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Graded codimensions of Lie superalgebra $b(2)$ ☆

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A R T I C L E   I N F O

A B S T R A C T

We study asymptotic behaviour of graded codimensions of Lie superalgebra $b(2)$. We prove that graded PI-exponent exists and is equal to $3 + 2\sqrt{3}$.

1. Introduction

We consider finite dimensional Lie superalgebras over a field of characteristic zero and study their $\mathbb{Z}_2$-graded identities. We pay main attention to numerical invariants of identities, in particular, to graded codimensions and their asymptotic behaviour.

It is well known that in case $\dim L < \infty$ both graded and ordinary codimensions are exponentially bounded [2]. One of the more important questions of the theory of numerical invariants of polynomial identities is: does a (graded) PI-exponent exist?

There are many papers where the existence of PI-exponent is proved for different classes of algebras. For example, if A is an associative PI-algebra or a finite dimensional Lie, Jordan or alternative algebra, then its PI-exponent exists and is a non-negative integer (see [4–6,11]). The existence of PI-exponent for any finite dimensional simple algebra was proved in [8]. It is not difficult to show that if PI-exponent of A exists then it is less than or equal to d provided that $d = \dim A < \infty$ (see for example [2]). In many important classes of algebras over an algebraically closed field (associative, Lie, Jordan, alternative) the equality $exp(A) = \dim A$ is equivalent to simplicity of A [4,5,11]. Recently [8] it was shown that $exp(L) < \dim L$ for any finite dimensional simple Lie superalgebra L of the type $b(t)$, $t \geq 3$ (in the notation of [9]). The existence of PI-exponent and similar inequality $exp(L) < \dim L$ for $b(2)$ was also proved in [8] although it is not simple superalgebra.

Graded codimensions of Lie superalgebras were studied much less. In particular, it is still unknown whether $exp^{gr}(L)$ exists even when L is a finite dimensional simple Lie superalgebra. In recent paper [10] it was proved that an upper graded PI-exponent of Lie superalgebra $b(t), t \geq 2$, is less than or equal to $t^2 - 1 + t\sqrt{t^2 - 1}$. In particular, this gives an upper bound for ordinary PI-exponent of $b(t)$. In the present paper we prove the existence of graded PI-exponent of $b(2)$. We also prove that $exp^{gr}(b(2)) = 3 + 2\sqrt{3}$. Note that it was recently announced that an ordinary PI-exponent does not exist in general non-associative case (see [12]).

2. Main constructions and definitions

Let L be a Lie superalgebra over a field F of characteristic zero, that is $L = L_0 \oplus L_1$ is a non-associative $\mathbb{Z}_2$-graded algebra satisfying two identical relations

$$xy + (-1)^{|x||y|} yx = 0,$$
$$x(yz) = (xy)z + (-1)^{|x||y|} y(xz) = 0$$

where $x, y, z \in L_0 \cup L_1$ and $|x| = 0$ if $x \in L_0$ while $|x| = 1$ if $x \in L_1$.

Elements from $L_0 \cup L_1$ are called *homogeneous* and we say that x is *even* if $x \in L_0$ or x is *odd* if $x \in L_1$.

Denote by $\mathcal{L}(X, Y)$ a free Lie superalgebra with infinite sets of even generators X and odd generators Y. A polynomial $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathcal{L}(X, Y)$ is said to be a *graded identity* of Lie superalgebra $L = L_0 \oplus L_1$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ whenever $a_1, \ldots, a_m \in L_0$, $b_1, \ldots, b_n \in L_1$.

Given positive integers $0 \leq k \leq n$, denote by $P_{k,n-k}$ the subspace of all multilinear polynomials $f = f(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in \mathcal{L}(X, Y)$ of degree k on even variables and of degree $n - k$ on odd variables. Denote by $Id^{gr}(L)$ the ideal of $\mathcal{L}(X, Y)$ of all graded

identities of L. Then $P_{k,n-k} \cap Id^{gr}(L)$ is the subspace of all multilinear graded identities of L of total degree n depending on k even variables and $n-k$ odd variables. Also denote by $P_{k,n-k}(L)$ the quotient

$$P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap Id^{gr}(L)}.$$

Then the graded $(k, n-k)$-codimension of L is

$$c_{k,n-k}(L) = \dim P_{k,n-k}(L)$$

and the total graded codimension of L is

$$c_n^{gr}(L) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(L).$$

It is known (see [2]) that if $\dim L < \infty$ then the sequence $\{c_n^{gr}(L)\}_{n \geq 1}$ is exponentially bounded and one can consider the related sequence $\sqrt[n]{c_n^{gr}(L)}$. The latter sequence has the lower and upper limits

$$\underline{exp}^{gr}(L) = \liminf_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}, \qquad \overline{exp}^{gr}(L) = \limsup_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}$$

called the *lower* and *upper graded* PI-*exponents* of L, respectively. If an ordinary limit exists, it is called an (*ordinary*) *graded* PI-*exponent* of L,

$$exp^{gr}(L) = \lim_{n \to \infty} \sqrt[n]{c_n^{gr}(L)}.$$

Symmetric groups and their representations play an important role in the theory of codimensions. One can find all details concerning application of representation theory of symmetric groups to study of polynomial identities in [1,3,7]. In case of graded identities one can consider $(S_k \times S_{n-k})$-action on multilinear graded polynomials. Namely, the subspace $P_{k,n-k} \subseteq \mathcal{L}(X, Y)$ has a natural structure of $(S_k \times S_{n-k})$-module where S_k acts on even variables $x_1, \ldots, x_k$ while S_{n-k} acts on odd variables $y_1, \ldots, y_{n-k}$. Clearly, $P_{k,n-k} \cap Id^{gr}(L)$ is the submodule under this action and we get an induced $S_k \times S_{n-k}$-action on $P_{k,n-k}(L)$. If G is a subgroup of $S_k \times S_{n-k}$ then G also acts naturally on $P_{k,n-k}(L)$. In particular,

$$c_{k,n-k}(L) \geq \dim M \tag{1}$$

for any subgroup $G \subseteq S_k \times S_{n-k}$ and for any G-submodule M of $P_{k,n-k}(L)$. We will use the relation (1) for getting a lower bound of $c_{k,n-k}(L)$.

3. Graded PI-exponent of $b(2)$

Recall the construction of Lie superalgebra $L = L_0 \oplus L_1 = b(2)$. Even component L_0 consists of all 4×4 matrices of the type

$$L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \,\middle|\, A \in M_2(F),\ \text{tr}(A) = 0 \right\},$$

where A is 2×2 traceless matrix and $t : A \to A^t$ is the usual transpose involution.

Odd component L_1 consists of 4×4 matrices

$$L_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \,\middle|\, \text{where } B^t = B,\ C^t = -C,\ B,C \in M_2(F) \right\}.$$

Also denote

$$L_1^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \,\middle|\, C^t = -C \in M_2(F) \right\},$$

$$L_1^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \,\middle|\, B^t = B \in M_2(F) \right\}.$$

Then $\dim L_1^- = 1$, $\dim L_1^+ = 3$.

As a vector space L is embedded into $M_4(F)$. Using ordinary associative matrix multiplication we can define super-Lie product on L as

$$\{x, y\} = xy - (-1)^{|x||y|} yx$$

for any homogeneous $x, y \in L_0 \cup L_1$ where $|x| = 0$ if $x \in L_0$ while $|x| = 1$, if $x \in L_1$.

Note that L_0 is a Lie algebra isomorphic to $sl_2(F)$. We will identify L_0 with $sl_2(F)$ and use the standard basis of $sl_2(F)$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, we will not use associative multiplication. This will allow us to omit super-Lie brackets, i.e. to write ab instead of $\{a, b\}$. We will also use the notation $ab \cdots c$ for the left-normed product $\{\{\ldots \{a, b\}, \ldots\}, c\}$.

We will also use the following agreement for denoting alternating sets of variables. If $f = f(x_1, \ldots, x_k, y_1, \ldots, y_n)$ is a multilinear polynomial and we apply to f the operator of alternation on variables $x_1, \ldots, x_k$, then we will write the same symbol (bar, double bar, tilde, double tilde, etc.) over the variables $x_1, \ldots, x_k$, that is

$$f(\bar{x}_1, \ldots, \bar{x}_k, y_1, \ldots y_n) = \sum_{\sigma \in S_k} (\text{sgn}\,\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, y_1, \ldots, y_n).$$

For example $\bar{x}a\bar{y} = xay - yax$, or

$$\bar{\bar{x}}_1 a\bar{\bar{x}}_2 b\bar{\bar{x}}_3 = \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma) x_{\sigma(1)} a x_{\sigma(2)} b x_{\sigma(3)}.$$

We will also use this notation for non-multilinear polynomials with repeating variables as follows

$$\bar{x}_1 \bar{x}_2 a\bar{\bar{x}}_1 \bar{\bar{x}}_2 = x_1 x_2 a\bar{\bar{x}}_1 \bar{\bar{x}}_2 - x_2 x_1 a_1 \bar{\bar{x}}_1 \bar{\bar{x}}_2$$

$$= x_1 x_2 a x_1 x_2 - x_1 x_2 a x_2 x_1 - x_2 x_1 a x_1 x_2 + x_2 x_1 a x_2 x_1.$$

Following this agreement we consider an alternating expression $h\bar{e}\bar{f}\,\bar{h}$ in the Lie algebra $L_0 = sl_2(F)$. Since $he = 2e$, $hf = -2f$, $ef = h$, it easily follows that

$$h\bar{e}\bar{f}\bar{h} = 8h$$

and

$$h \underbrace{\bar{e}\bar{f}\bar{h}\bar{\bar{e}}\bar{\bar{f}}\bar{\bar{h}} \cdots \tilde{e}\tilde{f}\tilde{h}}_{t \text{ alternating triples}} = 8^t h. \tag{2}$$

Consider a multilinear polynomial

$$g = g\big(x_0, x_1^1, x_2^1, x_3^1, \ldots, x_1^t, x_2^t, x_3^t\big) = Alt_1 Alt_2 \ldots Alt_t \big(x_0 x_1^1 \cdots x_3^1\big)$$

where Alt_j is the operator of alternation on x_1^j, x_2^j, x_3^j, $1 \le j \le t$, that is

$$g = x_0 \bar{x}_1^1 \bar{x}_2^1 \bar{x}_3^1 \cdots \tilde{x}_1^t \tilde{x}_2^t \tilde{x}_3^t.$$

The evaluation $\varphi : x_1^1, \ldots, x_1^t \to e, x_2^1, \ldots, x_2^t \to f, x_0, x_3^1, \ldots, x_3^t \to h$ gives us

$$\varphi(g) = 8^t h \tag{3}$$

in L_0 by (2). Moreover, if we denote the symmetrization on variables $x_i^1, \ldots, x_i^t$, $i = 1, 2, 3$ by Sym_i, then it follows from (3) and the definition of φ that

$$\varphi\big(Sym_1 Sym_2 Sym_3(g)\big) = (t!)^3 8^t h \tag{4}$$

in L_0 by virtue of (2). An element

$$g' = Sym_1 Sym_2 Sym_3(g) \tag{5}$$

with the fixed x_0 generates in $P_{3t+1,0}$ an irreducible S_{3t}-submodule with the character χ_λ, $\lambda = (t, t, t)$ where the permutation group S_{3t} acts on $x_1^1, x_2^1, x_3^1, \ldots, x_1^t, x_2^t, x_3^t$.

Given a partition $\mu = (\mu_1, \ldots, \mu_d) \vdash n$, we define the function

$$\Phi(\mu) = \frac{1}{z_1^{z_1} \cdots z_d^{z_d}},$$

where

$$z_1 = \frac{\mu_1}{n}, \quad \ldots, \quad z_d = \frac{\mu_d}{n}.$$

The value of $\Phi(\mu)$ is closely connected with $\deg \chi(\mu)$.

Lemma 1. *(See [8, Lemma 1].) Let $n \geq 100$. Then*

$$\frac{\Phi(\mu)^n}{n^{d^2+d}} \leq d_\mu \leq n\Phi(\mu)^n. \qquad \square$$

In particular, if $m = 3t$ and $\mu = (t, t, t)$ then

$$\deg \chi_\mu \geq m^{-12} 3^m. \tag{6}$$

In the next step we will construct an irreducible S_{6k}-submodule in $P_{1,6k+1} \not\subset Id^{gr}(L)$ where S_{6k} acts on some $6k$ odd variables. Denote

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in L_1^-.$$

Then $L_1^- = \mathrm{Span}\langle d \rangle$. It is not difficult to check that $L_1^+ L_1^- = L_0$. Hence there exist $a, b, c \in L_1^+$ such that

$$ad = e, \qquad bd = f, \qquad cd = h$$

where $\{e, f, h\}$ is the fixed basis of L_0. It follows that

$$h(\bar{a}d)(\bar{b}d)(\bar{c}d) = 8h$$

in L and

$$h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}d)(\bar{\bar{b}}d)(\bar{\bar{c}}d) = h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}d)(\bar{\bar{b}}d)(\bar{\bar{c}}d) = 64h.$$

Repeating this procedure and using (3) we obtain a multialternating expression

$$\begin{aligned}
H &= h(\bar{a}d)(\bar{b}d)(\bar{c}d)(\bar{\bar{a}}d)(\bar{\bar{b}}d)(\bar{\bar{c}}d)(\tilde{\bar{a}}d)(\tilde{b}d)(\tilde{c}d) \cdots (\tilde{\tilde{a}}\hat{d})(\tilde{\tilde{b}}d)(\tilde{\tilde{c}}d)(a\tilde{\tilde{d}})(bd)(cd) \\
&= 8^{k+1}h
\end{aligned} \tag{7}$$

depending on one h, $k+1$ elements a, b, c and $3(k+1)$ elements d. The element H on the left-hand side of (7) contains k alternating sets $\{a, b, c, d\}$. The first set consists of 1st a, 1st b, 1st c and 4th d. The second set consists of 2nd a, 2nd b, 2nd c and 7th d, and so on. The element H also contains $2(k+1)+1$ non-alternating entries d and four extra factors h, a, b, c out of alternating sets. This H is a value of the following multilinear polynomial: denote by

$$
\begin{aligned}
w &= w\big(x_0, y_1^1, y_2^1, y_3^1, z_1^1, z_2^1, z_3^1, \ldots, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}\big) \\
&= Alt_1 \ldots Alt_k \big(x_0 \big(y_1^1 z_1^1\big)\big(y_2^1 z_2^1\big)\big(y_3^1 z_3^1\big)\big(y_1^2 z_1^2\big)\big(y_2^2 z_2^2\big)\big(y_3^2 z_3^2\big)\cdots \\
&\quad \times \big(y_1^{k+1} z_1^{k+1}\big)\big(y_2^{k+1} z_2^{k+1}\big)\big(y_3^{k+1} z_3^{k+1}\big)\big)
\end{aligned}
$$

where x_0 is an even variable, all $y_\beta^\alpha, z_\beta^\alpha$ are odd and Alt_j is the operator of alternation on $y_1^j, y_2^j, y_3^j, z_1^{j+1}$, $j = 1, \ldots, k$. Then $\varphi(w) = H$ where φ is an evaluation of the form

$$
\varphi(x_0) = h, \qquad \varphi\big(y_1^j\big) = a, \qquad \varphi\big(y_2^j\big) = b, \qquad \varphi\big(y_3^j\big) = c, \qquad \varphi\big(z_i^j\big) = d,
$$
$$
j = 1, \ldots, k+1, \ i = 1, 2, 3.
$$

Also denote by Sym_1, Sym_2, Sym_3 the symmetrization on the sets $\{y_1^1, \ldots, y_1^k\}$, $\{y_2^1, \ldots, y_2^k\}$, $\{y_3^1, \ldots, y_3^k\}$, respectively, and by Sym_4 the symmetrization on $\{z_1^2, z_2^2, z_3^2, \ldots, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}\}$. If

$$
w' = Sym_1 Sym_2 Sym_3 Sym_4(w)
$$

then

$$
\varphi\big(w'\big) = (3k)!(k!)^3 \varphi(w) = (3k)!(k!)^3 8^{k+1} h
$$

in L. In particular, w' is not an identity of L.

Now let the permutation group S_{6k} act on the set $\{y_i^j, z_i^{j+1} \mid 1 \leq j \leq k,\ i = 1, 2, 3\}$ and let $x_0, z_1^1, z_2^1, z_3^1, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}$ be fixed. Then w' generates an irreducible S_{6k}-submodule in $P_{1,6(k+1)}$ corresponding to the partition $\mu = (3k, k, k, k) \vdash r = 6k$ with

$$
\Phi(\mu) = 2\sqrt{3}.
$$

Hence

$$
\deg \chi_\mu \geq r^{-20} \big(2\sqrt{3}\big)^r \tag{8}
$$

by Lemma 1.

Now let

$$u = w'\left(g', y_1^1, y_2^1, y_3^1, \cdots, z_1^{k+1}, z_2^{k+1}, z_3^{k+1}\right)$$

where g' is taken from (5). If $m = 3t, r = 6k$, group S_m acts on even variables x_i^j from g' whereas S_r acts on odd variables $\{y_i^j, z_i^j\}$ (except $z_1^1, z_2^1, z_3^1, y_1^{k+1}, y_2^{k+1}, y_3^{k+1}$) then u is not a graded identity of L as follows from (4), (6) and (7) and it generates an irreducible $S_m \times S_r$-submodule M in $P_{m+1,r+6}$ with the character $\chi_{\lambda,\mu}$ where $\lambda = (t, t, t)$, $\mu = (3k, k, k, k)$. Hence by (6), (8)

$$\dim M = \deg \chi_\lambda \deg \chi_\mu \geq \frac{1}{(m+r)^{32}} 3^m (2\sqrt{3})^r$$

and then by (1)

$$c_{m+1,r+6}(L) = \dim P_{m+1,r+6}(L) \geq \dim M \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}. \tag{9}$$

The inequality (9) means that we have proved the following lemma.

Lemma 2. *Let $t, r \geq 1$ be arbitrary integers and $m = 3t, r = 6k$. Then*

$$c_{m+1,r+6}(L) \geq \frac{3^m (2\sqrt{3})^r}{(m+r)^{32}}. \qquad \square$$

Now we will find a lower bound for nth graded codimension of L for the special case of n.

Lemma 3. *Let $n - 7$ be a multiple of* 6. *Then*

$$c_n^{gr}(L) \geq \frac{1}{3^{18} n^{38}} (3 + 2\sqrt{3})^n.$$

Proof. Let $q = n - 7$. Then applying Lemma 2 we obtain

$$c_n^{gr}(L) = \sum_i \binom{n}{i} c_{i,n-i}(L) \geq \sum_{j=0}^{q/6} \binom{n}{6j+1} c_{1+6j,q+6-6j}(L)$$

$$\geq \frac{1}{q^{32}} \sum_{j=0}^{q/6} \binom{n}{6j+1} 3^{6j} (2\sqrt{3})^{q-6j} \geq \frac{A}{n^{33}}$$

where

$$A = \sum_{j=0}^{q/6} \binom{n}{6j} 3^{6j} (2\sqrt{3})^{q-6j}$$

since

$$\binom{n}{i} \le n \binom{n}{i+1}.$$

Now, since given $0 \le j < \frac{q}{6}$, we have

$$\binom{q}{6j+i} 3^{6j+i}(2\sqrt{3})^{q-6j-i} < (3q)^5 \binom{q}{6j} 3^{6j}(2\sqrt{3})^{q-6j}$$

for all $1 \le i \le 5$. It follows that

$$A > \frac{6}{(3n)^5} \sum_{i=0}^{q} \binom{q}{i} 3^i (2\sqrt{3})^{q-i} = \frac{6}{(3n)^5}(3+2\sqrt{3})^q > \frac{2}{3^{18}n^5}(3+2\sqrt{3})^n.$$

Hence

$$c_n^{gr}(L) > \frac{(3+2\sqrt{3})^n}{3^{18}n^{38}}. \qquad \square$$

Now we consider the case when $n-7$ is not a multiple of 6, that is $n-7 \equiv i \pmod 6$ with $1 \le i \le 5$.

Lemma 4. *Let $t,r,i \ge 1$ be arbitrary integers, $m = 3t$, $r = 6k$ and $i \le 5$. Then*

$$c_{m+1+i,r+6}(L) \ge \frac{3^m(2\sqrt{3})^r}{(m+r)^{32}}.$$

Proof. The proof is similar to the proof of Lemma 2. We only need to change the polynomial $u = w(g', y_1^1, \ldots, z_3^{k+1})$ to $u' = ux_1 \cdots x_i$ and consider an evaluation φ with the same values on x_0, x_j^i, y_j^i, z_j^i as in Lemma 2 and $\varphi(x_1) = e$, $\varphi(x_2) = \ldots = \varphi(x_i) = h$ if $i \ge 2$. Then $\varphi(u') = \pm 2^i \varphi(u) \ne 0$ and we are done. $\square$

Slightly modifying arguments of Lemma 3 and using Lemma 4, we get the following result for arbitrary $1 \le i \le 5$.

Lemma 5. *Let $n-7 \equiv i \pmod 6$, $1 \le i \le 5$. Then*

$$c_n^{gr}(L) \ge \frac{1}{3^{18}n^{43}}(3+2\sqrt{3})^n. \qquad \square$$

Now we are ready to prove the main result of the paper.

Theorem 1. *Graded PI-exponent of Lie superalgebra $L = b(2)$ exists and is equal to*

$$exp^{gr}(L) = 3 + 2\sqrt{3}.$$

Proof. By Lemmas 3 and 5

$$\underline{exp}^{gr}(L) = \liminf_{n \to \infty} \sqrt[n]{c_n^{gr}(L)} \geq 3 + 2\sqrt{3}.$$

On the other hand, $\overline{exp}^{gr}(b(t)) \leq t^2 - 1 + t\sqrt{t^2 - 1}$ for all $t \geq 2$ as proved in [10]. Hence the limit

$$exp^{gr}\big(b(2)\big) = \lim_{n \to \infty} \sqrt[n]{c_n^{gr}\big(b(2)\big)}$$

exists and

$$exp^{gr}(L) = 3 + 2\sqrt{3}. \qquad \square$$

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