non-negative integer m_λ is the multiplicity of χ_λ in $\chi_n(A)$.

Obviously, it follows from (1) that

$$c_n(A) = \sum_{\lambda\vdash n} m_\lambda \deg \chi_\lambda.$$

Another important numerical characteristic is the nth *colength* of A defined by

$$l_n(A) = \sum_{\lambda\vdash n} m_\lambda$$

with m_λ taken from (1). In particular, if the sequence $\{l_n(A)\}$ is polynomially bounded as a function of n while some of $\deg \chi_\lambda$ with $m_\lambda \neq 0$ are exponentially large, the principal part of the asymptotic of $\{c_n(A)\}$ is defined by the largest value of $\deg \chi_\lambda$ with nonzero multiplicity.

For studying the asymptotic of codimensions, it is convenient to use the following functions. Let $0 \le x_1, \ldots, x_d \le 1$ be real numbers such that $x_1 + \cdots + x_d = 1$. Denote

$$\Phi(x_1, \ldots, x_d) = \frac{1}{x_1^{x_1} \cdots x_d^{x_d}}.$$

If $d = 2$, then instead of $\Phi(x_1, x_2)$ we will write

$$\Phi_0(x) = \frac{1}{x^x(1-x)^{1-x}}.$$

We assume that some of $x_1, \ldots, x_d$ can have zero values. In this case, we assume that $0^0 = 1$.

Given $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$, we define

$$\Phi(\lambda) = \frac{1}{(\frac{\lambda_1}{n})^{\frac{\lambda_1}{n}} \cdots (\frac{\lambda_d}{n})^{\frac{\lambda_d}{n}}}. \tag{2}$$

For partitions $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ with $k < d$, we also consider $\Phi(\lambda)$ as in (2), assuming $\lambda_{k+1} = \cdots = \lambda_d = 0$.

The relationship between $\deg \chi_\lambda$ and $\Phi(\lambda)$ is given by the following lemma.

Lemma 1 (See [10, Lemma 1]). *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition of n. If $k \leq d$ and $n \geq 100$, then*

$$\frac{\Phi(\lambda)^n}{n^{d^2+d}} \leq \deg \chi_\lambda \leq n\Phi(\lambda)^n.$$

Now we investigate how the value of $\Phi(x_1, \ldots, x_d)$ increases after adding one extra variable.

Lemma 2. *Let*

$$\Phi(x_1, \ldots, x_d) = \frac{1}{x_1^{x_1} \cdots x_d^{x_d}}, \qquad 0 \leq x_1, \ldots, x_d, \qquad x_1 + \cdots + x_d = 1,$$

and let $\Phi(z_1, \ldots, z_d) = a$ for some fixed $z_1, \ldots, z_d$. Then

$$\max_{0 \leq t \leq 1} \{\Phi(y_1, \ldots y_d, 1-t) | y_1 = tz_1, \ldots, y_d = tz_d\} = a + 1.$$

Moreover, the maximal value is achieved if $t = \frac{a}{a+1}$.

Proof. Consider

$$g(t) = \ln \Phi^{-1}(tz_1, \ldots, tz_d, 1-t).$$

Then

$$g(t) = t \ln t + (1-t) \ln(1-t) - t \ln a.$$

Hence its derivative is equal to

$$g'(t) = \ln \frac{t}{(1-t)a}$$

and $g'(t) = 0$ if and only if $t = (1-t)a$, that is $t = \frac{a}{a+1}$. It is not difficult to check that g has the minimum at this point.

Now we compute the value of g:

$$g\left(\frac{a}{a+1}\right) = \frac{a}{a+1} \ln \frac{a}{a+1} + \frac{1}{a+1} \ln \frac{1}{a+1} - \frac{a}{a+1} \ln a = \ln B,$$

where

$$B = \left(\frac{a}{a+1}\right)^{\frac{a}{a+1}} \left(\frac{1}{a+1}\right)^{\frac{1}{a+1}} a^{-\frac{a}{a+1}} = \frac{1}{a+1}.$$

Hence $\Phi_{max} = B^{-1} = a + 1$ and we have completed the proof. □

The following lemma shows what happens with $\Phi(\lambda)$ when we insert an extra row in Young diagram D_λ.

Lemma 3. *Let γ be a positive real number and let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a partition of n such that $\frac{\lambda_1}{n}, \ldots, \frac{\lambda_d}{n} \geq \gamma$. Then for any $\varepsilon > 0$ there exist $n' = kn$ and a partition $\mu \vdash n'$, $\mu = (\mu_1, \ldots, \mu_{d+1})$ such that for some integers $1 \leq i \leq d+1$ and $q \geq 1$ the following conditions hold*:

1) $\mu_j = q\lambda_j$ *for all* $j \leq i - 1$;
2) $\mu_{j+1} = q\lambda_j$ *for all* $j \geq i$; *and*
3) $|\Phi(\lambda) - \Phi(\mu) + 1| < \varepsilon$.

Moreover, k does not depend on λ and n.

Proof. Denote

$$z_1 = \frac{\lambda_1}{n}, \ldots, z_d = \frac{\lambda_d}{n}$$

and $a = \Phi(z_1, \ldots, z_d) = \Phi(\lambda)$. By Lemma 2,

$$\Phi(tz_1, \ldots, tz_d, 1-t) = a + 1 \tag{3}$$

if $t = \frac{a}{a+1}$. It is not difficult to check that $1 \leq \Phi(x_1, \ldots, x_d) \leq d$, and hence $\frac{1}{d+1} \leq 1 - t \leq \frac{1}{2}$.

Note that $\Phi = \Phi(x_1, \ldots, x_{d+1})$ can be viewed as a function of d independent indeterminates $x_1, \ldots, x_d$. Conditions $0 < \gamma \leq x_1, \ldots, x_d$ and $\frac{1}{d+1} \leq x_{d+1} \leq \frac{1}{2}$ define a compact domain Q in $\mathbb{R}^d$ since $x_{d+1} = 1 - x_1 - \cdots - x_d$. Since Φ is continuous on Q, there exists an integer k such that

$$|\Phi(x_1, \ldots, x_d, x_{d+1}) - \Phi(x'_1, \ldots, x'_d, x'_{d+1})| < \varepsilon$$

as soon as $|x_i - x'_i| < \frac{1}{k}$ for all $i = 1, \ldots, d$. Clearly, k does not depend on n and λ. Then there exists a rational number $t_0 = \frac{q}{k} < 1$ such that $|t - t_0| < \frac{1}{k}$ and

$$|\Phi(t_0z_1, \ldots, t_0z_d, 1-t_0) - a - 1| < \varepsilon. \tag{4}$$

Denote $y_0 = 1 - t_0$. Then $t_0z_i \leq 1 - t_0 = y_0 \leq t_0z_{i-1}$ for some i (or $y_0 > t_0z_1$, or $y_0 < t_0z_d$).

Now we set $n' = kn$,

$$\mu_1 = q\lambda_1, \ldots, \mu_{i-1} = q\lambda_{i-1},$$
$$\mu_{i+1} = q\lambda_i, \ldots, \mu_{d+1} = q\lambda_d,$$

and $\mu_i = n(k-q)$. Then $\mu = (\mu_1, \ldots, \mu_{d+1})$ is a partition of n' and

$$\Phi(\mu) = \Phi(t_0 z_1, \ldots, t_0 z_d, 1 - t_0).$$

In particular, $|\Phi(\lambda) - \Phi(\mu) - 1| < \varepsilon$ by (3) and (4), and we have completed the proof of the lemma. □

3. ALGEBRAS OF INFINITE WORDS

In this section we recall some constructions and algebras from [3] and their properties. These algebras will be used for constructing unital algebras.

Let $K = (k_1, k_2, \ldots)$ be an infinite sequence of integers $k_i \geq 2$. Then the algebra $A(K)$ is defined by its basis

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \ldots, \tag{5}$$

where

$$Z_i = \{z_j^{(i)} | 1 \leq j \leq k_i, \ i = 1, 2, \ldots\} \tag{6}$$

with the multiplication table

$$z_1^{(i)} a = z_2^{(i)}, \ldots, z_{k_i - 1}^{(i)} a = z_{k_i}^{(i)}, z_{k_i}^{(i)} b = z_1^{(i+1)} \tag{7}$$

for all $i = 1, 2, \ldots$. All remaining products are assumed to be zero.

It is easy to verify (see also [3]) that A satisfies the identity $x_1(x_2 x_3) = 0$ and if $m_\lambda \neq 0$ in (1) then $\lambda = (\lambda_1)$ or $\lambda = (\lambda_1, \lambda_2)$ or $\lambda = (\lambda_1, \lambda_2, 1)$. Denote by $W_n^{(d)}$, $d \leq n$, the subspace of the free algebra $F\{X\}$ of all homogeneous polynomials of degree n on $x_1, \ldots, x_d$. Given a PI-algebra A, we define

$$W_n^{(d)}(A) = \frac{W_n^{(d)}}{W_n^{(d)} \cap Id(A)}.$$

Recall that the height $h(\lambda)$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ is equal to d. We will use the following result from [3].

Lemma 4 ([3, Lemma 4.1]). *Let A be a PI-algebra with nth cocharacter $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. Then for every $\lambda \vdash n$ with $h(\lambda) \leq d$, we have that $m_\lambda \leq \dim W_n^{(d)}(A)$.*

Now let $w = w_1 w_2 \ldots$ be an infinite word in the alphabet $\{0, 1\}$. Given an integer $m \geq 2$, let $K_{m,w} = \{k_i\}$, $i = 1, 2, \ldots$, be the sequence defined by

$$k_i = \begin{cases} m, & if \ w_i = 0 \\ m + 1, & if \ w_i = 1, \end{cases} \tag{8}$$

and write $A(m, w) = A(K_{m,w})$.

Recall that the complexity $Comp_w(n)$ of an infinite word w is the number of distinct subwords of w of the length n (see [12], Chapter 1). Slightly modifying the proof of Lemma 4.2 from [3] we obtain:

Lemma 5. *For any $m \geq 2$ and for any infinite word w, the following inequalities hold*:

$$\dim W_n^{(d)}(A(m,w)) \leq d(m+1)nComp_w(n)$$

and

$$l_n(A(m,w)) \leq n^3 \dim W_n^{(3)}(A(m,w)).$$

Now we fix the algebra $A(m,w)$ by choosing the word w. Obviously, $Comp_w(n) \leq T$ for any infinite periodic word with period T. It is well known (see [12]) that $Comp_w(n) \geq n+1$ for any aperiodic word w. In the case when $Comp_w(n) = n+1$ for all $n \geq 1$, the word w is said to be *Sturmian*. It is also known that for any Sturmian or periodic word the limit

$$\pi(w) = \lim_{n\to\infty} \frac{w_1 + \cdots + w_n}{n} > 0$$

always exists (we always assume that a periodic word is nonzero). This limit $\pi(w)$ is called the *slope* of w. For any real number $\alpha \in (0,1)$, there exists a word w with $\pi(w) = \alpha$ and w is Sturmian or periodic depending on whether α is irrational or rational, respectively. Moreover,

$$exp(A(m,w)) = \Phi_0(\beta) = \frac{1}{\beta^\beta(1-\beta)^{1-\beta}}$$

for Sturmian or periodic word w, where $\beta = \frac{1}{m+\alpha}$, $\alpha = \pi(w)$ (see [3], Theorem 5.1). As a consequence, for any real $1 \leq \alpha \leq 2$ there exists an algebra A such that $exp(A) = \alpha$.

Finally, for any periodic word w and for any $m \geq 2$ there exists a finite dimensional algebra $B(m,w)$ satisfying the same identities as $A(m,w)$. In particular, for any rational $0 < \beta \leq \frac{1}{2}$, there exists a finite dimensional algebra B with

$$exp(B) = \Phi_0(\beta) = \frac{1}{\beta^\beta(1-\beta)^{1-\beta}}.$$

4. ALGEBRA WITH ADJOINED UNIT

We fix a Sturmian or periodic word w and $m \geq 2$ and consider the algebra $A = A(m,w)$. Denote by $A^\sharp$ the algebra obtained from A by adjoining external unit element 1. Our main goal is to prove that $exp(A^\sharp)$ exists and that

$$exp(A^\sharp) = exp(A) + 1.$$

First we find a polynomial upper bound for the colength of $A^\sharp$. We start with a remark concerning an arbitrary algebra B. Recall that, given an algebra B, $W_n^{(d)}(B)$

is the dimension of the space of homogeneous polynomials on $x_1, \ldots, x_d$ of total degree n modulo ideal $Id(B)$.

Lemma 6. *Let B be an arbitrary algebra. Suppose that* $\dim W_n^{(d)}(B) \le \alpha n^T$ *for some natural T, $\alpha \in \mathbb{R}$ and for all $n \ge 1$. Then*

$$\dim W_n^{(d)}(B^\sharp) \le \alpha(n+1)^{T+d+1}.$$

Proof. Denote by $F\{X\}^\sharp$ absolutely free algebra generated by X with adjoined unit element. First note that a multihomogeneous polynomial $f(x_1, \ldots, x_d)$ is an identity of $B^\sharp$ if all multihomogeneous on $x_1, \ldots, x_d$ components of $f(1+x_1, \ldots, 1+x_d)$ are identities of B.

Clearly, the number of multihomogeneous polynomials on $x_1, \ldots, x_d$ of total degree k, linearly independent modulo $Id(B)$, does not exceed $\dim W_k^{(d)}(B)$. On the other hand, the number of multihomogeneous components of total degree k in a free algebra $F\{x_1, \ldots, x_d\}$ does not exceed $(k+1)^d$. Take now

$$N = (k+1)^d \sum_{k=0}^{n} \dim W_k^{(d)}(B) + 1$$

assuming that $\dim W_0^{(d)}(B) = 1$. Clearly,

$$N \le 1 + (n+1)^d \alpha \sum_{k=0}^{n} k^T < \alpha(n+1)^{T+d+1}.$$

Given homogeneous polynomials $f_1, \ldots, f_{N+1}$ on $x_1, \ldots, x_d$ of degree n, consider their linear combination $f = \lambda_1 f_1 + \cdots + \lambda_{N+1} f_{N+1}$ with unknown coefficients $\lambda_1, \ldots, \lambda_{N+1}$. The assumption that some multihomogeneous component of $f(1+x_1, \ldots, 1+x_d)$ is an identity of $B^\sharp$ is equivalent to some linear equation on $\lambda_1, \cdots, \lambda_{N+1}$. Hence the condition that all multihomogeneous components of $f(1+x_1, \ldots, 1+x_d)$ are identities of B leads to at most N linear equations on $\lambda_1, \cdots, \lambda_{N+1}$. It follows that $f_, \ldots, f_{N+1}$ are linearly dependent modulo $Id(B^\sharp)$, and we have completed the proof. □

Lemma 7. *Let $A = A(m, w)$ where $m \ge 2$ and w is periodic or a Sturmian word. Then*

$$l_n(A^\sharp) \le 4(m+1)(n+1)^{12}$$

for all sufficiently large n.

Proof. First note that the cocharacter of $A^\sharp$ lies in the strip of width 4: that is, if $m_\lambda \ne 0$ in the decomposition

$$\chi_n(A^\sharp) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{9}$$

then $h(\lambda) \leq 4$. The number of partitions of n of type $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $1 \leq k \leq 4$ is less than n^4. By Lemma 5,

$$\dim W_n^{(4)}(A) \leq 4(n+1)Comp_w(n). \tag{10}$$

If w is a Sturmian word, then $Comp_w(n) = n + 1$. If w is periodic, then its complexity is finite and hence $Comp_w(n) \leq n + 1$ for all sufficiently large n in (10). In particular,

$$\dim W_n^{(4)}(A) \leq 4(m+1)(n+1)^2 \leq 4(m+1)n^2$$

for all sufficiently large n. Applying Lemmas 4, 5, and 6, we obtain

$$m_\lambda \leq \dim W_n^{(4)}(A^\sharp) \leq 4(m+1)(n+1)^8$$

for all $m_\lambda \neq 0$ in (9) and then

$$l_n(A^\sharp) = \sum_{\lambda \vdash n} m_\lambda \leq 4(m+1)n^4(n+1)^8 \leq .4(m+1)(n+1)^{12}.$$

□

In the next step, we shall find an upper bound for $\Phi(\lambda)$, provided that $m_\lambda \neq 0$ in the nth cocharacter of $A^\sharp$.

Lemma 8. *For any $\varepsilon > 0$, there exists n_0 such that $m_\lambda = 0$ in (9) if $n > n_0$ and*

$$\frac{\lambda_3}{\lambda_1} \geq \frac{\beta}{1-\beta} + \varepsilon$$

where $\beta = \frac{1}{m+\alpha}$ and α is the slope of w.

Proof. First let $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1) \vdash n$. Inequality $m_\lambda \neq 0$ means that there exists a multilinear polynomial g of degree n depending on one alternating set of four variables, $\lambda_3 - 1$ alternating sets of three variables and some extra variables, and g is not an identity of $A^\sharp$. That is, there exists an evaluation $\varphi : F\{X\}^\sharp \to A^\sharp$ such that $\varphi(g) \neq 0$ and the set $\{\varphi(x_1), \ldots, \varphi(x_n)\}$ contains at least λ_3 basis elements $b \in A$ and at most λ_1 elements $a \in A$. Obviously, $\varphi(g) = 0$ if $\{\varphi(x_1), \ldots, \varphi(x_n)\}$ does not contain exactly one element $z_j^{(i)} \in A$.

Any nonzero product of basis elements of A is the left-normed product of the type

$$z_j^{(i)} a^{k_1} b^{l_1} \cdots a^{k_t} b^{l_t},$$

where $k_1, \ldots, k_t, l_1, \ldots, l_t$ are equal to 0 or 1. More precisely, this product can be written in the form

$$z_j^{(i)} f(a, b), \tag{11}$$

where

$$f(a, b) = a^{t_0} b a^{t_1} b \cdots b a^{t_k} b a^{t_{k+1}}$$

is an associative monomial on a and b and

$$t_0 = m + w_i - j, t_1 = m + w_{i+1} - 1, \ldots, t_k = m + w_{i+k} - 1, t_{k+1} \leq m + w_{i+k+1} - 1.$$

In particular, $\deg_b f = k + 1$ and

$$\deg_a f = t_0 + t_1 + \cdots + t_{k+1} \geq t_1 + \cdots + t_k = (m-1)k + w_{i+1} + \cdots + w_{i+k}.$$

The total degree of monomial (11) (i.e., the number of factors) is

$$n = (m+1)k + w_i + \cdots + w_{i+k} + t_{k+1} - j + 1.$$

Hence, $(m+1)k \geq n - (1+k) - m - 1$ and $k \geq \frac{n-m-2}{m+2}$. In particular, k grows with increasing n.

It is known that

$$\frac{w_{i+1} + \cdots + w_{i+k}}{k} \geq \alpha - \frac{C}{k}$$

for some constant C (see [3], Proposition 5.1 or [12], Section 2.2). This implies that

$$\deg_a f > (m-1)k + k(\alpha - \delta),$$

where $\delta = \frac{C}{k}$ and

$$\frac{\deg_b f}{\deg_a f} < \frac{1 + \frac{1}{k}}{m - 1 + \alpha - \delta}.$$

Since $\varphi(g) \neq 0$, at least one monomial of the type (11) in $\varphi(g)$ is nonzero. Therefore,

$$\frac{\lambda_3}{\lambda_1} \leq \frac{\deg_b f}{\deg_a f} < \frac{1 + \frac{1}{k}}{m - 1 + \alpha - \delta}. \tag{12}$$

Since δ is an arbitrary small positive real number, one can choose n_0 such that

$$\frac{\lambda_3}{\lambda_1} < \frac{1 + \frac{1}{k}}{m - 1 + \alpha - \delta} < \frac{1}{m - 1 + \alpha} + \frac{\varepsilon}{2} \tag{13}$$

for all $n \geq n_0$. Combining (12) and (13), we conclude that

$$\frac{\lambda_3}{\lambda_1} < \frac{1}{m - 1 + \alpha} + \frac{\varepsilon}{2} \tag{14}$$

provided that $m_\lambda \neq 0$ in (9) and $n \geq n_0$. Note that $\frac{\beta}{1-\beta} = \frac{1}{m-1+\alpha}$ and hence we have completed the proof of the lemma in the case when $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1)$.

Slightly modifying previous arguments, we get the proof of the inequality (14) for a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with three parts. The only difference is that non-identical polynomial g depends on at least λ_3 skewsymmetric sets of variables of order 3, but after evaluation, one of these variables can be replaced by $z_j^{(i)}$, and we get the inequality

$$\frac{\lambda_3 - 1}{\lambda_1} \leq \frac{\deg_b f}{\deg_a f}$$

instead of (12). Taking into account that $\lambda_1 \to \infty$ if $n \to \infty$ we get the same conclusion, and thus complete the proof. □

For the lower bound of codimensions of $A^\sharp$, we need the following results.

Let $A = A(m, w)$ be an algebra defined by an integer $m \geq 2$ and by an infinite word $w = w_1 w_2 \ldots$ in the alphabet $\{0, 1\}$. Then

$$z_1^{(1)} a^{i_1} b a^{i_2} b \cdots a^{i_r} b = z_1^{r+1} \tag{15}$$

if $i_1 = m - 1 + w_1, i_2 = m - 1 + w_2, \ldots, i_r = m - 1 + w_r$. Otherwise, the left-hand side of (15) is zero.

Lemma 9. *Let $\lambda = (j, \lambda_2, \lambda_3, 1)$ be a partition of $n = j + mr + w_1 + \cdots + w_r + 1$ with $j \geq \lambda_2 = (m-1)r + w_1 + \cdots + w_r$, $\lambda_3 = r$, or let $\lambda = (\lambda_1, j, \lambda_3, 1)$ be a partition of the same n with $\lambda_1 = (m-1)r + w_1 + \cdots + w_r > j \geq \lambda_3 = r$. Then $m_\lambda \neq 0$ in* (9).

Proof. Recall that, given S_n-module M, the multiplicity of χ_λ in the character $\chi(M)$ is nonzero if $e_{T_\lambda} M \neq 0$ for some Young tableaux T_λ of shape D_λ. The essential idempotent $e_{T_\lambda} \in FS_n$ is equal to

$$e_{T_\lambda} = \left(\sum_{\sigma \in R_{T_\lambda}} \sigma \right) \left(\sum_{\tau \in C_{T_\lambda}} (sgn\ \tau) \tau \right).$$

Here R_{T_λ} is the row stabilizer of T_λ, i.e., the subgroup of all $\sigma \in S_n$ permuting indices only inside rows of T_λ, and C_{T_λ} is the column stabilizer of T_λ.

First, let $\lambda_1 = j \geq \lambda_2$. Denote $n_0 = mr + w_1 + \cdots + w_r + 1$, and consider the Young tableaux T_λ of the following type. Into the boxes of the 1st row of D_λ we place $n_0 + 1, \ldots, n_0 + j$ from left to right. Into the third row, we insert $j_1 = i_1 + 2, \ldots, j_r = i_1 + \cdots + i_r + r + 1$. (In fact, $j_1, \ldots, j_r$ are the positions of b in the product (15)). Into the second row, we insert from left to right $j_1 - 1, \ldots, j_r - 1, i_{r+1}, \ldots, i_{\lambda_2}$ where $\{i_{r+1}, \ldots, i_{\lambda_2}\} = \{2, \ldots, n_0\} \setminus \{j_1 - 1, j_1, \ldots, j_r - 1, j_r\}$ and into the unique box of the 4th row we put 1.

Then

$$e_{T_\lambda}(x_1, \ldots, x_n) = Sym_1 Sym_2 Sym_3 Alt_1 \cdots Alt_{\lambda_2}(x_1, \ldots, x_n),$$

where we have as follows:

- Alt_1 is the alternation on $\{1, j_1 - 1, j_1, n_0 + 1\}$;
- Alt_k is the alternation on $\{j_k - 1, j_k, n_0 + k\}$ if $2 \leq k \leq r$;
- Alt_k is the alternation on $\{i_k, n_0 + k\}$ if $r < k \leq \lambda_2$;

- Sym_1 is the symmetrization on $\{n_0+1, \ldots, n_0+j\}$;
- Sym_2 is the symmetrization on $\{j_1, \ldots, j_r\}$;
- Sym_3 is the symmetrization on $\{2, \ldots, n\} \setminus \{j_1, \ldots, j_r\}$.

After an evaluation

$$\varphi(x_1) = z_1^{(1)}, \varphi(x_{n_0+1}) = \cdots = \varphi(x_{n_0+j}) = 1 \in A^\sharp, \varphi(x_{j_1}) = \cdots = \varphi(x_{j_r}) = b$$

and

$$\varphi(x_i) = a \ if \ i \neq 1, j_1, \ldots, j_r, n_0+1, \ldots, n_0+j,$$

we have

$$\varphi(e_{T_\lambda}(x_1 \cdots x_n)) = j!r!(n_0-r-1)!z_1^{(r+1)} \neq 0;$$

hence $m_\lambda \neq 0$ in (9).

Similarly, filling up the second row of T_λ by $n_0+1, \ldots, n_0+j$ in the case when $\lambda_1 = (m-1)+w_1+\cdots+w_r > j \geq \lambda_3 = r$, we prove that $e_{T_\lambda}(x_1 \cdots x_n)$ is not an identity of $A^\sharp$. □

Recall that, given $0 \leq \beta \leq 1$,

$$\Phi_0(\beta) = \Phi(\beta, 1-\beta) = \frac{1}{\beta^\beta (1-\beta)^{1-\beta}}.$$

Lemma 10. *Let $A = A(m, w)$ be an algebra defined for an integer $m \geq 2$ and a Sturmian or periodic word w with slope α. Let also $\beta = \frac{1}{m+\alpha}$. Then for any $\varepsilon > 0$ there exist a constant C, positive integers $n_1 < n_2 \ldots$, and partitions $\lambda^{(i)} \vdash n_i$ such that for some large enough i_0 the following properties hold*:

1) $|\Phi(\lambda^{(i)}) - \Phi_0(\beta) - 1| < \varepsilon$ for all $i \geq i_0$;
2) $n_{i+1} - n_i < C$ *for all* $i \geq i_0$; *and*
3) $m_\lambda^{(i)} \neq 0$ *in* $\chi_{n_i}(A^\sharp)$ *for all* $i \geq i_0$.

Proof. Note that $\beta < \frac{1}{2}$ since $\alpha > 0$. First, take an arbitrary $r \geq 1$, $n = mr + w_1 + \cdots + w_r$ and $\lambda = (\lambda_1, \lambda_2)$, where $\lambda_1 = (m-1)r + w_1 + \cdots + w_r$, $\lambda_2 = r$. We set

$$x_1 = \frac{\lambda_1}{n} = \frac{m-1+\frac{w_1+\cdots+w_r}{r}}{m+\frac{w_1+\cdots+w_r}{r}},$$

$$x_2 = \frac{\lambda_2}{n} = \frac{1}{m+\frac{w_1+\cdots+w_r}{r}}.$$

As it was mentioned in the proof of Lemma 8 (see also [3], Proposition 5.1 or [12], Section 2.2), there exists a constant C_1 such that

$$\left|\frac{w_1+\cdots+w_r}{r} - \alpha\right| < \frac{C_1}{r}. \tag{16}$$

Hence for any $\varepsilon_1 > 0$, we can find r_0 such that

$$|\Phi(\lambda) - \Phi_0(\beta)| < \varepsilon_1 \tag{17}$$

for all $r \geq r_0$, since $\Phi(z_1, z_2)$ is a continuous function and $(x_1, x_2) \to (1-\beta, \beta)$ when $r \to \infty$.

Now using Lemmas 2 and 3, given $\varepsilon_2 > 0$, we insert one extra row into D_λ, that is, we construct a partition $\mu = (\mu_1, \mu_2, \mu_3)$ of $n_0 = nk$ such that

$$|\Phi(\lambda) - \Phi(\mu) - 1| < \varepsilon_2. \tag{18}$$

We have three options. Either μ_1 is a new row, that is, $(\mu_2, \mu_3) = (q\lambda_1, q\lambda_2)$; or μ_2 is a new row, that is, $(\mu_1, \mu_3) = (q\lambda_1, q\lambda_2)$; or μ_3 is a new row, that is, $(\mu_1, \mu_2) = (q\lambda_1, q\lambda_2)$.

First, we exclude the third case. Suppose that $(\mu_1, \mu_2) = (q\lambda_1, q\lambda_2)$. Recall that by Lemma 2, the maximal value of $\Phi(tz_1, tz_2, 1-t)$ is achieved if

$$t = \frac{\Phi(z_1, z_2)}{1 + \Phi(z_1, z_2)}.$$

Since $\Phi(z_1, z_2) < 2$ if $\beta < \frac{1}{2}$, we obtain that $1 - t > \frac{1}{3}$. For Lemma 3, this means that the new row of D_μ cannot be the third row; that is, the case $(\mu_1, \mu_2) = (q\lambda_1, q\lambda_2)$ is impossible.

Now let $(\mu_2, \mu_3) = (q\lambda_1, q\lambda_2)$. We exchange μ to μ' in the following way. We set $\mu_2' = qr(m-1) + w_1 + \cdots + w_{qr}$ and take $\mu' = (\mu_1, \mu_2', \mu_3)$. Then $\mu' \vdash n'$ where

$$n' - n_0 = \mu_2' - \mu_2 = w_1 + \cdots + w_{qr} - q(w_1 + \cdots + w_r).$$

Using again inequality (16), we get

$$|n' - n_0| < C_1(q+1). \tag{19}$$

Inequality (19) also shows that $\mu_1 \geq \mu_2' \geq \mu_3$ if n is sufficiently large and our construction of partition μ is correct.

Clearly, $|\Phi(\mu) - \Phi(\mu')| \to 0$ if $n \to \infty$ and

$$|\Phi(\mu) - \Phi(\mu')| < \varepsilon_3 \tag{20}$$

for any fixed $\varepsilon_3 > 0$, for all sufficiently large r (and n). Starting from this sufficiently large r, we denote $n_r = n' + 1$ and take $\lambda^{(r)} \vdash n_r$, $\lambda^{(r)} = (\mu_1, \mu', \mu_3, 1)$. All preceding $n_1, \ldots, n_{r-1}$ and $\lambda^{(1)}, \ldots, \lambda^{(r-1)}$, we choose in an arbitrary way.

Since $\mu_3 = qr$, by Lemma 9 the multiplicity of the irreducible character $\lambda^{(r)}$ in $\chi_{n_r}(A^\sharp)$ is not equal to zero and $|n_r - kn| < C_2 = C_1(q+1) + 1$ by (19), since $n_0 = nk$. It is not difficult to see that in this case

$$|\Phi(\mu') - \Phi(\lambda^{(r)})| < \varepsilon_4 \tag{21}$$

for any fixed $\varepsilon_4 > 0$ if r (and the corresponding n) is sufficiently large. Combining (17), (18), (20), and (21), we see that $\lambda^{(r)}$ satisfies conditions (1) and (3) of the lemma.

Finally, consider the difference between n_r and n_{r+1}, provided that all $n_{r+1}, n_{r+2}, \dots$ are constructed by the same procedure. That is, we take

$$\bar{n} = m(r+1) + w_1 + \cdots + w_{r+1} + 1$$

and obtain n_{r+1}, satisfying the same condition

$$|n_{r+1} - k\bar{n}| < C_2.$$

On the other hand, $\bar{n} - kn = k(m + w_{r+1}) \leq k(m+1)$ and $|kn - n_r| < C_2$. Hence we have

$$|n_{r+1} - n_r| < C = 2C_2 + k(m+1).$$

This latter inequality completes the proof of the lemma if $(\mu_2, \mu_3) = (q\lambda_1, q\lambda_2)$. Arguments in the case $(\mu_1, \mu_3) = (q\lambda_1, q\lambda_2)$ are the same. □

5. THE MAIN RESULT

Now we are ready to prove the main result of the paper.

Theorem 1. *Let $w = w_1 w_2 \dots$ be a Sturmian or periodic word, and let $A = A(m, w)$, $m \geq 2$, be an algebra defined by m and w in (5)–(8). If $A^\sharp$ is the algebra obtained from A by adojining an external unit, then PI-exponent of $A^\sharp$ exists and*

$$exp(A^\sharp) = 1 + exp(A).$$

Proof. Let $\alpha = \pi(w)$ be the slope of w, and let $\beta = \frac{1}{m+\alpha}$. Recall that $exp(A) = \Phi_0(\beta)$, where

$$\Phi_0(\beta) = \frac{1}{\beta^\beta (1-\beta)^{1-\beta}}$$

([3]). First, we prove that for any $\delta > 0$, there exists N such that

$$\Phi(\lambda) < \Phi_0(\beta) + 1 + \delta \tag{22}$$

as soon as λ is a partition of $n \geq N$, with $m_\lambda \neq 0$ in $\chi_n(A^\sharp)$.

By Lemma 8, for any $\varepsilon > 0$, there exists n_0 such that

$$\frac{\lambda_3}{\lambda_1} < \frac{\beta}{1-\beta} + \varepsilon \tag{23}$$

if $n \geq n_0$, $\lambda \vdash n$ and $m_\lambda \neq 0$. If $\lambda = (n)$ or $\lambda = (\lambda_1, \lambda_2)$, then by the hook formula for dimensions of irreducible S_n-representations it follows that $\deg \chi_\lambda \leq 2^n$. Then by Lemma 1,

$$\Phi(\lambda) \leq 2\sqrt[n]{n^6}$$

and (22) holds for all sufficiently large n, since $1 \leq \Phi_0(\beta) \leq 2$.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Denote $\mu = (\lambda_1, \lambda_3) \vdash n'$, where $n' = n - \lambda_2$. If $x_1 = \frac{\lambda_1}{n'}, x_2 = \frac{\lambda_3}{n'}$, then

$$\Phi(\mu) = \Phi(x_1, x_2) = \Phi_0(x_2)$$

and

$$x_2 \leq \varphi(\varepsilon) = \frac{\beta + (1-\beta)\varepsilon}{1 + (1-\beta)\varepsilon}$$

which follows from (23). Since

$$\lim_{n\to\infty} \varphi(\varepsilon) = \beta$$

and Φ_0 is continuous, there exist N and ε such that $\Phi(\mu) < \Phi_0(\beta) + \delta$ for all $n > N$. Then by Lemma 2,

$$\Phi(\lambda) \leq \Phi(\mu) + 1 < \Phi_0(\beta) + 1 + \delta.$$

Now consider the case $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1)$. Excluding the second row of diagram D_λ, we get a partition $\mu = (\mu_1, \mu_2, 1) = (\lambda_1, \lambda_3, 1)$ of $n' = n - \lambda_2$ with

$$\frac{\mu_2}{\mu_1} < \frac{\beta}{1-\beta} + \varepsilon.$$

Consider also partition $\mu' = (\mu_1, \mu_2)$ of $n' - 1$. As before, given $\delta > 0$, one can find n_0 such that

$$\Phi(\mu') < \Phi_0(\beta) + \frac{\delta}{2}$$

provided that $n \geq n_0$.

Since Φ is continuous, for all sufficiently large n (and n'), we have

$$\Phi(\mu) < \Phi_0(\beta) + \delta.$$

Applying again Lemma 2, we get (22). It now follows from (22) and Lemmas 1 and 7 that

$$c_n(A^\sharp) = \sum_{\lambda\vdash n} m_\lambda \deg \chi_\lambda \leq (\Phi_0(\beta) + 1 + \delta)^n l_n(A^\sharp) \leq 4(m+1)(n+1)^{13}(\Phi_0(\beta) + 1 + \delta)^n.$$

Hence

$$\overline{exp}(A^\sharp) = \limsup_{n\to\infty} \sqrt[n]{c_n(A^\sharp)} \leq \Phi_0(\beta) + \delta + 1$$

for any $\delta > 0$, that is,

$$\overline{exp}(A^\sharp) \leq \Phi_0(\beta) + 1 = exp(A) + 1. \tag{24}$$

Now we find a lower bound for codimensions. Since

$$c_n(A^\sharp) \geq \deg \chi_\lambda \geq \frac{\Phi(\lambda)^n}{n^{20}},$$

by Lemma 1 if $m_\lambda \neq 0$ in $\chi_n(A^\sharp)$, then by Lemma 10 for any $\varepsilon > 0$ there exists a sequence $n_1 < n_2 < \ldots$ such that

$$c_{n_i}(A^\sharp) \geq \frac{1}{n_i^{20}}(\Phi_0(\beta) + 1 - \varepsilon)^{n_i}, \ i = 1, 2, \ldots$$

and $n_{i+1} - n_i < C = const$, for all $i \geq 1$. Note that the sequence $\{c_n(R)\}$ is nondecreasing for any unital algebra R. Then

$$\underline{exp}(A^\sharp) = \liminf_{n\to\infty} \sqrt[n]{c_n(A^\sharp)} \geq \Phi_0(\beta) + 1. \tag{25}$$

Now (24) and (25) complete the proof of the theorem. □

Corollary 1. *For any real numbers $\gamma \in [2, 3]$ there exists an algebra A with* 1 *such that $exp(A) = \gamma$.*

As it was mentioned in the preliminaries, PI-exponents of finite dimensional algebras form a dense subset of the interval $[1, 2]$. Hence we get the following corollary.

Corollary 2. *For any real numbers $\beta < \gamma \in [2, 3]$, there exists a finite dimensional algebra B with* 1 *such that $\beta \leq exp(B) \leq \gamma$. In particular, PI-exponents of finite dimensional unital algebras form a dense subset of the interval* $[2, 3]$.

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