ISSN 0001-4346, Mathematical Notes, 2014, Vol. 95, No. 4, pp. 487–499. © Pleiades Publishing, Ltd., 2014.
Original Russian Text © M. V. Zaitsev, D. Repovš, 2014, published in Matematicheskie Zametki, 2014, Vol. 95, No. 4, pp. 538–553.

A Four-Dimensional Simple Algebra with Fractional PI-Exponent

M. V. Zaitsev[1*] **and D. Repovš**[2**]

[1] *Moscow State University, Moscow, Russia*
[2] *University of Ljubljana, Ljubljana, Slovenia*

Received April 2, 2012; in final form, June 3, 2013

Abstract—Numerical characteristics of identities of finite-dimensional nonassociative algebras are studied. The main result is the construction of a four-dimensional simple unitary algebra with fractional PI-exponent strictly less than its dimension.

DOI: 10.1134/S0001434614030213

Keywords: *finite-dimensional nonassociative algebra, identities of a nonassociative algebra, PI-exponent, fractional PI-exponent.*

1. INTRODUCTION

This paper studies identities of nonassociative algebras over a field F of characteristic zero. All necessary information about identities in algebras can be found in the books [1]–[3].

Each algebra A can be associated with an integer sequence $\{c_n(A)\}$, $n = 1, 2, \dots,$ called the *sequence of codimensions* (all necessary definitions are given below). If this sequence grows exponentially, as, e.g., in the finite-dimensional case, there arises the question of the existence of the limit $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$. It has been proved that such a limit exists and is integer for all associative PI-algebras [4], [5], for finite-dimensional Lie algebras [6], [7], for finite-dimensional Jordan and alternative algebras [8], [9], and for a whole series of other algebras. In the infinite-dimensional case, this limit, called the PI-*exponent*, may take both fractional and integer values even in the class of Lie algebras (see [10]–[12]); in the class of Lie superalgebras, finite-dimensional algebras with fractional exponent have recently been found [13].

In the general case, the PI-exponent of a finite-dimensional algebra does not exceed the dimension of this algebra and can be arbitrarily close to 1 [14], [15]. However, for two-dimensional algebras, the PI-exponent either takes one of the values 2 and 1 (in this case, $c_n(A) \le n + 1$) or vanishes (in this case, $c_n(A) = 0$ for all sufficiently large n) [16]. For three-dimensional algebras, the question of whether the exponent is integer remains open; however, it is known that either $c_n(A) \ge 2^n$ asymptotically, or this sequence is polynomially bounded. However, for a unitary three-dimensional algebra, the PI-exponent always exists and is integer [17]. Until recently, the least known dimension of an algebra with fractional exponent was 5 [18], and that of a Lie superalgebra was 7 [13]. In the abstract [19], the existence of a four-dimensional commutative algebra with fractional exponent was announced.

The question of the existence and the value of the PI-exponent of a simple finite-dimensional algebra over an algebraically closed field occupies a special place. In most of the studied classes of algebras (associative [20], Lie [6], Jordan, alternative, and some other classes [8], [9]), the PI-exponent of an algebra is equal to the dimension of this algebra. The first examples of simple algebras for which the PI-exponent is strictly less than dimension were given in [13]. The least dimension of an algebra among those presented in [13] is 17. But the question of whether their PI-exponents are integer remains open.

In this paper, we construct an example of a four-dimensional simple unitary algebra with fractional PI-exponent (Theorem 1).

*E-mail: zaicevmv@mail.ru
**E-mail: dusan.repovs@guest.arnes.si

2. BASIC NOTIONS AND DEFINITIONS

Throughout the paper, F denotes a field of characteristic zero, and all algebras are considered over F. We follow the convention of omitting parentheses from left-normed products, i.e., write abc instead of $(ab)c$.

By $F\{X\}$ we denote a free nonassociative algebra over F with a countable set X of generators. Recall that, given an algebra A over F, a nonassociative polynomial $f = f(x_1, \dots, x_n)$ from $F\{X\}$ is called an *identity of* A if

$$f(a_1, \dots, a_n) = 0 \qquad \text{for any} \quad a_1, \dots, a_n \in A.$$

All identities of A form an ideal in $F\{X\}$, which we denote by $\mathrm{Id}(A)$.

Let $P_n = P_n(x_1, \dots, x_n)$ denote the subspace of $F\{X\}$ consisting of all multilinear polynomials in $x_1, \dots, x_n$. Then $P_n \cap \mathrm{Id}(A)$ is the subspace of all multilinear identities of the algebra A in n variables.

Recall that the *nth codimension of identities* of an algebra A (or simply the *nth codimension of* A) is defined as

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}\,.$$

In a variety of cases, the sequence $\{c_n(A)\}$ grows no faster than the exponential of n, i.e.,

$$c_n(A) < a^n$$

for some $a > 1$. In these cases, the sequence of codimensions satisfies the conditions

$$0 \le \sqrt[n]{c_n(A)} \le a,$$

which suggests the following definition.

Definition. The *lower* PI-*exponent* of an algebra A is the lower limit

$$\underline{\exp}(A) = \varliminf_{n\to\infty} \sqrt[n]{c_n(A)}\,.$$

The upper limit

$$\overline{\exp}(A) = \varlimsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

is called the *upper* PI-*exponent* of A. If $\underline{\exp}(A)$ and $\overline{\exp}(A)$ are equal, i.e., the sequence $\{\sqrt[n]{c_n(A)}\}$ has an ordinary limit, then this limit is called the PI-*exponent* of the algebra A and denoted by $\exp(A)$; thus,

$$\exp(A) = \overline{\exp}(A) = \underline{\exp}(A).$$

On the space P_n, the symmetric group S_n naturally acts as

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)});$$

when endowed with this action, P_n becomes a module over the group ring FS_n, or, briefly, an S_n-module. The intersection $P_n \cap \mathrm{Id}(A)$ is an S_n-submodule in P_n. We recall that the degree of the character $\chi(M)$ of an S_n-module M is the dimension of M, i.e., $\deg \chi(M) = \dim M$. Moreover, the irreducible characters of such a module, that is, the characters of its irreducible representations, are uniquely determined by partitions $\lambda \vdash n$ of the number n, where

$$\lambda = (\lambda_1, \dots, \lambda_k), \qquad \lambda_1 \ge \dots \ge \lambda_k > 0, \qquad \lambda_1 + \dots + \lambda_k = n$$

(all necessary information on the representation theory of symmetric groups can be found in [21] and on its application in the theory of identities, in [1], [2]).

Any S_n-module M decomposes into a sum of irreducible modules; in terms of characters, this can be written as

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the character of the irreducible module corresponding to λ and m_λ is its multiplicity in the decomposition of M. This implies

$$\deg \chi(M) = \sum_{\lambda \vdash n} m_\lambda \deg \chi_\lambda.$$

In particular, for the S_n-module $P_n/P_n \cap \mathrm{Id}(A)$, we obtain

$$\chi_n(A) = \chi\left(\frac{P_n}{P_n \cap \mathrm{Id}(A)}\right) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1}$$

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda \deg \chi_\lambda. \tag{2}$$

Decomposition (1) is called the *nth cocharacter of* A, and relation (2) makes it possible to estimate $c_n(A)$. We say that the character $\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ of the module M *lies in a strip of width* d if all partitions λ with nonzero multiplicities m_λ include at most d parts, i.e., $\lambda = (\lambda_1, \dots, \lambda_k)$ and $k \le d$. It is known that, for such partitions, we have $\deg \chi_\lambda \le d^n$.

It is also known (see, e.g., [2, Theorem 4.6.2] and [15]) that, if A is a finite-dimensional F-algebra with $\dim A = d$, then its cocharacter $\chi_n(A)$ lies in a strip of width d and, moreover, the sum

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

(see (1)), which is called the *nth colength*, in bounded by a polynomial function in n. To be more precise, according to Theorem 1 from [15], we have

$$l_n(A) \le d(n+1)^{d^2+d}. \tag{3}$$

This means, in particular, that, in the case of exponentially growing codimensions, a key role is played by the maximal dimensions $\deg \chi_\lambda$. They can be conveniently estimated by using the functions $\Phi(\lambda)$ defined below, which depend on partitions $\lambda \vdash n$. Let

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash n, \qquad \text{where} \quad k \le d, \quad \lambda_1 + \dots + \lambda_k = n.$$

We set

$$\Phi(\lambda) = \frac{1}{(\lambda_1/n)^{\lambda_1/n} \dots (\lambda_d/n)^{\lambda_d/n}}.$$

(If k is strictly less than d, then the corresponding multipliers 0^0 are equal to 1).

In [13], the following relationship between a value of the function $\Phi(\lambda)$ and the degree $\deg \chi(\lambda)$ of the corresponding character was mentioned (see Lemma 1). Given $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, where $n \ge 100$ and $k \le d$, we have

$$\frac{\Phi(\lambda)^n}{n^{d^2+d}} \le \deg \chi_\lambda \le n\Phi(\lambda)^n. \tag{4}$$

Now, let A be any finite-dimensional algebra over F. Consider its nth cocharacter (1) and let $\Phi_{\max}^{(n)}$ denote the maximal value of $\Phi(\lambda)$ over all those $\lambda \vdash n$ for which $m_\lambda \ne 0$ in (1). Combining relations (2), (3), and (4), we obtain the following assertion.

Lemma 1. *If* $\dim A = d$, *then*

$$\frac{1}{n^{d^2+d}} (\Phi_{\max}^{(n)})^n \le c_n(A) \le (n+1)^{d^2+d+1} (\Phi_{\max}^{(n)})^n$$

for all $n \ge 100d$.

We also need the following property of the function $\Phi(\lambda)$. If $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_q)$ are two partitions of the same integer n, then the *Young diagram* D_λ corresponding to the partition λ is the table whose first row contains λ_1 boxes, second row contains λ_2 boxes, and so on. In a similar way, the diagram D_μ corresponding to the partition $\mu \vdash n$ is constructed. We say D_μ is obtained from D_λ by *pushing down one box* if there exist i and j, $1 \le i < j \le q$, such that $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$, and $\mu_p = \lambda_p$ for all other $1 \le p \le q$. Given $\lambda = (\lambda_1, \dots, \lambda_q) \vdash n$ and $\mu = (\mu_1, \dots, \mu_q, 1) \vdash n$, we say that D_μ is obtained from D_λ by pushing down one box if one of the rows in D_μ is shorter by one box than the corresponding row in D_λ and all of the remaining rows (except the last one) are of the same length.

Lemma 2. *If D_μ is obtained from D_λ by pushing down one box, then $\Phi(\mu) \ge \Phi(\lambda)$.*

Proof. Suppose that $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_{q'})$ are two partitions of n and $q' = q$ or $q + 1$. Then

$$\Phi(\lambda)^n = \frac{n^n}{\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q}}. \tag{5}$$

If $q = q'$, then the denominator in the analogous expression for $\Phi(\mu)^n$ is obtained from the denominator in (5) by replacing one product of the form $a^a b^b$, $a \ge b + 2$, by $(a-1)^{a-1}(b+1)^{b+1}$. In this case, we have $\Phi(\mu)^n \ge \Phi(\lambda)^n$, because the function $f(x) = x^x (c-x)^{c-x}$ decreases in the interval $(c/2; 0)$. If $q' = q + 1$, then we replace the factor a^a in the denominator in (5) by $(a-1)^{a-1} \cdot 1^1 < a^a$ and again obtain $\Phi(\mu)^n > \Phi(\lambda)^n$. This inequality, together with $\Phi(\lambda), \Phi(\mu) > 0$, implies the assertion of the lemma. ☐

3. A FOUR-DIMENSIONAL ALGEBRA AND ITS COCHARACTER

Consider a four-dimensional vector space W with basis $\{e_{-1}, e_0, e_1, e_2\}$. Let us define multiplication on this space as follows:

(a) $e_i e_0 = e_0 e_i = e_i$ for all $-1 \le i \le 2$;

(b) $e_i e_j = 0$ if $i, j \ne 0$ and either one of the inequalities $i > j$ and $i + j < -1$ or the inequality $i + j > 2$ holds;

(c) $e_i e_j = e_{i+j}$ otherwise.

It is easy to see that W is a simple algebra with unit e_0 and that the decomposition

$$W = W_{-1} \oplus W_0 \oplus W_1 \oplus W_2,$$

where $W_i = \langle e_i \rangle$ for $i = -1, \dots, 2$, is a $\mathbb{Z}$-grading on W.

Consider the cocharacter

$$\chi_n(W) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{6}$$

of the algebra W. Since $\dim W = 4$, this cocharacter lies in a strip of width 4, i.e., $\lambda_5 = 0$ for any partition $\lambda \vdash n$, provided that $m_\lambda \ne 0$ in (6).

Recall that a *Young tableau* T_λ is a diagram D_λ whose boxes are filled in with the numbers $1, \dots, n$. Any irreducible module over the group algebra $R = FS_n$ is isomorphic to the minimal left ideal Re_{T_λ}, where e_{T_λ} is the element R defined as follows.

We refer to the subgroup of all permutations in S_n which leave the symbols $1, \dots, n$ in their rows as the *row stabilizer* and denote this subgroup by R_{T_λ}; to the subgroup which leaves these symbols in their columns we refer as the *column stabilizer* and denote it by C_{T_λ}. We set

$$R(T_\lambda) = \sum_{\sigma \in R_{T_\lambda}} \sigma, \qquad C(T_\lambda) = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau, \qquad e_{T_\lambda} = R(T_\lambda)C(T_\lambda).$$

The element e_{T_λ} is a quasi-idempotent of the group ring FS_n, i.e., $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, $\alpha \neq 0$, and $FS_n e_{T_\lambda}$ is an irreducible S_n-module with character χ_λ. Moreover, if M is an FS_n-module and

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

then $m_\lambda \neq 0$ if and only if $e_{T_\lambda} M \neq 0$.

Now, let f be a multilinear polynomial of degree n generating an irreducible FS_n-submodule M in P_n with character χ_λ. We can assume that $e_{T_\lambda} f \neq 0$ for some Young tableau T_λ. Then the polynomial $g = C(T_\lambda) f$ generates M as an R-module as well. The variables of g are divided into m disjoint subsets as

$$\{x_1, \dots, x_n\} = X_1 \cup \dots \cup X_m,$$

where m is the number of columns in the Young diagram D_λ and each X_j is the set of variables whose numbers are written in the jth column. Moreover, g is alternating with respect to each of the sets $X_1, \dots, X_m$. In other words, any irreducible FS_n-submodule of P_n with character χ_λ is generated by a multilinear polynomial depending on m disjoint skew-symmetric sets of variables of cardinalities $\lambda'_1, \dots, \lambda'_m$, where $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ is the partition conjugate to λ (i.e., $\lambda'_1, \dots, \lambda'_m$ are the heights of the columns of D_λ).

To obtain stronger (than $\lambda_5 = 0$) constraints on the cocharacter of W, we introduce yet another numerical characteristic of partitions. Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. We write -1 in all boxes of the first row, 0 in all boxes of the second, etc., so that all boxes of the kth row are filled in with $k-2$. We define the *weight of the diagram* D_λ as the sum of all numbers in D_λ. We also refer to this number as the *weight of the partition* λ and denote it by $wt(\lambda)$. In other words,

$$wt(\lambda) = \sum_{i=1}^{k} (i-2)\lambda_i.$$

Lemma 3. *If $m_\lambda \neq 0$ in the decomposition* (6)*, then*

- $\lambda_5 = 0$;
- $wt(\lambda) \leq 2$*, i.e.,* $\lambda_1 - \lambda_3 - 2\lambda_4 \geq -2$.

Proof. We have already proved the equality $\lambda_5 = 0$. Let us prove the second assertion. Suppose that $m_\lambda \neq 0$, i.e., there exists an irreducible FS_n-submodule in P_n with character χ_λ which is not contained in the ideal of identities of W. As mentioned above, this means that there exists a multilinear polynomial $g = g(x_1, \dots, x_n)$ which is alternating with respect to the variables from each of the columns T_λ and does not vanish identically on W. Hence there exists a permutation $\varphi\colon X \to \{e_{-1}, e_0, e_1, e_2\}$ for which $\varphi(g) \neq 0$. By virtue of the antisymmetry of g, instead of variables from one column, we must substitute different basis elements of the algebra W. In particular, if $x_{i_1}, \dots, x_{i_4}$ are variables from the same column of height 4, then the total weight of the elements $\varphi(x_{i_1}), \dots, \varphi(x_{i_4})$ in the $\mathbb{Z}$-grading of W is equal to $-1 + 0 + 1 + 2 = 2$. Therefore, the least possible weight of $\varphi(g)$ in the $\mathbb{Z}$-grading is

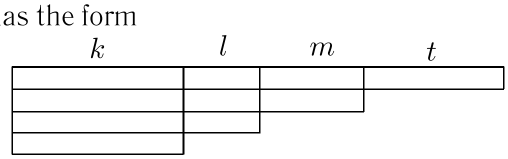
$$wt(\lambda) = -\lambda_1 + \lambda_3 + 2\lambda_4.$$

Since all components of the algebra W_k with $k \geq 3$ are zero, it follows that $-\lambda_1 + \lambda_3 + 2\lambda_4 \leq 2$, which proves the lemma. $\square$

In what follows, we need a sufficient condition for the multiplicity m_λ to not vanish. Consider a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Let us rewrite it in the form

$$\lambda = (k + l + m + t, k + l + m, k + l, k).$$

The diagram of this partition has the form



Here $n = 4k + 3l + 2m + t$, the weight of the partition λ is equal to $-m - t + 2k$, and the necessary condition for m_λ to be nonzero will be

$$m + t \geq 2k - 2.$$

Lemma 4. *If $m + t \geq 2k$ and $m \leq 2k$, then $m_\lambda \neq 0$ in the decomposition* (6).

Proof. To prove the lemma, it suffices to construct a multilinear polynomial f of degree n depending on

- k alternating sets of variables $\{x_1^{(i)}, \dots, x_4^{(i)}\}$, $1 \leq i \leq k$;
- l alternating sets $\{y_1^{(i)}, y_2^{(i)}, y_3^{(i)}\}$, $1 \leq i \leq l$;
- m alternating sets $\{z_1^{(i)}, z_2^{(i)}\}$, $1 \leq i \leq m$;
- t variables $u_1^{(i)}$, $1 \leq i \leq t$,

and such that the symmetrization of this polynomial with respect to the sets

$$\{x_1^{i_1}, y_1^{i_2}, z_1^{i_3}, u_1^{i_4}\}, \qquad \{x_2^{i_1}, y_2^{i_2}, z_2^{i_3}\}, \qquad \{x_3^{i_1}, y_3^{i_2}\}, \qquad \{x_1^{i_4}\},$$

where $1 \leq i_1 \leq k$, $1 \leq i_2 \leq l$, $1 \leq i_3 \leq m$, and $1 \leq i_4 \leq t$, yields a polynomial not vanishing identically on W.

It is convenient to label variables in alternating sets by the same mark over the symbols of these variables. For example,

$$\bar{x}_1\bar{x}_2\bar{x}_3 = \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)},$$
$$\bar{a}_1\widetilde{b}_1\bar{a}_2\widetilde{b}_2 = a_1b_1a_2b_2 - a_1b_2a_2b_1 - a_2b_1a_1b_2 + a_2b_2a_1b_1,$$
$$(\bar{x}\bar{\bar{x}})(\bar{y}\bar{\bar{y}}) = (xx)(yy) - (yx)(xy) - (xy)(yx) + (yy)(xx).$$

We use this convention not only for variables but also for elements of the algebra W. For instance,

$$\bar{e}_{-1}(\bar{e}_1\bar{e}_2) = e_{-1}(\bar{e}_1\bar{e}_2) - e_1(\bar{e}_{-1}\bar{e}_2) - e_2(\bar{e}_1\bar{e}_{-1}) = 0 - e_1(e_{-1}e_2) + e_2(e_{-1}e_1)$$
$$= -e_1^2 + e_2e_0 = -e_2 + e_2 = 0.$$

First, we construct an expression alternating with respect to e_{-1}, e_0, e_1, and e_2 and contains two occurences of the basis element e_{-1}. We set

$$f_1 = e_{-1}[\bar{e}_{-1}((\bar{e}_0e_{-1})(\bar{e}_1\bar{e}_2))].$$

This element is the polynomial

$$x_{-1}[\bar{x}_{-1}((\bar{x}_0x_{-1})(\bar{x}_1\bar{x}_2))]$$

in the variables x_{-1}, x_0, x_1, and x_2, which is equal to

$$f_1 = e_{-1}[\bar{e}_{-1}((e_0e_{-1})(\bar{e}_1\bar{e}_2))] = e_{-1}[\bar{e}_{-1}(e_{-1}(\bar{e}_1\bar{e}_2))]$$
$$= e_{-1}[e_{-1}(e_{-1}(\bar{e}_1\bar{e}_2)) - e_1(e_{-1}(\bar{e}_{-1}\bar{e}_2)) - e_2(e_{-1}(\bar{e}_1\bar{e}_{-1}))]$$
$$= e_{-1}[0 - e_1(e_{-1}(e_{-1}e_2)) + e_2(e_{-1}(e_{-1}e_1))]$$
$$= e_{-1}[-e_1(e_{-1}e_1) + e_2(e_{-1}e_0)]$$
$$= e_{-1}[-e_1e_0 + e_2e_{-1} = e_{-1}[-e_1 + 0] = -e_0.$$

Let

$$g(x_{-1}, x_0, x_1, x_2, y, z) = y[\bar{x}_{-1}((\bar{x}_0z)(\bar{x}_1\bar{x}_2))].$$

Then the left-normed degree f_1^k is the value of a symmetrization of the polynomial

$$g(x_{-1}^{(1)}, x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, y_1^{(1)}, z_1^{(1)}) \cdots g(x_{-1}^{(k)}, x_0^{(k)}, x_1^{(k)}, x_2^{(k)}, y_1^{(k)}, z_1^{(k)}),$$

which is alternating with respect to $\{x_{-1}^{(i)}, x_0^{(i)}, x_1^{(i)}, x_2^{(i)},\}$, $i = 1, \dots, k$. The symmetrization is over the four sets

$$\{x_{-1}^{(1)}, \dots, x_{-1}^{(k)}, y_1^{(1)}, z_1^{(1)}, \dots, y_1^{(k)}, z_1^{(k)}\}, \qquad \{x_0^{(1)}, \dots, x_0^{(k)}\},$$
$$\{x_1^{(1)}, \dots, x_1^{(k)}\}, \qquad \{x_2^{(1)}, \dots, x_2^{(k)}\}.$$

Similarly, the 3-alternated polynomial takes the value

$$f_2 = \bar{e}_{-1}\bar{e}_0\bar{e}_1 = (\bar{e}_{-1}\bar{e}_0)e_1 + (\bar{e}_0\bar{e}_1)e_{-1} + (\bar{e}_1\bar{e}_{-1})e_0 = 0 + 0 - e_{-1}e_1e_0 = -e_0.$$

We also set

$$f_3 = e_{-1}[\bar{e}_{-1}((\bar{e}_0\widetilde{e}_{-1})(\bar{e}_1\bar{e}_2))]\widetilde{e}_0.$$

This expression contains the alternating set e_{-1}, e_0, e_1, e_2 labeled by an overbar and the alternating set e_{-1}, e_0 labeled by a tilde. Since $W_+e_{-1} = 0$, where $W_+ = W_1 \oplus W_2$, it follows that

$$f_3 = f_1e_0 = f_1 = -e_0.$$

Finally, let

$$f_4 = \bar{\bar{e}}_{-1}[[\bar{e}_{-1}((\bar{e}_0\widetilde{e}_{-1})(\bar{e}_1\bar{e}_2))]\widetilde{e}_0\bar{\bar{e}}_0].$$

Here we have one alternating set $\{e_{-1}, e_0, e_1, e_2\}$ and two alternating sets $\{e_{-1}, e_0\}$. Let

$$a = [\bar{e}_{-1}((\bar{e}_0\widetilde{e}_{-1})(\bar{e}_1\bar{e}_2))]\widetilde{e}_0.$$

Then the degree of a in the $\mathbb{Z}$-grading is 1; therefore, it follows from the condition $W_+e_{-1} = 0$ and the calculations performed above for f_1 that $a = -e_1$ and

$$f_4 = e_{-1}(ae_0) - e_0(ae_{-1}) = e_{-1}a = -e_0.$$

First, consider the special case where $m = 2k$, $t = 0$. We set

$$f = f(\underbrace{e_{-1}, \dots, e_{-1}}_{\alpha}, \underbrace{e_0, \dots, e_0}_{\beta}, \underbrace{e_1, \dots, e_1}_{\gamma}, \underbrace{e_2, \dots, e_2}_{\delta}) = (f_2^l)(f_4)^k. \tag{7}$$

In the expression for f, the element e_{-1} occurs in k alternating sets of order 4, in $2k = m$ alternating sets of order 2, and in l alternating sets of order 3. There are no occurrences of e_{-1} of nonalterable type in f. The total degree α of f in e_{-1} is equal to $k + l + m$. The element e_0 occurs k times in 4-alternated sets, $m = 2k$ times in 2-alternated sets, and l times in 3-alternated sets; thus, the total number of occurrences of e_0 is $k + l + m$. Finally, e_1 occurs in k alternating sets of order 4 and in l alternating sets of order 3, and e_2 occurs in k alternating sets of order 4. This means that f is a value of the polynomial generating the irreducible module corresponding to the partition

$$\lambda = (k + l + m + t, k + l + m, k + l, k), \qquad \text{where} \quad m = 2k, \quad t = 0.$$

Therefore, $m_\lambda \neq 0$ for this partition, because $f = (-e_0)^{k+l} = \pm e_0$.

Consider the more general case in which $m = 2k$ and $t > 0$. We set $n_0 = n - t$ and construct the polynomial f specified in (7) for the partition

$$\lambda_0 = (k + l + m, k + l + m, k + l, k) \vdash n_0.$$

We have

$$\begin{aligned} g &= f(\underbrace{e_{-1} + e_0, \dots, e_{-1} + e_0}_{\alpha}, e_0, \dots, e_2)\underbrace{(e_{-1} + e_0) \cdots (e_{-1} + e_0)}_{t} \\ &= f(e_{-1} + e_0, \dots, e_{-1} + e_0, e_0, \dots, e_2) + f', \end{aligned}$$

where $f' = tf(e_{-1}, \dots, e_{-1}, e_0, \dots, e_2)e_{-1} \in W_{-1}$, because $(W_{-1} \oplus W_1 \oplus W_2)e_{-1} = 0$. Moreover,

$$f(e_{-1} + e_0, \dots, e_{-1} + e_0, e_0, \dots, e_2) = f(e_{-1}, \dots, e_{-1}, e_0, \dots, e_2) + f'',$$

where $f'' \in W_1 \oplus W_2$. It follows that $g = \pm e_0 + f' + f'' \neq 0$, and g is a value of the polynomial generating the irreducible FS_n-module with character χ_λ. Therefore, $m_\lambda \neq 0$ in (6).

Now, suppose that $m = 2q < 2k$ and $t \neq 0$. Consider the product

$$(f_1)^{k-q}(f_2)^l(f_4)^q = f_0 = f_0(\underbrace{e_{-1},\dots,e_{-1}}_{\alpha},\underbrace{e_0,\dots,e_0}_{\beta},\underbrace{e_1,\dots,e_1}_{\gamma},\underbrace{e_2,\dots,e_2}_{\delta}).$$

As above, we have $\delta = k$, e_2 occurs in the alternating sets of order 4, $\gamma = k + l$, $\beta = k + l + m$, and e_1 and e_0 occur in alternating sets; moreover, e_1 occurs k times in sets of order 4 and l times in sets of order 3. The numbers of occurrences of e_0 in sets of orders 4, 3, and 2 are k, l, and m, respectively. The element e_{-1} occurs in alternating sets of orders 4, 3, and 2 as well, and the numbers of occurrences are $k - q + q = k$, l, and $2q = m$, respectively. In addition, this element e_{-1} also occurs $2(k - q)$ times outside alternating sets at the expense of the factor f_1^{k-q}. In particular,

$$\alpha = k + l + m + t_0,$$

where $t_0 = 2(k - q)$, i.e., $m + t_0 = 2k$.

Consider the expressions

$$f_0' = f_0(\underbrace{e_{-1}+e_0,\dots,e_{-1}+e_0}_{\alpha},\underbrace{e_0,\dots,e_0}_{\beta=k+l+m},\underbrace{e_1,\dots,e_1}_{\gamma=k+l},\underbrace{e_2,\dots,e_2}_{\delta=k}),$$

and

$$f = f_0'\underbrace{(e_{-1}+e_0)\dots(e_{-1}+e_0)}_{t-t_0}.$$

As in the preceding case, we have

$$f = f_0 + f' + f'',$$

where $f_0 = \pm e_0$, $f' \in W_{-1}$, and $f'' \in W_+$, i.e., $f \neq 0$. The element f is a nonzero value of the polynomial corresponding to the partition

$$\lambda = (k + l + m + t, k + l + m, k + l, k) \qquad \text{with} \quad m = 2q < 2k.$$

Therefore, for such partitions λ, the multiplicity in (6) does not vanish either.

Finally, for odd $m = 2q + 1 < 2k$, we take the product

$$f_0 = f_1^{k-q-1} f_2^l f_3 f_4^q,$$

replace e_{-1} by $e_{-1} + e_0$ in this product, and set

$$f = f_0(e_{-1}+e_0,\dots,e_{-1}+e_0,e_0,\dots,e_0,\dots,e_2,\dots e_2)\underbrace{(e_{-1}+e_0)\dots(e_{-1}+e_0)}_{t-t_0},$$

where $t_0 = 2(k - q) - 1$. In the expression for f_0, the element $e_{-1}' = e_{-1} + e_0$ occurs in k four-element alternating sets, l three-element sets, and $m = 2q + 1$ two-element sets; outside the alternated sets, this element occurs t times. For e_0, e_1, and e_2, the same conditions as in the preceding case hold. Since $f_0 = \pm e_0$ and $f = f_0 + f'$, where $f' \in W_{-1} \oplus W_1 \oplus W_2$, it follows that $f \neq 0$. In other words, the multiplicity m_λ is also nonzero for $\lambda = (k + l + m + t, k + l + m, k + l, k)$ with odd $m < 2k$, provided that $m + t \geq 2k$. This proves the lemma for $k \neq 0$.

If $k = 0$, then $m = 0$, and the partition λ has the form $\lambda = (l + t, l, l)$. For this partition, the multilinear polynomial is constructed in a similar way. First, we take the polynomial

$$f = f_2^l = f(e_{-1},\dots,e_{-1},e_0,\dots,e_0,e_1,\dots,e_1)$$

of degree l in each of the basis elements e_{-1}, e_0, e_1. Then, we set

$$f' = f(e_{-1}+e_0,\dots,e_{-1}+e_0,e_0,\dots,e_1)\underbrace{(e_{-1}+e_0)\dots(e_{-1}+e_0)}_{t}.$$

As previously, f' is not an identity in W and generates an S_n-module with character χ_λ. ◻

4. ESTIMATES OF THE PI-EXPONENT

The simplicity of the algebra W implies the existence of its PI-exponent (see [13, Theorem 3]). To estimate the PI-exponent and, in particular, prove that it is noninteger, we introduce the following quantity.

Let $\lambda = (\lambda_1, \dots, \lambda_4) \vdash n$, and let m_λ be the multiplicity of λ in the cocharacter (6). We set

$$a_n = \max\{\Phi(\lambda) \mid m_\lambda \neq 0\}.$$

By virtue of Lemma 1, we have

$$\overline{\exp(A)} = \varlimsup_{n\to\infty} a_n, \qquad \underline{\exp(A)} = \varliminf_{n\to\infty} a_n \tag{8}$$

for any four-dimensional algebra.

Consider the sequence $\{a_n\}$ for the algebra W. According to (8) and [13, Theorem 3], the sequence $\{a_n\}$ has a limit as $n \to \infty$. We need one more property of this sequence.

Lemma 5. *Let $a_n = \Phi(\lambda^{(0)})$. Then the partition $\lambda^{(0)}$ can be chosen so that $wt(\lambda^{(0)}) \geq 0$ for sufficiently large n.*

Proof. Let $\lambda^{(0)}$ be one of the points of maximum of the function $\Phi(\lambda)$ which determine a_n. It can be assumed that, for all such $\lambda \vdash n$ with $\Phi(\lambda) = a_n$, this partition is of maximum weight. As above, we write $\lambda^{(0)}$ in the form

$$\lambda^{(0)} = (k+l+m+t, k+l+m, k+l, k).$$

Suppose that $wt(\lambda^{(0)}) = -m - t + 2k < 0$, i.e., $m + t > 2k$.

First, note that $k \neq 0$ for $\lambda^{(0)}$. Indeed, it is easy to see that $\Phi(\lambda^{(0)}) \leq 3$ for $k = 0$. At the same time, for the partition $\lambda = (3p, p, p, p)$, we have

$$\Phi(\lambda) = \left(\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{6}\right)^{3/6}\right)^{-1} = \sqrt{12} > 3.4.$$

Since the sequence $\{a_n\}$ converges and, for any n, there exists a p with $|n - 6p| \leq 5$, it follows that $a_n > 3$ for all sufficiently large n, and $k \neq 0$.

Now, note that, transferring one box in the diagram D_λ from the second row to the third, we obtain D_μ, where

$$\mu = (k'+l'+m'+t', k'+l'+m', k'+l', k') \qquad \text{with} \quad k' = k, \quad t' = t+1, \quad m' = m-2.$$

Therefore, either $m' - 2k' > 0$ (in which case $m' + t' - 2k' > 0$), $m' - 2k' = 0$, or -1. In the last two cases, we have $m' + t' - 2k' \geq 0$. Thus, for $m \geq 2$, pushing down one or several boxes, we obtain a partition μ of higher weight for which $\Phi(\mu) \geq \Phi(\lambda^{(0)})$ by Lemma 2. Moreover, μ satisfies the assumptions of Lemma 4 and, therefore, $m_\mu \neq 0$ in (6). The maximality of the weight of $\lambda^{(0)}$ implies $m \leq 1$.

Thus, $t \geq 2k \geq 2$, and, moving one box of $D_{\lambda^{(0)}}$ from the first row to the second, we obtain a diagram D_μ for which $\Phi(\mu) \geq \Phi(\lambda^{(0)})$ and $wt(\mu) > wt(\lambda^{(0)})$. Moreover, μ again satisfies the conditions of Lemma 4, and $m_\mu \neq 0$. It follows that $m + t - 2k \leq 0$ for $\lambda^{(0)}$, which completes the proof of the lemma. □

We have already mentioned that if a diagram D_μ is obtained from a diagram D_λ by pushing down one box, then $\Phi(\mu) \geq \Phi(\lambda)$. Now we estimate this deviation.

Lemma 6. *Let $\lambda = (\lambda_1, \dots, \lambda_q)$ and $\mu = (\mu_1, \dots, \mu_{q'})$ be two partitions of n with $q' = q$ or $q + 1$. Suppose that D_μ is obtained from D_λ by pushing down one box. Then*

$$\Phi(\lambda) \geq \frac{1}{n^{(q^2+3q+4)/n}}\,\Phi(\mu).$$

Proof. The procedure of pushing down a box can be performed in two steps. First, we cut out a box from D_λ and obtain $D_{\lambda'}$, where $\lambda' \vdash n-1$; then, attaching one box to $D_{\lambda'}$, we obtain D_μ. According to Lemma 6.2.4 in [2], we have

$$\deg \chi_{\lambda'} \le \deg \chi_\lambda \le n \deg \chi_{\lambda'}, \qquad \deg \chi_{\lambda'} \le \deg \chi_\mu \le n \deg \chi_{\lambda'},$$

which readily implies

$$\deg \chi_\lambda \ge \frac{1}{n} \deg \chi_\mu. \tag{9}$$

Using (9) and (4), we obtain

$$\Phi(\lambda)^n \ge \frac{1}{n} \deg \chi_\lambda \ge \frac{1}{n^2} \deg \chi_\mu \ge \frac{1}{n^{(q+1)^2+q+1+2}} \Phi(\mu)^n,$$

which proves the lemma. □

Below we prove yet another relation between the values of the function Φ at various partitions.

Lemma 7. *Suppose that the Young diagram of a partition*

$$\lambda = (\lambda_1, \dots, \lambda_d) \vdash (n-1)$$

is obtained from a diagram D_μ by deleting one box. Then

$$\Phi(\lambda) \le n^{(d^2+d+2)/n} \Phi(\mu)$$

for $n \ge d$.

Proof. By virtue of (4), we have

$$\Phi(\lambda)^{n-1} \le (n-1)^{d^2+d} \deg \chi_\lambda \le n^{d^2+d} \deg \chi_\lambda, \qquad \deg \chi_\mu \le n \Phi(\mu)^n.$$

On the other hand, $\deg \chi_\lambda \le \deg \chi_\mu$ according to [2, Lemma 6.2.4]. Since the maximum value of $\Phi(\lambda)$ is d, it follows that

$$\Phi(\lambda) \le n^{(d^2+d+2)/n} \Phi(\mu). \quad \square$$

Let us define one more sequence related to W. For $n \ge 6$, we set

$$b_n = \max\{\Phi(\lambda) \mid \lambda = (\lambda_1, \dots, \lambda_4) \vdash n,\ m_\lambda \ne 0,\ \lambda_1 - \lambda_3 = 2\lambda_4\}$$

if n has a partition λ with $\lambda_1 - \lambda_3 = 2\lambda_4$ for which $m_\lambda \ne 0$ in (6). Otherwise, we set $b_n = \min\{b_{n-1}, a_n\}$. Note that, according to Lemma 4, if $n = 6k$, then the partition $\lambda = (3k, k, k, k)$ satisfies the required conditions.

Lemma 8. *The following relations hold*:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \exp(W).$$

Proof. According to [13, Theorem 3], the PI-exponent of any finite-dimensional simple algebra exists; therefore, the limit $\lim_{n\to\infty} a_n = \exp(W)$ exists as well, as follows from (8). Thus, to prove the lemma, it suffices to find a function $\psi = \psi(n)$ such that $\lim_{n\to\infty} \psi(n) = 1$ and

$$\psi(n) a_n \le b_n \le a_n \tag{10}$$

for all sufficiently large n.

Fix n and take a partition $\lambda \vdash n$ for which $\Phi(\lambda) = a_n$. By Lemma 5, we can choose λ so that $wt(\lambda) \ge 0$; by Lemma 3, $wt(\lambda)$ is then equal to 0, 1, or 2.

If $wt(\lambda) = 0$, then $b_n = a_n$. Suppose that $wt(\lambda) = 1$. Let us write λ as

$$\lambda = (k+l+m+t, k+l+m, k+l, k).$$

Then $m+t=2k-1$. If $m \neq 0$, then we can transfer one box from the second to the first row in the diagram D_λ and obtain a diagram D_μ for $\mu=(k+l+m'+t', k+l+m', k+l, k)$, $m'=m-1$, $t'=t+2$, with $wt(\mu)=0$. Then, by Lemma 4, the multiplicity of μ in $\chi_n(W)$ is nonzero. As mentioned in the proof of Lemma 5, the partition λ has a nonzero component k. Therefore, by virtue of Lemma 6, we have

$$b_n \geq \Phi(\mu) \geq \frac{\Phi(\lambda)}{n^{32/n}} = \frac{a_n}{n^{32/n}}. \tag{11}$$

If $m=0$ but $l>0$ and $t>0$, then a partition μ with weight zero can be obtained by moving one box of D_λ from the third to the second row, and we again obtain inequality (11) for b_n. The case where $m=0$, $l>0$, and $t=0$ is impossible, because $m+t=2k-1$.

The only partition λ with $wt(\lambda)=1$ for which the transfers specified above cannot be done is $(3k-1, k, k, k)$. But Lemma 7 implies that, for this λ, we have

$$\Phi(\lambda) \leq n^{22/n}\Phi(\mu), \tag{12}$$

where $\mu=(3k, k, k, k)$. Since $\Phi(\mu)=\sqrt{12}<3.48$, we obtain

$$\Phi(\lambda) < n^{22/n}\cdot 3.48.$$

Note that any partition of the form $\rho=(3q, 3q, q, q)$ satisfies the assumptions of Lemma 4, and

$$\Phi(\rho) = \frac{8}{\sqrt[4]{27}} > 3.5;$$

hence $\Phi(\lambda)$ cannot satisfy inequality (12) for sufficiently large n, i.e., $\lambda \neq (3k-1, k, k, k)$, and if $wt(\lambda)=1$, then inequality (11) holds.

Now, suppose that $wt(\lambda)=2$. Then we twice move a box one row upward in the diagram D_λ. This cannot be done only if either $\lambda=(3k-2, k, k, k)$, $\lambda=(q, q, q, 1)$, or the first transfer of one box upward results in the partition $\mu=(3k-1, k, k, k)$. The first and the third possibility are excluded for the same reason as in the case of $wt(\lambda)=1$, namely, because such partitions cannot maximize $\Phi(\lambda)$; the second possibility cannot occur because if $\lambda=(q, q, q, 1)\vdash n$, $\mu=(q, q, q)\vdash(n-1)$, and $\Phi(\mu)=3$, then $\deg\chi_\lambda \leq n\deg\chi_\mu$.

In the remaining cases, twice applying Lemma 6, we obtain

$$b_n \geq \frac{a_n}{n^{64/n}}. \tag{13}$$

Relations (11) and (13) imply the required condition (10), which proves the lemma. □

To state and prove the main results of this paper, we extend the domain of the function Φ. For any $0 \leq x_1, \dots, x_4 \leq 1$, we set

$$\Phi(x_1, \dots, x_4) = \frac{1}{x_1^{x_1}\cdots x_4^{x_4}}. \tag{14}$$

Inside the domain of Φ, consider the closed subset T determined by the conditions

$$\begin{cases} x_1 \geq x_2 \geq x_3 \geq x_4, \\ x_1+x_2+x_3+x_4=1, \\ x_1-x_3=2x_4. \end{cases} \tag{15}$$

Theorem 1. *The* PI-*exponent of the algebra* W *exists and is equal to*

$$\exp(W) = \max\{\Phi(x_1, \dots, x_4) \mid (x_1, \dots, x_4) \in T\}. \tag{16}$$

In particular, $\exp(W) \approx 3.610718614$.

Proof. The existence of the exponent has already been mentioned and follows from the simplicity of W. We have

$$\exp(W) = b = \lim_{n\to\infty} b_n$$

by Lemma 8. It remains to show that $b = M$, where

$$M = \max\{\Phi(x_1, \dots, x_4) \mid (x_1, \dots, x_4) \in T\}.$$

Let $Z = (z_1, \dots, z_4)$ be a point of maximum of Φ on T. Clearly, we can choose a point $A = (a_1, \dots, a_4) \in T$ with rational coefficients arbitrarily close to Z. Let m denote the common denominator of the rational numbers $a_1, \dots, a_4$. Then $\lambda_1 = a_1 m, \dots, \lambda_4 = a_4 m$ are nonnegative integers, and $\lambda_1 \geq \dots \geq \lambda_4$. In other words, $\lambda = (\lambda_1, \dots, \lambda_4)$ is a partition of m satisfying the condition $\lambda_1 - \lambda_3 = 2\lambda_4$. Moreover, for any $t = 1, 2, \dots$, the partition $t\lambda = (t\lambda_1, \dots, t\lambda_4)$ of $n_t = tm$ satisfies the same condition. It follows that

$$b_{n_t} \geq \Phi(t\lambda) = \Phi(\lambda). \tag{17}$$

Since the sequence $\{b_i\}$ converges and $\Phi(\lambda)$ in (17) can be made arbitrarily close to M, it follows that $b \geq M$. The reverse inequality is obvious. Thus, we have proved the relation $b = M$.

To fully complete the proof, we must justify the approximate estimate of $\exp(W)$. In [11], an example of an infinite-dimensional Lie algebra L for which

$$3.1 < \underline{\exp}(L) \leq \overline{\exp}(L) < 3.9$$

was constructed. In the recent paper [12], it was proved that the ordinary PI-exponent of L exists, i.e., $\underline{\exp}(L) = \overline{\exp}(L)$. Moreover, it turned out that

$$\exp(L) = \max\{\Phi(x_1, \dots, x_4) \mid (x_1, \dots, x_4) \in T\},$$

where Φ is the function defined by (14) and the domain T is determined by (15). It was also shown in [12] that

$$M = \Phi(\beta_1, \dots, \beta_4),$$

where β_4 is a positive root of the equation $16t^3 - 24t^2 + 11t - 1 = 0$, $\beta_4 \approx 0.276953179$, and

$$\beta_3 = 2\beta_4 - 4\beta_4^2, \qquad \beta_2 = \frac{\beta_3^2}{\beta_4}, \qquad \beta_1 = \frac{\beta_3^3}{\beta_4^2}.$$

This implies

$$\exp(W) = \exp(L) \approx 3.610718614,$$

which completes the proof of the theorem. □

Corollary 1. *There exist finite-dimensional simple unitary algebras with fractional exponent strictly less than their dimension.*

Corollary 2. *The least dimension of a unitary algebra with fractional* PI*-exponent is* 4.

Proof. Theorem 1 implies the existence of four-dimensional unitary algebras with fractional PI-exponent. The nonexistence of such algebras in dimensions 2 and 3 follows from results of [17]. □

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (grant no. 13-01-00234a) and by Slovenian Research Agency (grant no. P1-0292-0101).

REFERENCES

1. Yu. A. Bahturin [Yu. A. Bakhturin], *Identical Relations in Lie Algebras* (Nauka, Moscow, 1985; VNU Science Press, Utrecht, 1987).
2. A. Giambruno and M. Zaicev, *Polynomial Identities and Asymptotic Methods*, in *Math. Surveys Monogr.* (Amer. Math. Soc., Providence, RI, 2005), Vol. 122.
3. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings that Are Nearly Associative*, in *Monographs in Modern algebra* (Nauka, Moscow, 1978) [in Russian].
4. A. Giambruno and M. Zaicev, "On codimension growth of finitely generated associative algebras," Adv. Math. **140** (2), 145–155 (1998).
5. A. Giambruno and M. Zaicev, "Exponential codimension growth of PI algebras: an exact estimate," Adv. Math. **142** (2), 221–243 (1999).
6. A. Giambruno, A. Regev, and M. V. Zaicev, "Simple and semisimple Lie Algebras and codimension growth," Trans. Amer. Math. Soc. **352** (4), 1935–1946 (2000).
7. M. V. Zaitsev, "Integrality of exponents of codimension growth of finite-dimensional Lie algebras," Izv. Ross. Akad. Nauk Ser. Mat. **66** (3), 23–48 (2002) [Russian Acad. Sci. Izv. Math. **66** (3), 463–487 (2002)].
8. A. Giambruno and M. Zaicev, "Codimension growth of special simple Jordan algebras," Trans. Amer. Math. Soc. **362** (6), 3107–3123 (2010).
9. A. Giambruno, I. Shestakov, and M. Zaicev, "Finite dimensional nonassociative algebras and codimension growth," Adv. Appl. Math. **47** (1), 125–139 (2011).
10. S. P. Mishchenko, A. Regev, and M. Zaicev, "Integrality of exponents of some abelian-by-nilpotent varieties of Lie algebras," Comm. Algebra **28** (9), 4105–4130 (2000).
11. M. V. Zaicev and S. P. Mishchenko, "An example of a variety of Lie algebras with a fractional exponent," J. Math. Sci. **93** (6), 977–982 (1999).
12. A. B. Verevkin, M. V. Zaitsev, and S. P. Mishchenko, "A sufficient condition for coincidence of lower and upper exponents of the variety of linear algebras," Vestnik Moskov. Univ. Ser. I Mat. Mekh., No. 2, 36–39 (2011) [Moscow Univ. Math. Bull. **66** (2), 86–89 (2011)].
13. A. Giambruno and M. Zaicev, "On codimension growth of finite-dimensional Lie superalgebras," J. London Math. Soc. (2) **85**, 534–548 (2012).
14. A. Giambruno, S. Mishchenko, and M. Zaicev, "Algebras with intermediate growth of the codimensions," Adv. Appl. Math. **37** (3), 360–377 (2006).
15. A. Giambruno, S. Mishchenko, and M. Zaicev, "Codimensions of algebras and growth functions," Adv. Math. **217** (3), 1027–1052 (2008).
16. A. Giambruno, S. Mishchenko, and M. Zaicev, "Codimension growth of two dimensional algebras," Proc. Amer. Math. Soc. **135** (11), 3405–3415 (2007).
17. M. V. Zaitsev, "Identities of unitary finite-dimensional algebras," Algebra Logika **50** (5), 563–594 (2011) [Algebra Logic **50** (5), 381–404 (2011)].
18. A. Giambruno, S. Mishchenko, and M. Zaicev, "Polynomial identities of algebras of small dimension," Comm. Algebra **37** (6), 1934–1948 (2009).
19. S. S. Mishchenko, "Codimension growth of commutative and noncommutative algebras," in *Algeba and Mathematical Logic*, Proceedings of international conference dedicated to the 100th birthday of Professor V. V. Morozov and Youth Workshop-Conference "Modern Problems of Algebra and Mathematical Logic", Kazan, Russia, 2011 (Kazan. Feder. Univ., Kazan, 2011), pp. 144–145 [in Russian].
20. A. Regev, "Codimensions and trace codimensions of matrices are asymtotically equal," Israel J. Math. **47** (2-3), 246–250 (1984).
21. G. D. James, *The Representation Theory of the Symmetric Groups*, in Lecture Notes in Math. (Springer-Verlag, Berlin–New York–Heidelberg, 1978; Nauka, Moscow, 1982), Vol. 682.