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Graded Identities of Some Simple Lie Superalgebras

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Abstract We study $\mathbb{Z}_2$-graded identities of Lie superalgebras of the type $b(t), t \geq 2$, over a field of characteristic zero. Our main result is that the n-th codimension is strictly less than $(\dim b(t))^n$ asymptotically. As a consequence we obtain an upper bound for ordinary (non-graded) PI-exponent for each simple Lie superalgebra $b(t), t \geq 3$.

Keywords Polynomial identity · Lie superalgebra · Codimensions · Exponential growth · Fractional PI-exponent

Mathematics Subject Classifications (2010) Primary 17B01, 16P90; Secondary 16R10

1 Introduction

In this paper we study numerical invariants of identities of Lie superalgebras. One of the main numerical characteristics of the identities of an algebra A over a field F of characteristic zero is the sequence of codimensions $\{c_n(A)\}, n = 1, 2, \ldots,$ and

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its asymptotic behaviour. Many deep and interesting results in this area were proved during the last few decades (see, for example, [1]) both in the associative and the non-associative cases. In particular, in many classes of algebras (associative [2], Lie [3–5], Jordan, alternative and some others [6]) it was proved that if A is a finite dimensional algebra, $\dim A = d$, and F is algebraically closed then PI-exponent $exp(A)$ is equal to d if and only if A is simple. In general $exp(A) \leq \dim A$ as it was observed in [7, 8]. Recently (see [9]) it was shown that $exp(L) < \dim L$ provided that L is a simple Lie superalgebra of the type $b(t), t \geq 3$, (we use notations from [10] for simple Lie superalgebras). Unfortunately, an upper bound $\alpha = \alpha(t) < \dim b(t)$ was not found for $exp(b(t))$ in [9].

Since any Lie superalgebra L is $\mathbb{Z}_2$-graded, one can consider graded codimensions $c_n^{gr}(L)$ and graded PI-exponent $exp^{gr}(L)$. Graded codimensions and graded PI-exponents of Lie superalgebras were studied earlier in several papers (see, for example, [11–13]). Existence and integrality of graded exponents were proved for some classes of Lie superalgebras. On the other hand, there are no known examples where $exp^{gr}(L)$ is fractional.

There are some relations between graded and non-graded identities, codimensions and PI-exponents. In particular,

$$c_n(A) \leq c_n^{gr}(A) \tag{1}$$

(see [14] or [7]) for any finite dimensional G-graded algebra A where G is a finite group. Hence when A is finite dimensional and simple and $exp(A) = \dim A$ it follows from Eq. 1 and [7] that $exp^{gr}(A)$ exists and is equal to $\dim A$.

First series of examples with $exp(A) \neq \dim A$ where A is a finite dimensional simple algebra is given by simple Lie superalgebras $b(t), t \geq 3$, of the dimension $\dim b(t) = 2t^2 - 1$ [9]. It is important to study asymptotics of $c_n^{gr}(b(t))$ and to compare it with the asymptotics of $c_n(b(t))$. The main result of this paper says that the (upper) graded PI-exponent of $b(t)$ is less than or equal to $t^2 - 1 + t\sqrt{t^2-1}$. As a consequence of this result and Eq. 1 we obtain an upper bound for ordinary PI-exponent of $b(t)$, $exp(b(t)) \leq t^2 - 1 + t\sqrt{t^2-1}$. In particular, the difference $\dim b(t) - exp(b(t))$ is at leasi $t^2 - t\sqrt{t^2-1}$ which is a decreasing function of t with limit $\frac{1}{2}$.

2 Preliminaries

Let A be an algebra over a field F of characteristic zero. Recall that A is said to be $\mathbb{Z}_2$-graded algebra if A has a vector space decomposition $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$, $A_0A_1 + A_1A_0 \subseteq A_1$. Usually elements of A_0 are called even while elements of A_1 are called odd. Any element of $A_0 \cup A_1$ is called homogeneous. In particular, a Lie superalgebra L is a $\mathbb{Z}_2$-graded algebra $L = L_0 \oplus L_1$ satisfying the following two relations

$$xy - (-1)^{|x||y|}yx = 0,$$

$$x(yz) = (xy)z + (-1)^{|x||y|}y(xz)$$

where x, y, z are homogeneous elements and $|x| = 0$ if x is even while $|x| = 1$ if x is odd.

Denote by $\mathcal{L}(X, Y)$ a free Lie superalgebra with infinite sets of even generators X and odd generators Y. A polynomial $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathcal{L}(X, Y)$ is said to be a graded identity of Lie superalgebra $L = L_0 \oplus L_1$ if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ whenever $a_1, \ldots, a_m \in L_0, b_1, \ldots, b_n \in L_1$.

Given positive integers $0 \le k \le n$, denote by $P_{k,n-k}$ the subspace of all multilinear polynomials $f = f(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in \mathcal{L}(X, Y)$ of degree k in even variables and of degree $n-k$ in odd variables. Denote by $Id^{gr}(L)$ an ideal of $\mathcal{L}(X, Y)$ of all graded identities of L. Then $P_{k,n-k} \cap Id^{gr}(L)$ is the subspace of all multilinear graded identities of L of total degree n depending on k even variables and $n-k$ odd variables. Denote also by $P_{k,n-k}(L)$ the quotient

$$P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap Id^{gr}(L)}.$$

Then the graded $(k, n-k)$-codimension of L is

$$c_{k,n-k}(L) = \dim P_{k,n-k}(L)$$

and the total graded codimension of L is

$$c_n^{gr}(L) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(L). \tag{2}$$

If the sequence $\{c_n^{gr}(L)\}_{n\ge1}$ is exponentially bounded then one can consider the related bounded sequence $\sqrt[n]{c_n^{gr}(L)}$. The latter sequence has the following lower and upper limits

$$\underline{exp}^{gr}(L) = \liminf_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}, \qquad \overline{exp}^{gr}(L) = \limsup_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}$$

called the lower and upper PI-exponents of L, respectively. If an ordinary limit exists, it is called an (ordinary) graded PI-exponent of L,

$$exp^{gr}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}$$

Symmetric groups and their representations play an important role in the theory of codimensions. In particular, in the case of graded identities one can consider the $S_k \times S_{n-k}$-action on multilinear graded polynomials. Namely, the subspace $P_{k,n-k} \subseteq \mathcal{L}(X, Y)$ has a natural structure of $S_k \times S_{n-k}$- module where S_k acts on even variables $x_1, \ldots, x_k$ while S_{n-k} acts on odd variables $y_1, \ldots, y_{n-k}$. Clearly, $P_{k,n-k} \cap Id^{gr}(L)$ is the submodule under this action and we get an induced $S_k \times S_{n-k}$-action on $P_{k,n-k}(L)$. The character $\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L))$ is called $(k, n-k)$ cocharacter of L. By Maschke's Theorem this character can be decomposed into the sum of irreducible characters

$$\chi_{k,n-k}(L) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu}\chi_{\lambda,\mu} \tag{3}$$

where λ and μ are partitions of k and $n-k$, respectively (all details concerning representations of symmetric groups can be found in [15]).

Recall that an irreducible $S_k \times S_{n-k}$-module with the character $\chi_{\lambda,\mu}$ is the tensor product of S_k-module with the character χ_λ and S_{n-k}-module with the character

χ_μ. In particular, the dimension $\deg \chi_{\lambda,\mu}$ of this module is the product $d_\lambda d_\mu$ where $d_\lambda = \deg \chi_\lambda$, $d_\mu = \deg \chi_\mu$. Taking into account multiplicities $m_{\lambda,\mu}$ in Eq. 3 we get the relation

$$c_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu. \tag{4}$$

A number of irreducible components in the decomposition of $\chi_{k,n-k}(L)$, i.e. the sum

$$l_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu}$$

is called the $(k, n-k)$-colength of L. If $\dim L < \infty$ then by Ado Theorem (see [10, Theorem 1.4.1]), L has a faithful finite dimensional graded representation. Hence L has an embedding $L \subset A = A_0 \oplus A_1$ as a Lie superalgebra where A is a finite dimensional associative superalgebra. Given $0 \le k \le n$, consider the graded $(k, n-k)$-cocharacter of A:

$$\chi_{k,n-k}(A) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} \overline{m}_{\lambda,\mu} \chi_{\lambda,\mu}.$$

Then by [16],

$$\sum_{k=0}^{n} \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} \overline{m}_{\lambda,\mu} \le q(n)$$

for some polynomial $q(n)$. Following the argument of the proof of [3, Lemma 3.2] we obtain that

$$m_{\lambda,\mu} \le \overline{m}_{\lambda,\mu}.$$

Hence in the finite dimensional case the total colength is polynomially bounded, that is, for any L, $\dim L < \infty$, there exists a polynomial $f(n)$ such that

$$\sum_{k=0}^{n} l_{k,n-k}(L) \le f(n).$$

It follows that

$$c_{k,n-k}(L) \le f(n) d_\lambda^{\max} d_\mu^{\max} \tag{5}$$

where $d_\lambda^{\max}$, $d_\mu^{\max}$ are maximal possible dimensions of S_k- and S_{n-k}-representations, respectively, such that $m_{\lambda,\mu} \ne 0$. We will use relation (5) for finding an upper bound for $\overline{exp}^{gr}(L)$.

3 Dimensions of some S_m-Representations

In this section we prove some technical results which we will use later. Fix an integer $t \ge 2$ and consider an irreducible S_m-representation with the character χ_μ, $\mu = (\mu_1, \ldots, \mu_d)$, $d \le t^2$. For convenience we will write $\mu = (\mu_1, \ldots, \mu_{t^2})$ even in the case $d < t^2$ assuming $\mu_{d+1} = \ldots = \mu_{t^2} = 0$.

We define the following function of a partition $\mu \vdash m$

$$\Phi(\mu) = \frac{1}{\left(\frac{\mu_1}{m}\right)^{\frac{\mu_1}{m}} \cdots \left(\frac{\mu_{t^2}}{m}\right)^{\frac{\mu_{t^2}}{m}}} \qquad (6)$$

In Eq. 6 we assume that $0^0 = 1$ if some of μ_j are equal to zero. The value of $\Phi(\mu)^m$ is equal to d_μ up to a polynomial factor. More precisely, we have the following relation:

Lemma 1 [9, Lemma 1] *Let $m \geq 100$. Then*

$$\frac{\Phi(\mu)^m}{m^{t^4+t^2}} \leq d_\mu \leq m\Phi(\mu)^m.$$

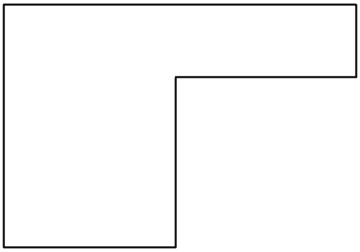
Now let λ and μ be two partitions of m with the corresponding Young diagrams D_λ, D_μ. We say that D_μ is obtained from D_λ by pushing down one box if there exist $1 \leq i < j \leq t^2$ such that $\mu_i = \lambda_i - 1$, $\mu_j = \lambda_j + 1$ and $\mu_k = \lambda_k$ for all remaining k.

Lemma 2 (see [9, Lemma 3], [17, Lemma 2]) *Let D_μ be obtained from D_λ by pushing down one box. Then $\Phi(\mu) \geq \Phi(\lambda)$.*

Now we define the weight of partition $\mu = (\mu_1, \ldots, \mu_{t^2})$ as follows:

$$wt\ \mu = -\left(\mu_1 + \cdots + \mu_{\frac{t^2-t}{2}}\right) + \left(\mu_{\frac{t^2-t}{2}+1} + \cdots + \mu_{t^2}\right).$$

Recall (see [1]) that the hook partition $h(d, l, k)$ is a partition with the Young diagram of the shape



Here the first d rows have length $l + k$ and remaining k rows have length l. We slightly modify this notion and say that a partition $\mu = (\mu_1, \ldots, \mu_{t^2}) \vdash m$ is a hook $h(s, r)$ if $\mu_1 = \ldots = \mu_{\frac{t^2-t}{2}} = s$ and $\mu_{\frac{t^2-t}{2}+1} = \cdots = \mu_{t^2} = r < s$.

The following observation is elementary.

Lemma 3 *Let m be a multiple of $t(t^2 - 1)$. Then there exists a hook partition $\mu = h(s, r)$ of m with $s = r\frac{t+1}{t-1}$ and $wt\ \mu = 0$.*

Proof Let $m = it(t^2 - 1)$. If we take $\mu = h(r, s)$ with $s = (t + 1)i$, $r = (t - 1)i$ then the number of boxes in the first $\frac{t^2-t}{2}$ rows, that is $\mu_1 + \cdots + \mu_{\frac{t^2-t}{2}}$, equals to

$$s\frac{t^2 - t}{2} = it\frac{(t - 1)(t + 1)}{2} = \frac{m}{2}.$$

Similarly, the number of boxes in all remaining rows of D_μ equals to

$$r\frac{t^2+t}{2} = it\frac{(t-1)(t+1)}{2} = \frac{m}{2}.$$

Hence $wt\ \mu = 0$ and we are done. □

Lemma 4 *Let m be a multiple of $t(t^2-1)$ and let $\mu = h(s,r)$ be the hook partition with zero weight as in Lemma 3. Then $\Phi(\mu) = t\sqrt{t^2-1}$.*

Proof Since

$$\frac{\mu_1}{m} = \ldots = \frac{\mu_{\frac{t^2-t}{2}}}{m} = \frac{s}{m}, \qquad \frac{\mu_{\frac{t^2-t}{2}+1}}{m} = \ldots = \frac{\mu_{t^2}}{m} = \frac{r}{m}$$

and $m = rt(t+1), s = r\frac{t+1}{t-1}$, we have

$$\frac{r}{m} = \frac{1}{t(t+1)}, \qquad \frac{s}{m} = \frac{1}{t(t-1)}.$$

Hence

$$\Phi(\mu) = \frac{1}{\left(\frac{1}{t(t+1)}\right)^{\frac{t^2+t}{2t(t+1)}}\left(\frac{1}{t(t-1)}\right)^{\frac{t^2-t}{2t(t-1)}}} = \left(t^2(t+1)(t-1)\right)^{\frac{1}{2}} = t\sqrt{t^2-1}.$$

□

For an arbitrary partition of weight zero we have the following.

Lemma 5 *Let m be a multiple of $t(t^2-1)$ and let ν be a partition of m with $wt\ \nu = 0$. Then $\Phi(\nu) \le t\sqrt{t^2-1}$.*

Proof The Young diagram D_ν of ν consists of two parts. The first one $\overline{\nu}$ contains first $\frac{t^2-t}{2}$ rows and the second part $\overline{\overline{\nu}}$ contains all remaining rows. Pushing down boxes inside $\overline{\nu}$ and $\overline{\overline{\nu}}$ separately we get new partition $\nu' \vdash m$ with $wt\ \nu' = 0$ maximally close to hook partition. That is, first $0 < i \le \frac{t^2-t}{2}$ rows of $D_{\nu'}$ have the length a and rows $i+1, \ldots, \frac{t^2-t}{2}$ (in case $i < \frac{t^2-t}{2}$) have the length $a-1$. Similarly,

$$\nu'_{\frac{t^2-t}{2}+1} = \ldots = \nu'_{\frac{t^2-t}{2}+j} = b, \quad \nu'_{\frac{t^2-t}{2}+j+1} = \ldots = \nu'_{t^2} = b-1$$

for some j. But under our assumption m admits a hook partition by Lemma 3, hence $\frac{m}{2}$ is a multiple of $\frac{t^2-t}{2}$. It follows that $i = \frac{t^2-t}{2}$. Similarly, $j = \frac{t^2+t}{2}$ and $\nu' = h(a,b)$. Finally note that if m admits a hook partition $\mu = h(r,s)$ of weight zero then μ is uniquely defined. Hence $a = b\frac{t+1}{t-1}$ and $\Phi(\nu') = t\sqrt{t^2-1}$ by Lemma 4. By applying Lemma 2 we complete the proof. □

The main goal of this section is to get a similar upper bound for $\Phi(\mu)$ for any $\mu \vdash m$ without any restriction on m and with $wt(\mu) \le 1$.

First we prove an easy technical result.

Lemma 6 *Let $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of n such that $\lambda_1 - \lambda_s \geq 2s$. Then by pushing down one or more boxes in D_λ one can get a partition $\mu = (\mu_1, \dots \mu_s) \vdash n$ with $\mu_s = \lambda_s + 1$ and $\mu_1 > \lambda_1 - s$. Similarly, one can get $\nu = (\nu_1, \dots \nu_s) \vdash n$ with $\nu_1 = \lambda_1 - 1$ and $\nu_s < \lambda_s + s$.*

Proof First we find μ. If $s = 2$ then the statement is obvious. Suppose $s > 2$. Then we push down boxes in D_λ using only rows $2, 3, \dots, s$. If we get on some step the diagram D_μ with $\mu_s = \lambda_s + 1$ then we proof is completed. Otherwise we will get a diagram $D_{\bar{\mu}}$ where $\bar{\mu}_1 = \lambda_1$, $\bar{\mu}_2 = \cdots = \bar{\mu}_t = p + 1$, $\bar{\mu}_{t+1} = \cdots = \bar{\mu}_s = p$ for some p and some $2 \leq t \leq s$. Moreover, $p = \lambda_s$ if $t < s$ or $p + 1 = \lambda_s$ if $t = s$. In this case we can cut $s - 1$ boxes from the first row of $D_{\bar{\mu}}$ and the glue one box to each row $2, \dots, s$ in $D_{\bar{\mu}}$. Then the partition $\mu = (\mu_1, \dots, \mu_s)$, $\mu_1 = \bar{\mu}_1 - s + 1$, $\mu_j = \bar{\mu}_j + 1$, $j = 2, \dots, s$, satisfies all conditions and we are done.

Similarly, if we push down boxes only in rows $1, \dots, s - 1$ in D_λ then either we will get a partition $\nu = (\nu_1, \dots, \nu_s)$ with $\nu_1 = \lambda_1 - 1$, $\nu_s = \lambda_s$ on some step or we will get a partition $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_s)$ such that$\bar{\nu}_1 = \cdots = \bar{\nu}_t = p + 1$, $\bar{\nu}_{t+1} = \cdots = \bar{\nu}_{s-1} = p$, $\bar{\nu}_s = \lambda_s$ for some $1 \leq t \leq s - 1$. In the latter case we push down one box from each row $1, \dots, s - 1$ to the last row of $D_{\bar{\nu}}$. Then we get the required $\nu \vdash n$ and the proof is completed. □

Now we consider partitions with t^2 components whose weight cannot be increased by pushing down boxes in the Young diagram.

Lemma 7 *Let $\mu = (\mu_1, \dots, \mu_{t^2})$ be a partition whose weight cannot be increased by pushing down boxes. Then $\mu_1 - \mu_{t^2} \leq 4t^2$ and $wt(\mu) \geq -2t^4$.*

Proof Denote $p = \frac{t^2+t}{2}$, $q = \frac{t^2-t}{2}$ for brevity. Clearly, $\mu_q \leq \mu_{q+1} + 1$. If $\mu_1 - \mu_q \geq 3q$ then by pushing down boxes we can get a partition $\mu' = (\mu'_1, \dots, \mu'_q)$ with $\mu'_q = \mu_q + 2$ by Lemma 6. Hence $\mu_1 - \mu_q < 3q$. Similarly, we can get $(\mu''_{q+1}, \dots, \mu''_{q+p})$ with $\mu''_{q+1} = \mu_{q+1} - 2$ provided that $\mu_{q+1} - \mu_{q+p} \geq 3p$. Therefore $\mu_{q+1} - \mu_{q+p} < 3p$. Finally we obtain

$$\mu_1 - \mu_{q+p} < 3p + 3q + 1 = 3t^2 + 1 < 4t^2.$$

For proving the second part of our lemma we split D_μ into two parts D_1 and D_2 where D_1 consists of the first q rows of D_μ while D_2 consists of the last p rows of D_μ. By our assumption we cannot cut one box from D_1 and glue it to D_2. Denote by a and b the number of boxes in D_1, D_2, respectively. Denote also $\mu_{p+q} = x$. By the first part of the lemma $\mu_1 \leq 4t^2 + x$. Hence $a \leq (4t^2 + x)q$. Obviously, $b \geq px$. Hence

$$wt(\mu) = b - a \geq x(p - q) - 4t^2 q \geq -4t^2 \frac{t^2 - t}{2} \geq -2t^4$$

and we complete the proof. □

Next lemma shows how to reduce this problem to the case $wt\ \mu = 0$ and $m = jt(t^2 - 1)$.

Lemma 8 *Let $\mu = (\mu_1, \ldots, \mu_{t^2})$ be a partition of m and let $wt\ \mu \le 1$. Then there exist an integer $m_0 \ge m$ and a partition $\nu \vdash m_0$ such that*

(1) $m_0 - m \le 6t^6$,
(2) $wt\ \nu = 0$,
(3) *m_0 is a multiple of $t^2(t-1)$,*
(4)

$$\Phi(\mu) \le (m + 6t^6)^{\left(\frac{t^4+t^2+2}{m}\right)^{6t^6}} \Phi(\nu).$$

Proof First we reduce the question to the case $wt\ \mu = 0$. If $wt\ \mu = 1$ then we can add one extra box to the first row of D_μ and get a partition of zero weight.

Let $wt(\mu) < 0$. By Lemmas 2 and 7 we can suppose that $wt(\mu) \ge -2t^4$. If we add one box to each of rows $1, 2, \ldots, t^2 - t + 1$ of D_μ we get the Young diagram D_ρ of partition $\rho \vdash m + t^2 - t + 1$ with $wt(\rho) = wt(\mu) + 1$. Applying this procedure at most $2t^4$ times we get $\rho' \vdash m_0'$ with $wt(\rho') = 0$ where

$$m_0' \le m + (t^2 - t + 1) \cdot 2t^4 \le m + 4t^6.$$

If m_0' is a multiple of $t(t^2-1)$ then there is nothing to do. Otherwise there exists $0 < i < t(t^2-1)$ such that $m_0' + i$ is a multiple of $t(t^2-1)$. Note that m_0' is even since it admits a partition of weight zero. Hence i is also even.

First we enlarge $D_{\rho'}$ to $D_{\mu'}$ by adding $\frac{t^2-1}{2}$ boxes to all $t^2 - t$ first rows. Then also $wt\ \mu' = 0$. Since $\mu'_{t^2-t} - \mu'_{t^2-t+1} \ge \frac{t^2-1}{2}$, we can glue $\frac{i}{2} < t\frac{t^2-1}{2}$ boxes to the last t rows of $D_{\mu'}$ and get $D_{\mu''}$. Finally, we glue $\frac{i}{2}$ boxes to the first row of $D_{\mu''}$ and obtain the diagram D_ν such that $wt\ \nu = 0$. Denote by m_0 the number of boxes of D_ν. As follows from our procedure, an upper bound for m_0 is

$$m + 4t^6 + (t^2 - t)\frac{t^2-1}{2} + t(t^2-1) < 6t^6 + m.$$

It is shown in [17, Lemma 7] that if $\lambda \vdash n-1, \lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda' \vdash n, \lambda' = (\lambda'_1, \ldots, \lambda'_d)$ and D_λ is obtained from $D_{\lambda'}$ by cutting one box then

$$\Phi(\lambda) \le n^{\frac{d^2+d+2}{n}} \Phi(\lambda').$$

Hence

$$\Phi(\mu) \le (m + 6t^6)^{(\frac{t^4+t^2+2}{m})^{6t^6}} \Phi(\nu)$$

and we complete the proof. □

As a corollary of Lemmas 5 and 8 we immediately obtain

Lemma 9 *Let $\mu = (\mu_1, \ldots, \mu_{t^2})$ be a partition of m and let $wt\ \mu \le 1$. Then there exists a polynomial $g(m)$ such that $\Phi(\mu) \le g(m)^{\frac{1}{m}} t\sqrt{t^2-1}$.*

4 Graded Codimensions of Lie Superalgebras of Type $b(t)$

In this section we use notations from [10]. Recall that $L = b(t), t \geq 2$, is a Lie superalgebra of $2t \times 2t$ matrices of the type

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where $A, B, C \in M_t(F)$, $B^T = B$, $C^T = -C$ and $tr A = 0$. Here the map $X \to X^T$ is the transpose involution. Decomposition $L = L_0 \oplus L_1$ is defined by setting

$$L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \mid A \in M_t(F), tr(A) = 0 \right\},$$

and

$$L_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B^T = B, C^T = -C \in M_t(F) \right\}.$$

Super-Lie product on L is given by

$$[x, y] = xy - (-1)^{|x||y|} yx$$

for homogeneous $x, y \in L_0 \cup L_1$.

It is not difficult to see that also L has $\mathbb{Z}$-grading

$$L = L^{(-1)} \oplus L^{(0)} \oplus L^{(1)} \tag{7}$$

where $L^{(0)} = L_0$,

$$L^{(-1)} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C^T = -C \in M_t(F) \right\}, \tag{8}$$

$$L^{(1)} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B^T = B \in M_t(F) \right\} \tag{9}$$

and $L^{(n)} = 0$ for all $n \neq 0, \pm 1$. In particular, $L^{(-1)} \oplus L^{(1)} = L_1$ and $\dim L^{(0)} = t^2 - 1$, $\dim L^{(-1)} = \frac{t(t-1)}{2}$, $\dim L^{(1)} = \frac{t(t+1)}{2}$.

Let $\chi_{k,n-k}(L)$ be $(k, n-k)$-cocharacter of L. Consider its decomposition (3) into irreducible components.

Lemma 10 *Let $m_{\lambda,\mu} \neq 0$ in Eq. 3. Then D_λ lies in the strip of width $t^2 - 1$, that is, $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $d \leq t^2 - 1$. In particular, $d_\lambda \leq \alpha(k)(t^2 - 1)^k$ for some polynomial $\alpha(k)$.*

Proof Denote $A = FS_k$. Recall that, given a partition $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash k$, the irreducible S_k-module corresponding to λ is isomorphic to the minimal left ideal generated by an essential idempotent e_{T_λ} constructed in the following way.

Let T_λ be Young tableau, that is Young diagram D_λ filled up by integers $1, \dots, k$. Denote by R_{T_λ} and C_{T_λ} row and column stabilizers in S_k of T_λ, respectively. Then

$$R(T_\lambda) = \sum_{\sigma \in R_{T_\lambda}} \sigma \ , \quad C(T_\lambda) = \sum_{\tau \in C_{T_\lambda}} (\text{sgn}\tau)\tau$$

and

$$e_{T_\lambda} = R(T_\lambda)C(T_\lambda).$$

It is known that $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, $0 \neq \alpha \in \mathbb{Q}$, and an irreducible FS_k-module M has the character χ_λ if and only if $e_{T_\lambda} M \neq 0$. In particular, if M is an irreducible $FS_k \times FS_{n-k}$-submodule in $P_{k,n-k}(L)$ with the character $\chi_{\lambda,\mu}$ then M can be generated by a multilinear polynomial of the type $e_{T_\lambda}\varphi(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ with even $x_1, \dots, x_k$ and odd $y_1, \dots, y_{n-k}$ (since M is the direct sum of isomorphic irreducible S_k-modules with characters χ_λ). From the relation $e_{T_\lambda}^2 = \alpha e_{T_\lambda} \neq 0$ it follows that the polynomial

$$\psi(x_1, \dots, x_k, y_1, \dots, y_{n-k}) = C(T_\lambda)e_{T_\lambda}\varphi(x_1, \dots, x_k, y_1, \dots, y_{n-k})$$

also generates M.

Suppose now that $d > t^2 - 1$. Then D_λ contains at least one column of height d greater than $t^2 - 1 = \dim L_0$. In this case ψ depends on at least one alternating set of even variables of order greater than $\dim L_0$. Standard arguments show that in this case ψ is an identity of L, a contradiction. Hence $d \leq t^2 - 1$. Now by [1, Lemma 6.2.5] there exists a polynomial $\alpha(k)$ such that $d_\lambda \leq \alpha(k)(t^2 - 1)^k$ and we complete the proof. □

Lemma 11 *Let $m_{\lambda,\mu} \neq 0$ in Eq. 3. Then $wt\ \mu \leq 1$.*

Proof As in the previous lemma an irreducible $FS_k \times FS_{n-k}$-submodule M of $P_{k,n-k}(L)$ with the character $\chi_{\lambda,\mu}$ can be generated by

$$\psi = \psi(x_1, \dots, x_k, y_1, \dots, y_{n-k}) = C(T_\mu)e_{T_\mu}\varphi$$

for some multilinear polynomial φ. The set of variables $\{y_1, \dots, y_{n-k}\}$ can be split into disjoint union

$$\{y_1, \dots, y_{n-k}\} = Y_1 \cup \ldots \cup Y_p$$

where $p = \mu_1$, every set Y_j consists of odd indeterminates with the indices from the j-th column of T_μ. In particular, ψ is alternating on any subset Y_j, $1 \leq j \leq p$, and we cannot substitute the same basis elements of L instead of distinct variables from the same column of T_μ, otherwise the value of ψ will be zero. Hence the minimal degree in $\mathbb{Z}$-grading Eqs. 7, 8 and 9 of the value of ψ on L is equal to $q = wt\ \mu$. So, if $q > 1$ then ψ is an identity of L since $L^{(q)} \oplus L^{(q+1)} \oplus \cdots = 0$, a contradiction. □

Now we are ready to prove the main result of the paper.

Theorem 1 *Let L be a Lie superalgebra of the type $b(t), t \geq 2$, over a field F of characteristic zero. Then there exists a polynomial $h = h(n)$ such that*

$$c_n^{gr}(L) \leq h(n) \left(t^2 - 1 + t\sqrt{t^2 - 1}\right)^n.$$

In particular,

$$\overline{exp}^{gr}(L) \leq t^2 - 1 + t\sqrt{t^2 - 1} < 2t^2 - 1 = \dim L.$$

Proof Consider the inequality (5) for $c_{k,n-k}(L)$. By Lemma 10, $d_\lambda^{\max} \leq \alpha(k)(t^2-1)^k$ and by Lemma 11 we have $wt\ \mu \leq 1$ where $d_\mu = d_\mu^{\max}$. Then by Lemmas 1 and 9,

$$d_\mu^{\max} \leq (n-k)g(n-k)\left(t\sqrt{t^2-1}\right)^{n-k}.$$

Hence

$$c_{k,n-k}(L) \leq f(n)(n-k)\alpha(k)g(n-k)(t^2-1)^k\left(t\sqrt{t^2-1}\right)^{n-k}.$$

Clearly one can take a polynomial $h' = h'(n)$ such that $\alpha(k)g(n-k) \leq h'(n)$ for all $k = 0, \ldots, n$. Then

$$c_{k,n-k}(L) \leq h(n)(t^2-1)^k\left(t\sqrt{t^2-1}\right)^{n-k}$$

where $h(n) = nf(n)h'(n)$. Now by Eq. 2

$$c_n^{gr}(L) \leq h(n)\sum_{k=0}^{n}\binom{n}{k}(t^2-1)^k\left(t\sqrt{t^2-1}\right)^{n-k} = h(n)\left(t^2-1+t\sqrt{t^2-1}\right)^n.$$

Obviously,

$$\overline{exp}^{gr}(L) = \limsup_{n\to\infty}\sqrt[n]{c_n^{gr}(L)} \leq t^2 - 1 + t\sqrt{t^2-1}$$

and we complete the proof of Theorem 1. □

As a consequence of Theorem 1 we get an upper bound for ordinary PI-exponent of $L = b(t), t \geq 2$.

Theorem 2 *Let L be a Lie superalgebra of the type* $L = b(t)$, $t \geq 2$, *over a field of characteristic zero. Then* $exp(L) \leq t^2 - 1 + t\sqrt{t^2-1}$.

Proof The statement easily follows from Theorem 1, the inequality $c_n^{gr} \leq c_n(L)$ [7, 14] and from the existence of $exp(L)$ [9]. □

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