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ON IDENTITIES OF INFINITE DIMENSIONAL LIE SUPERALGEBRAS

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Abstract. We study codimension growth of infinite dimensional Lie superalgebras over an algebraically closed field of characteristic zero. We prove that if a Lie superalgebra L is a Grassmann envelope of a finite dimensional simple Lie algebra, then the PI-exponent of L exists and is a positive integer.

1. Introduction

We shall consider algebras over a field F of characteristic zero. One of the approaches in the investigation of associative and non-associative algebras is to study numerical invariants associated with their identical relations. Given an algebra A, we can associate to it the sequence of its codimensions $\{c_n(A)\}_{n\in\mathbb{N}}$ (all notions and definitions will be given in the next section).

This sequence gives some information not only about the identities of A but also about the structure of A. For example, A is nilpotent if and only if $c_n(A) = 0$ for all large enough n. If A is an associative non-nilpotent F-algebra, then A is commutative if and only if $c_n(A) = 1$ for all $n \geq 1$.

For an associative algebra A with a non-trivial polynomial identity, the sequence $c_n(A)$ is exponentially bounded by the celebrated Regev Theorem [20], while $c_n(A) = n!$ if A does not satisfy any non-trivial polynomial identity. In the non-associative case the sequence of codimensions may have even faster growth. For example, if A is an absolutely free algebra, then

$$c_n(A) = a_n n!$$

where

$$a_n = \frac{1}{2}\binom{2n-2}{n-1}$$

is the Catalan number, i.e. the number of all possible arrangements of brackets in the word of length n.

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For a Lie algebra L the sequence $\{c_n(L)\}_{n\in\mathbb{N}}$ is in general not exponentially bounded, even if L satisfies non-trivial Lie identities (see for example [18]). Nevertheless, a class of Lie algebras with exponentially bounded codimensions is sufficiently wide. It includes, in particular, all finite dimensional algebras [1, 11], Kac-Moody algebras [23, 24], infinite dimensional simple Lie algebras of Cartan type [15], Virasoro algebra, and many others.

In the case when $\{c_n(A)\}_{n\in\mathbb{N}}$ is exponentially bounded, the upper and lower limits of the sequence $\{\sqrt[n]{c_n(A)}\}_{n\in\mathbb{N}}$ exist and a natural question arises: does the ordinary limit

$$\lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exist? In the case of existence we call this limit $exp(A)$ or the PI-exponent of A.

Amitsur conjectured in the 1980's that for any associative P.I. algebra such a limit exists and that it is a non-negative integer. This conjecture was confirmed first for verbally prime P.I. algebras in [4, 21] and later for the general case in [8, 9]. For Lie algebras a series of positive results was obtained for finite dimensional algebras [6, 7, 25], for algebras with nilpotent commutator subalgebras [17], for affine Kac-Moody algebras [23, 24], and for some other classes (see [16]). For Lie superalgebras there exist only partial results [26, 27, 30, 31].

On the other hand, it was shown in [28] that there exists a Lie algebra L with

$$3.1 < \liminf_{n\to\infty} \sqrt[n]{c_n(L)} \le \limsup_{n\to\infty} \sqrt[n]{c_n(L)} < 3.9.$$

This algebra L is soluble and almost nilpotent; i.e. it contains a nilpotent ideal of finite codimension. In the general non-associative case there exists, for any real number $\alpha > 1$, an algebra A_α such that

$$\lim_{n\to\infty} \sqrt[n]{c_n(A_\alpha)} = \alpha$$

(see [5]). Note also that by a recent result [12] there exist finite dimensional Lie superalgebras with a fractional limit $\sqrt[n]{c_n(L)}$.

In the present paper we shall study Grassmann envelopes of finite dimensional simple Lie algebras. Our main result is the following theorem:

Theorem 1. *Let $L_0 \oplus L_1$ be a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic zero with some $\mathbb{Z}_2$-grading. Also, let $\widetilde{L} = L_0 \otimes G_0 \oplus L_1 \otimes G_1$ be the Grassmann envelope of L. Then the limit*

$$exp(\widetilde{L}) = \lim_{n\to\infty} \sqrt[n]{c_n(\widetilde{L})}$$

exists and is a positive integer. Moreover, $exp(\widetilde{L}) = \dim L$.

Another result of our paper concerns graded identities. Since any Lie superalgebra L is $\mathbb{Z}_2$-graded, one can consider $\mathbb{Z}_2$-graded identities of L and the corresponding graded codimensions $c_n^{gr}(L)$. We shall prove that graded codimensions have similar properties.

Theorem 2. *Let $L = L_0 \oplus L_1$ be a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic zero with some $\mathbb{Z}_2$-grading. Also, let $\widetilde{L} = L_0 \otimes G_0 \oplus L_1 \otimes G_1$ be a Grassmann envelope of L. Then the limit*

$$exp^{gr}(\widetilde{L}) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(\widetilde{L})}$$

exists and is a non-negative integer. Moreover, $exp^{gr}(\widetilde{L}) = \dim L$.

In other words, both PI-exponent $exp(\widetilde{L})$ and graded PI-exponent $exp^{gr}(\widetilde{L})$ exist, they are integers, and they coincide. Note that for an arbitrary $\mathbb{Z}_2$-graded algebra the growth of ordinary codimensions and graded codimensions may differ. For example, if $A = M_k(F) \otimes F\mathbb{Z}_2$ with the canonical $\mathbb{Z}_2$-grading induced from group algebra $F\mathbb{Z}_2$, where $M_k(F)$ is a full $k \times k$ matrix algebra, then $exp(A) = k^2$, while $exp^{gr}(A) = 2k^2$ (see [10] for details). In the Lie case one can take $L = L_0 \oplus L_1$ to be a two dimensional metabelian algebra with $L_0 = \langle e\rangle, L_1 = \langle f\rangle$ and with only one non-trivial product $[e, f] = f$. Then $c_n(L) = n - 1$ for all $n \geq 2$; hence $exp(L) = 1$. On the other hand, $exp^{gr}(L) = 2$.

2. The main constructions and definitions

Let A be an arbitrary non-associative algebra over a field F and let $F\{X\}$ be an absolutely free F-algebra with a countable generating set X. A polynomial $f = f(x_1, \ldots, x_n)$ is said to be an identity of A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. The set of all identities of L forms a T-ideal $Id(A)$ in $F\{X\}$, that is, an ideal which is stable under all endomorphisms of $F\{X\}$. Denote by $P_n = P_n(x_1, \ldots, x_n)$ the subspace of all multilinear polynomials on $x_1, \ldots, x_n$ in $F\{X\}$. Then $P_n \cap Id(A)$ is a subspace of all multilinear identities of A of degree n. In the case when char $F = 0$, the T-ideal $Id(A)$ is completely determined by the subspaces $\{P_n \cap Id(A)\}, n = 1, 2, \ldots$.

For estimating how many identities an algebra A can have, one can define the so-called n-th codimension of the identities of A or, for short, the codimension of A:

$$c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)},\ n = 1, 2, \ldots\ .$$

As was mentioned above, the class of associative and non-associative algebras with exponentially bounded sequence $\{c_n(A)\}$ is sufficiently wide. In the case when $c_n(A) < a^n$ for some real a, one can define the lower and the upper PI-exponents of A as follows:

$$\underline{exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

and the ordinary PI-exponent as follows:

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \tag{1}$$

provided that $\underline{exp}(A) = \overline{exp}(A)$.

For $\mathbb{Z}_2$-graded algebras one can also consider graded identities. Let X and Y be two infinite sets of variables and let $F\{X \cup Y\}$ be an absolutely free algebra generated by $X \cup Y$. If we suppose that all elements of X are even and all elements of Y are odd, i.e. $\deg(x) = 0, \deg(y) = 1$ for any $x \in X, y \in Y$, then $F\{X \cup Y\}$ can be naturally endowed by a $\mathbb{Z}_2$-grading. A polynomial $f = f(x_1, \dots, x_m, y_1, \dots, y_n) \in F\{X \cup Y\}$ is said to be a graded identity of a superalgebra $A = A_0 \oplus A_1$ if $f(a_1, \dots, a_m, b_1, \dots, b_n) = 0$, for all $a_1, \dots, a_m \in A_0, b_1, \dots, b_n \in A_1$. Fix $0 \le k \le n$ and denote by $P_{k,n-k}$ the subspace of $F\{X \cup Y\}$ spanned by all multilinear polynomials in $x_1, \dots, x_k \in X$, $y_1, \dots, y_{n-k} \in Y$. Then $P_{k,n-k} \cap Id(A)$ is the set of all multilinear polynomial identities of the superalgebra $A = A_0 \oplus A_1$ in k even and $n-k$ odd variables.

One of the equivalent definitions of graded codimensions of A is

$$c_n^{gr}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(A),$$

where

$$c_{k,n-k}(A) = \dim \frac{P_{k,n-k}}{P_{k,n-k} \cap Id(A)}.$$

Starting from a $\mathbb{Z}_2$-graded algebra of some class (Lie, Jordan alternative, etc.), one can construct a $\mathbb{Z}_2$-graded algebra of a different class using the notion of a Grassmann envelope – they play an exceptional role in PI-theory. For example, any variety of associative algebras is generated by the Grassmann envelope of some finite dimensional associative superalgebra [14]. In the Lie case any so-called special variety is generated by the Grassmann envelope of a finitely generated Lie superalgebra [22].

We recall this construction for the Lie and the super Lie cases. Let G be the Grassmann algebra generated by 1 and the infinite set $\{e_1, e_2, \dots\}$ satisfying the following relations: $e_i e_j = -e_j e_i, i, j = 1, 2, \dots$. It is known that G has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$, where

$$G_0 = Span\langle e_{i_1} \cdots e_{i_n} | n = 2k, k = 0, 1, \dots \rangle,$$

$$G_1 = Span\langle e_{i_1} \cdots e_{i_n} | n = 2k+1, k = 0, 1, \dots \rangle.$$

Given a Lie algebra L with a $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$, its Grassmann envelope

$$G(L) = L_0 \otimes G_0 \oplus L_1 \otimes G_1 \subset L \otimes G$$

is a Lie superalgebra. Vice versa, if $L = L_0 \oplus L_1$ is a Lie superalgebra, then $G(L)$ is an ordinary Lie algebra with a $\mathbb{Z}_2$-grading.

3. Cocharacters of Grassmann envelopes

The main tool in studying codimension asymptotics is the representation theory of symmetric groups. We refer the reader to [13] for details. The symmetric group S_n acts naturally on multilinear polynomials in $F\{X\}$ as

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \tag{2}$$

Hence P_n is an FS_n-module and $P_n \cap Id(L)$, and also

$$P_n(L) = \frac{P_n}{P_n \cap Id(L)}$$

are FS_n-modules. The S_n-character $\chi(P_n(L))$ is called the n-th cocharacter of L and we shall write

$$\chi_n(L) = \chi(P_n(L)).$$

Recall that any irreducible FS_n-module corresponds to a partition λ of n, $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \cdots \geq \lambda_k$ are positive integers and $\lambda_1 + \cdots + \lambda_k = n$. By the Maschke Theorem, any finite dimensional FS_n-module M decomposes into a direct sum of irreducible components, and hence its character $\chi(M)$ has a decomposition

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

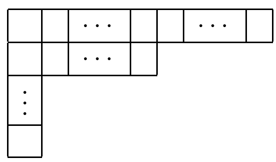
where m_λ are non-negative integers. In particular, for the algebra L we have

$$\chi(L) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{3}$$

Integers m_λ in (3) are called multiplicities of χ_λ in $\chi_n(L)$, and $d_\lambda = \deg \chi_\lambda = \chi_\lambda(1)$ are the dimensions of the corresponding irreducible representations. Therefore

$$c_n(L) = \dim P_n(L) = \sum_{\lambda \vdash n} m_\lambda d_\lambda. \tag{4}$$

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ one can construct the Young diagram D_λ containing λ_1 boxes in the first row, λ_2 boxes in the second row, and so on:

$D_\lambda =$ 

Given integers $k, l, d \geq 0$, we define the partition

$$h(k,l,d) = (\underbrace{l+d, \ldots, l+d}_{k}, \underbrace{l, \ldots, l}_{d})$$

of $n = kl + d(k+l)$. The Young diagram associated with $h(k,l,d)$ is hook-shaped, and we define $H(k,l)$, an infinite hook, as the union of all D_λ with $\lambda = h(k,l,d)$, $d = 1, 2, \ldots$. For short we shall say that a partition $\lambda \vdash n$ lies in the hook $H(k,l)$, $\lambda \in H(k,l)$, if $D_\lambda \subset H(k,l)$. In other words, $\lambda \in H(k,l)$ if $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\lambda_{k+1} \leq l$. According to this definition we shall say that the cocharacter of L lies in the hook $H(k,l)$ if $m_\lambda = 0$ in (3) as soon as $\lambda \notin H(k,l)$.

A special case of $H(k,l)$ is an infinite strip $H(k,0)$. In this case $\lambda \in H(k,0)$ if $\lambda_{k+1} = 0$.

The following fact is well-known, and we state it without proof.

Lemma 1. *Let L be a finite dimensional algebra,* $\dim L = d < \infty$. *Then $\chi_n(L)$ lies in the hook $H(d,0)$ for all $n \geq 1$.* □

Another important numerical invariant of the identities of L is the colength $l_n(L)$. By definition

$$l_n(L) = \sum_{\lambda \vdash n} m_\lambda, \tag{5}$$

where m_λ are taken from (3). It easily follows from (4) and (5) that

$$\max\{d_\lambda | m_\lambda \neq 0\} \leq c_n(L) \leq l_n(L) \cdot \max\{d_\lambda | m_\lambda \neq 0\}. \tag{6}$$

For studying graded identities of $L = L_0 \oplus L_1$ we need to act separately on even and odd variables. More precisely, the space $P_{k,n-k} = P_{k,n-k}(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ is an $S_k \times S_{n-k}$-module where symmetric groups S_k, S_{n-k} act on $x_1, \dots, x_k$ and $y_1, \dots, y_{n-k}$, respectively. Any irreducible $S_k \times S_{n-k}$-module is a tensor product of an S_k-module and an S_{n-k}-module and corresponds to the pair λ, μ of partitions, $\lambda \vdash k, \mu \vdash n-k$. As before, the subspace $P_{n-k} \cap Id(L)$ is an $S_k \times S_{n-k}$-stable subspace, and one can consider the quotient

$$P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap Id(L)}$$

as an $S_k \times S_{n-k}$-module. Its $S_k \times S_{n-k}$-character $\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L))$ is decomposed into irreducible components,

$$\chi_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{7}$$

and we define the $(k, n-k)$-colength of L as

$$l_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu}$$

with $m_{\lambda,\mu}$ taken from (7).

First, we prove some relations between graded and non-graded numerical invariants. We begin by recalling the correspondence between multilinear homogeneous polynomials in a free $\mathbb{Z}_2$-graded Lie algebra and in a free Lie superalgebra. Let $f = f(x_1, \dots, x_k, y_1, \dots, y_m)$ be a non-associative polynomial multilinear on $x_1, \dots, x_k,\ y_1, \dots, y_m$, where $x_1, \dots, x_k$ are supposed to be even indeterminates and $y_1, \dots, y_m$ are supposed to be odd. Then f is a linear combination of monomials from $P_{k,m}$. Let $M = M(x_1, \dots, x_k, y_1, \dots, y_m)$ be such a monomial. We fix positions of $y_1, \dots, y_m$ in M and write M for short in the following form:

$$M = X_0 y_{\sigma(1)} X_1 \cdots X_{m-1} y_{\sigma(m)} X_m,$$

where $X_0, \dots, X_m$ are some words (possibly empty) consisting of left and right brackets and indeterminates $x_1, \dots, x_k$. Now we define a monomial $\widetilde{M}$ on even indeterminates $x_1, \dots, x_k$ and odd indeterminates $y_1, \dots, y_m$ from the free Lie superalgebra as

$$\widetilde{M} = \mathrm{sgn}(\sigma) X_0 y_{\sigma(1)} X_1 \cdots X_{m-1} y_{\sigma(m)} X_m.$$

Extending this map $\tilde{}$ by linearity, we obtain a linear isomorphism $P_{k,m} \to P_{k,m}$ of two subspaces of a $\mathbb{Z}_2$-graded free Lie algebra and a free Lie superalgebra, respectively. Although the monomials in $P_{k,m}$ are not linearly independent, it easily follows from the Jacobi and the super-Jacobi identities that the map $\tilde{}$ is well-defined. Similarly, we can define the inverse map from a free Lie superalgebra to a free $\mathbb{Z}_2$-graded Lie algebra.

Following the same argument as in the associative case (see [10, Lemma 3.4.7]), we obtain the following result for any $\mathbb{Z}_2$-graded Lie algebra L and its Grassmann envelope $G(L) = G_0 \otimes L_0 \oplus G_1 \otimes L_1$.

Lemma 2. *Let $f \in P_{k,m}$ be a multilinear polynomial in a free Lie algebra L. Then*

- *f is a graded identity of L if and only if $\widetilde{f}$ is a graded identity of $G(L)$ and*
- *$\widetilde{\widetilde{f}} = f$.* □

The next lemma is an obvious generalization of Lemma 1.

Lemma 3. *Let $L = L_0 \oplus L_1$ be a finite dimensional Lie algebra,* $\dim L_0 = k, \dim L_1 = l$, *and let*

$$\chi_{q,n-q}(L) = \sum_{\substack{\lambda \vdash q \\ \mu \vdash n-q}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

be its $(q, n-q)$-graded cocharacter. If $m_{\lambda,\mu} \neq 0$, then $\lambda \in H(k,0)$ and $\mu \in H(l,0)$. $\square$

Using this remark we restrict the shape of the graded cocharacter of the Grassmann envelope $G(L)$.

Lemma 4. *Let $L = L_0 \oplus L_1$ be a finite dimensional Lie algebra,* $\dim L_0 = k, \dim L_1 = l$, *and let $\widetilde{L}$ be its Grassmann envelope. If*

$$\chi_{q,n-q}(\widetilde{L}) = \sum_{\substack{\lambda \vdash q \\ \mu \vdash n-q}} m_{\lambda,\mu}\chi_{\lambda,\mu} \tag{8}$$

and $m_{\lambda,\mu} \neq 0$ in (8), then $\lambda \in H(k,0)$ and $\mu \in H(0,l)$.

Proof. Suppose $m_{\lambda,\mu} \neq 0$ in (8) for some $\lambda \vdash q, \mu \vdash n-q$. Then there exists a multilinear polynomial $g = g(x_1, \dots, x_q, y_1, \dots, y_{n-q})$ such that

$$f = e_{T_\lambda} e_{T_\mu} g(x_1, \dots, y_{n-q})$$

is not a graded identity of $\widetilde{L}$, where $e_{T_\lambda} \in FS_q, e_{T_\mu} \in FS_{n-q}$ are essential idempotents generating minimal left ideals in FS_q, FS_{n-q}, respectively. Inclusion $\lambda \in H(k,0)$ immediately follows by Lemma 3 since L and $G(L)$ have the same cocharacters on even indeterminates. Since e_{T_λ} and e_{T_μ} commute, applying Lemma 4.8.6 from [10] we get

$$\widetilde{f} = a e_{T_\lambda} g,$$

where $a \in I_{\mu'}$. Here μ' is the conjugated to μ partition of $n-q$ and $I_{\mu'}$ is the minimal two-sided ideal of FS_{n-q} generated by $e_{T_{\mu'}}$. That is, $I_{\mu'}$ has the character $r \cdot \chi_{\mu'}$, where $r = d_{\mu'} = \deg \mu'$.

By Lemma 2, $\widetilde{f}$ is not a graded identity of $G(\widetilde{L})$. Since $\widetilde{\widetilde{h}} = h$ for any $h \in P_{q,n-q}$, we see that $\widetilde{f}$ is not a graded identity of L and $\mu' \in H(l,0)$ by Lemma 3. In other words, the number of rows of the Young diagram $D_{\mu'}$ does not exceed l. This number equals the number of columns of D_μ; hence $\mu \in H(0,l)$, and we are done. $\square$

Using the previous lemma we restrict the shape of the non-graded cocharacter of $G(L)$.

Lemma 5. *Let $L = L_0 \oplus L_1$ be a finite dimensional Lie algebra,* $\dim L_0 = k, \dim L_1 = l$, *and let*

$$\chi(\widetilde{L}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

be the n-th (non-graded) cocharacter of $\widetilde{L} = G(L)$. Then $m_\lambda \neq 0$ only if $\lambda \in H(k,l)$.

Proof. Suppose $f \in P_n$ is not an identity of $\widetilde{L}$. Since f is multilinear we may assume that $f(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}) \in P_{q,n-q}$ is not an identity of $\widetilde{L}$ for some $0 \le q \le n$. Moreover, we can consider only the case when a graded polynomial f generates in $P_{q,n-q}$ an irreducible $S_q \times S_{n-q}$-submodule M with the character (χ_λ, χ_μ), $\lambda \vdash q, \mu \vdash n-q$.

Now we lift the $S_q \times S_{n-q}$-action up to an S_n-action and consider a decomposition of FS_nM into irreducible components:

$$\chi(FS_nM) = \sum_{\nu \vdash n} m_\nu \chi_\nu.$$

Since by Lemma 4 λ lies in the hook $H(k,0)$, i.e. the horizontal strip of height k, and μ lies in $H(0,l)$, the vertical strip of width l, it follows from the Littlewood-Richardson rule for induced representations ([13, 2.8.13]; see also [10, Thm. 2.3.9]) that $m_\nu = 0$ as soon as $\nu \notin H(k,l)$, and we have completed the proof. ☐

Lemma 6. *Let $G(L) = \widetilde{L} = \widetilde{L}_0 \oplus \widetilde{L}_1$ be the Grassmann envelope of a finite dimensional Lie algebra $L = L_0 \oplus L_1$ with $\dim L_0 = k, \dim L_1 = l$. Then its colength sequence $\{l_n(\widetilde{L})\}$ is polynomially bounded.*

Proof. We use the notation $\{z_1, z_2, \ldots\}$ for non-graded indeterminates here since $\{x_1, x_2, \ldots\}$ were even variables in the previous statements.

Let

$$\chi(\widetilde{L}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{9}$$

be the n-th cocharacter of $\widetilde{L}$. By Lemma 5 we have $\lambda \in H(k,l)$ as soon as $m_\lambda \neq 0$ in (9). Fix $\lambda \vdash n$ with $m_\lambda = m \neq 0$ and consider the FS_n-submodule

$$W_1 \oplus \cdots \oplus W_m \subseteq P_n(\widetilde{L}) \tag{10}$$

with $\chi(W_i) = \chi_\lambda$, for all $i = 1, \ldots, m$.

We shall prove that

$$m \le (k+l)2^{2kl} n^{k^2+l^2} \tag{11}$$

in (10). Denote by $\lambda_1', \ldots, \lambda_l'$ the heights of the first l columns of the Young diagram D_λ. Clearly, it suffices to prove inequality (11) only for λ with $\lambda_k > l$ and $\lambda_l' > k$. Otherwise, $\lambda \in H(k', l')$ with $k' \le k, l' \le l$ and $k' + l' < k + l$.

Denote

$$\mu_1 = \lambda_1' - k, \ldots, \mu_l = \lambda_l' - k.$$

Then $\lambda_1 + \cdots + \lambda_k + \mu_1 + \cdots + \mu_l = n$.

It is well-known (see, for example, [29]) that one can choose multilinear $f_1 \in W_1, \ldots, f_m \in W_m$ such that $FS_nf_1 = W_1, \ldots, FS_nf_m = W_m$ and each $f_i, i = 1, \ldots, m$, is symmetric on k sets of indeterminates of orders $\lambda_1, \ldots, \lambda_k$ and is alternating on l sets of orders $\mu_1, \ldots, \mu_l$.

According to this decomposition into symmetric and alternating sets we rename $z_1, \ldots, z_n$ as follows:

$$\{z_1, \ldots, z_n\} = \{z_1^1, \ldots, z_{\lambda_1}^1, \ldots, z_1^k, \ldots, z_{\lambda_k}^k, \bar{z}_1^1, \ldots, \bar{z}_{\mu_1}^1, \ldots, \bar{z}_1^l, \ldots, \bar{z}_{\mu_l}^l\}, \tag{12}$$

where each f_i is symmetric on any set $\{z_1^j, \ldots, z_{\lambda_j}^j\}$, $j = 1, \ldots, k$, and is alternating on any set $\{\bar{z}_1^s, \ldots, \bar{z}_{\mu_s}^s\}$, $s = 1, \ldots, l$.

We shall find $\delta_1, \ldots, \delta_m \in F$ such that

$$f = \delta_1 f_1 + \cdots + \delta_m f_m$$

is an identity of $\widetilde{L}$ if (11) does not hold. Note that for any $\delta_1, \ldots, \delta_m \in F$ a polynomial f is also symmetric on each subset $\{z_1^i, \ldots, z_{\lambda_i}^i\}, 1 \leq i \leq k$, and alternating on each subset $\{\bar{z}_1^s, \ldots, \bar{z}_{\mu_s}^s\}$, $s = 1, \ldots, l$.

Let $E = \{e_1, \ldots, e_{k+l}\}$ be a homogeneous basis of L with $E_0 = \{e_1, \ldots, e_k\} \subset L_0$, $E_1 = \{e_{k+1}, \ldots, e_{k+l}\} \subset L_1$. Then f is an identity of $\widetilde{L}$ if and only if $\varphi(f) = 0$ for any evaluation $\varphi : Z \to \widetilde{L}$ such that $\varphi(z_i) = g_i \otimes a_i, 1 \leq i \leq n$, where a_i is a basis element from E and $g_i \in G$ has the same parity as a_i and $g_1 \cdots g_n \neq 0$ in G.

Note also that $\varphi(f) = 0$ implies $\varphi'(f) = 0$ for any evaluation φ' such that $\varphi'(z_i) = g_i' \otimes a_i, 1 \leq i \leq n$, provided that $g_1 \cdots g_n \neq 0$.

Using these two remarks we shall find an upper bound for the number of evaluations for asking the question whether f is an identity of $\widetilde{L}$ or not.

First consider one symmetric subset $Z_1 = \{z_1^1, \ldots, z_{\lambda_1}^1\}$. If $\varphi(z_i^1) = g \otimes e, \varphi(z_j^1) = h \otimes e$, for some $i \neq j$ with $e \in E_1$, then $\varphi(f) = 0$, as follows from the symmetry on Z_1. Hence we need to check only evaluations with at most $r \leq l$ odd values $\varphi(z_{i_1}^1) = g_1 \otimes e_{t_1}, \ldots, \varphi(z_{i_r}^1) = g_r \otimes e_{t_r}$, where $e_{t_1}, \ldots, e_{t_r} \in E_1$ are distinct. Since Z_1 is the symmetric set of variables, the result of evaluation φ does not depend (up to the sign) on the choice of $i_1, \ldots, i_r$. Hence we have $\binom{l}{r}$ possibilities.

Given $0 \leq r \leq l$, we estimate the number of evaluations of remaining $\lambda_1 - r$ variables in the even component of $\widetilde{L}$. First, let $r = 0$ and $\varphi(z_i^1) = g_i \otimes a_i, a_i \in E_0, 1 \leq i \leq \lambda_1$. If e_1 appears in the row $(a_1, \ldots, a_{\lambda_1})$ exactly α_1 times, e_2 appears α_2 times, and so on, then the result of such substitution depends only on $\alpha_1, \ldots, \alpha_k$ since f is symmetric on Z_1. Hence we have no more than $(\lambda_1 + 1)^k$ variants since $0 \leq \alpha_1, \ldots, \alpha_k \leq \lambda_1$. In particular, we need at most $(n+1)^k$ evaluations if $r = 0$.

Now let $r = 1$. We can replace by an odd element an arbitrary variable from Z_1 and get (up to the sign) the same value $\varphi(f)$ since f is symmetric on Z_1. Suppose, say, that $\varphi(z_{\lambda_1}^1) = h \otimes e, e \in E_1$, and $\varphi(z_1^1) = g_1 \otimes a_1, \ldots, \varphi(z_{\lambda_1 - 1}^1) = g_{\lambda_1 - 1} \otimes a_{\lambda_1 - 1}$, where all a_j are even. If $\alpha_1, \ldots, \alpha_k$ are the same integers as in the case $r = 0$, then the result of the substitution also depends only on $\alpha_1, \ldots, \alpha_k$. Hence for $r = 1$ we have at most

$$\binom{l}{1} \lambda_1^k \leq \binom{l}{1} (n+1)^k$$

variants for φ since $0 \leq \alpha_1, \ldots, \alpha_k \leq \lambda_1 - 1$.

Similarly, for general $0 \leq r \leq l$ we have at most

$$\binom{l}{r} (\lambda_1 + 1 - r)^k \leq \binom{l}{r} (n+1)^k$$

variants. Therefore, for evaluating all variables from Z_1 it suffices that

$$\sum_{r=0}^{l} \binom{l}{r} (n+1)^k = 2^l (n+1)^k$$

substitutions and for all symmetric variables we need at most

$$(2^l (n+1)^k)^k$$

substitutions.

Now consider the alternating set $Z_1' = \{\bar{z}_1^1, \ldots, \bar{z}_{\mu_1}^1\}$. If $\varphi(\bar{z}_i^1) = g \otimes e, \varphi(\bar{z}_j^1) = h \otimes e$, for some $i \neq j$ with the same $e \in E_0$, then $\varphi(f) = 0$. Hence we can choose only $0 \le r \le k$ distinct basis elements $b_1, \ldots, b_r \in E_0$ for values of $\bar{z}^1_{i_1}, \ldots, \bar{z}^1_{i_r}$ of the type $g_i \otimes b_i$. Up to the sign, the result of the substitution does not depend on $i_1, \ldots, i_r$, and we have only $\binom{k}{r}$ options.

Suppose now that all $\varphi(\bar{z}^1_i), 1 \le i \le r$, are fixed even values. Let

$$\varphi(\bar{z}^1_{r+1}) = g_1 \otimes b_1, \ldots, \varphi(\bar{z}^1_{\mu_1}) = g_{\mu_1 - r} \otimes b_{\mu_1 - r}, \quad b_1 \ldots, b_{\mu_1 - r} \in E_1.$$

Then (up to the sign) the result of φ depends only on the number of entries of $e_{k+1}, \ldots, e_{k+l}$ into the row $(b_1, \ldots, b_{\mu_1 - r})$. Hence we have at most $(\mu_1 - r + 1)^l$ variants for the substitution of odd variables. As in the symmetric case we have the following upper bound:

$$\sum_{r=0}^{k} \binom{k}{r} (n+1)^l = 2^k (n+1)^l$$

for one subset and $(2^k(n+1)^l)^l$ for all skew variables.

We have proved that one can find $T \le 2^{kl}(n+1)^{l^2+k^2}$ evaluations $\varphi_1, \ldots, \varphi_T$ such that the relations

$$\varphi_1(f) = \cdots = \varphi_T(f) = 0 \tag{13}$$

imply $\varphi(f) = 0$ for any evaluation φ; that is, f is an identity of $\widetilde{L}$. Recall that $f = \delta_1 f_1 + \cdots + \delta_m f_m$. Therefore for any evaluation φ the equality $\varphi(f) = 0$ can be viewed as a system of $k + l$ homogeneous linear equations in the algebra $\widetilde{L}$ on unknown coefficients $\delta_1, \ldots, \delta_m$. If (11) does not hold, then the system (13) has a non-trivial solution $\bar{\delta}_1, \ldots, \bar{\delta}_m$, and $f = \bar{\delta}_1 f_1 + \cdots + \bar{\delta}_m f_m$ is an identity of $\widetilde{L}$, a contradiction.

We have proved the inequality (11). From this inequality it follows that all multiplicities in (9) are bounded by $(k+l)2^{2kl}n^{k^2+l^2}$. Finally, note that the number of partitions $\lambda \in H(k,l)$ is bounded by n^{k+l}. Hence

$$l_n(\widetilde{L}) < (k+l)2^{2kl}n^{k^2+l^2+kl},$$

and we have thus completed the proof. □

As a corollary of previous results we obtain the following:

Proposition 1. *Let $L = L_0 \oplus L_1$ be a finite dimensional $\mathbb{Z}_2$-graded Lie algebra with $\dim L_0 = k, \dim L_1 = l$ and let $\widetilde{L} = G(L)$ be its Grassmann envelope. Then there exist constants $\alpha, \beta \in \mathbb{R}$ such that*

$$c_n(\widetilde{L}) \le \alpha n^{\beta} (k+l)^n.$$

In particular,

$$\overline{exp}(\widetilde{L}) = \limsup_{n \to \infty} \sqrt[n]{c_n(\widetilde{L})} \le k + l.$$

Proof. By [10, Lemma 6.2.5], there exist constants C and r such that

$$\sum_{\lambda \in H(k,l)} d_\lambda \le C n^r (k+l)^n$$

for all $n = 1, 2, \ldots$. In particular,

$$\max\{d_\lambda | \lambda \vdash n, \lambda \in H(k,l)\} \le C n^r (k+l)^n.$$

Now Lemma 6 and the inequality (6) complete the proof. □

4. Existence of PI-exponents

Proposition 2. *Let L be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic zero with some $\mathbb{Z}_2$-grading, $L = L_0 \oplus L_1$, $\dim L_0 = k, \dim L_1 = l$. Also let $\widetilde{L} = G(L)$ be its Grassmann envelope. Then there exist constants $\gamma > 0, \delta \in \mathbb{R}$ such that*

$$c_n(\widetilde{L}) \geq \gamma n^\delta (k+l)^n.$$

In particular,

$$\underline{exp}(\widetilde{L}) = \liminf_{n\to\infty} \sqrt[n]{c_n(\widetilde{L})} \geq k+l.$$

Proof. Denote $d = k + l = \dim L$. By [19, Theorem 12.1], for the adjoint representation of L there exists a multilinear asssociative polynomial $h = h(u_1^1, \dots, u_d^1, \dots, u_1^m, \dots, u_d^m)$ alternating on each subset of indeterminates $\{u_1^i, \dots, u_d^i\}$, $1 \leq i \leq m$, such that under any evaluation $\varphi : u_j^i \to ad\ b_j^i, b_j^i \in L$, the value $\varphi(h)$ is a scalar linear transformation of L and $\varphi(h) \neq 0$ for some h. It follows that for any integer $t \geq 1$ there exists a multilinear Lie polynomial

$$f_t = f_t(u_1^1, \dots, u_d^1, \dots, u_1^{mt}, \dots, u_d^{mt}, w)$$

alternating on each set $\{u_1^i, \dots, u_d^i\}$, $1 \leq i \leq mt$, such that $\varphi(f_t) \neq 0$ for some evaluation $\varphi : \{u_1^1, \dots, u_d^{mt}, w\} \to L_0 \cup L_1$. Since f_t is multilinear and alternating on each set $\{u_1^i, \dots, u_d^i\}$ and $d = \dim L_0 + \dim L_1$, it follows that for any $t \geq 1$ we get a graded multilinear polynomial

$$f_t = f_t(x_1^1, \dots, x_k^1, \dots, x_1^{mt}, \dots, x_k^{mt}, y_1^1, \dots, y_l^1, \dots, y_1^{mt}, \dots, y_l^{mt}, w)$$

which is not a graded identity of L and is alternating on each subset $\{x_1^i, \dots, x_k^i\}$ and on each subset $\{y_1^i, \dots, y_l^i\}$, $1 \leq i \leq mt$, where x_j^i's are even and y_j^i's are odd variables. The latter indeterminate w can be taken of arbitrary parity; say, $w = x_0$ is even.

Consider an $S_p \times S_q$-action on

$$P_{p+1,q} = P_{p+1,q}(x_0, x_1^1, \dots, x_k^{mt}, y_1^1, \dots, y_l^{mt}),$$

where $p = mtk, q = mtl$ and S_p, S_q act on $\{x_j^i\}$, $\{y_j^i\}$, respectively. It follows from Lemma 3 that the $S_p \times S_q$-character of the submodule generated by f in $P_{p+1,q}$ lies in the pair of strips $H(k, 0)$, $H(l, 0)$, that is,

$$\chi(F[S_p \times F_q]f) = \sum_{\substack{\lambda\vdash p\\ \mu\vdash q}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

with $m_{\lambda,\mu} = 0$, unless $\lambda \in H(k, 0), \mu \in H(l, 0)$. Hence λ is a partition of mtk with at most k rows. On the other hand, f depends on mt alternating subsets of even indeterminates of order k each. It is well-known that in this case $m_{\lambda,\mu} = 0$ if $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\lambda_1 \geq mt + 1$. It follows that only the rectangular partition

$$\lambda = (\underbrace{mt, \dots, mt}_{k}) \tag{14}$$

can appear in $F[S_p \times F_q]f$ with non-zero multiplicity. Similarly,

$$\mu = (\underbrace{mt, \dots, mt}_{l}) \tag{15}$$

if $m_{\lambda,\mu} \neq 0$. Hence we can assume that f has the form

$$f = e_{T_\lambda} e_{T_\mu} g(x_1^1, \ldots, y_l^{mt}, w)$$

with λ and μ of the types (14), (15), respectively.

By Lemma 2, the polynomial $\widetilde{f}$ is not an identity of the Lie superalgebra $\widetilde{L} = G(L)$, and by Lemma 4.8.6 from [10], the graded polynomial $\widetilde{f}$ generates in $P_{p+1,q}(\widetilde{L})$ an irreducible $S_p \times S_q$-submodule with the character $(\chi_\lambda, \chi_{\mu'})$, where

$$\mu' = (\underbrace{l, \ldots, l}_{mt})$$

is conjugated to a μ partition of mtl.

First we apply the Littlewood-Richardson rule and induce this $S_p \times S_q$-module up to an S_n-module. Then we induce the obtained S_n-module up to an S_{n+1}-module, where $n = p + q = mt(k + l)$. It follows from the Littlewood-Richardson rule that the induced S_{n+1}-module can contain only a simple submodule corresponding to partitions $\nu \vdash n + 1$ such that the Young diagram D_ν contains a subdiagram D_{ν_0}, where

$$\nu_0 = h(k, l, t_0) = (\underbrace{l + t_0, \ldots, l + t_0}_{k}, \underbrace{l, \ldots, l}_{t_0})$$

is a finite hook with $t_0 \geq l - k, mt - kl$. Since we are interested in an asymptotic of codimensions, we may assume that $mt - kl > l - k$ and then $t_0 = mt - kl$. In particular, ν_0 is a partition of $n_0 = (k+l)t_0 + kl$. Then $n+1-n_0 = (k+l-1)kl+1$, and by [10, Lemma 6.2.4]

$$d_{\nu_0} \leq d_\nu \leq n^c d_{\nu_0},$$

where $c = (k + l - 1)kl + 1$ and

$$d_{h(k,l,t_0)} \simeq a n_0^b (k + l)^{n_0} \quad \text{if } n_0 \to \infty$$

for some constants a, b by Lemma 6.2.5 from [10]. Here the relation $f(n) \simeq g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. Since $c_{n+1}(\widetilde{L}) \geq d_\nu$ we get the inequality

$$c_{n+1}(\widetilde{L}) \geq \alpha(n + 1)^\beta (k + l)^{n+1} \tag{16}$$

for all $n = m(k + l)t$, $t = 1, 2, \ldots$, for some constants $\alpha > 0$ and β.

Since the Lie algebra L is simple, the Grassmann envelope $\widetilde{L}$ is a centerless Lie superalgebra. It is not difficult to see that in this case $c_{r+1}(\widetilde{L}) \geq c_r(\widetilde{L})$, for all $r \geq 1$. Hence by (16) we have

$$c_{n+j}(\widetilde{L}) \geq \alpha(n + 1)^\beta (k + l)^{n+1}$$

for any $1 \leq j \leq m(k + l)$. Since $n = m(k + l)t$ one can find constants $\gamma > 0$ and δ such that

$$c_r(\widetilde{L}) \geq \gamma r^\delta (k + l)^r$$

for all positive integers r, and we have completed the proof. □

Theorem 1 now easily follows from Propositions 1 and 2.

Proof of Theorem 2. First we obtain an upper bound for $c_n^{gr}(\widetilde{L})$:

$$c_n^{gr}(\widetilde{L}) = \sum_{q=0}^{n} \binom{n}{q} c_{q,n-q}(\widetilde{L}),$$

where

$$c_{q,n-q}(\widetilde{L}) = \sum_{\substack{\lambda \vdash q \\ \mu \vdash n-q}} m_{\lambda,\mu} d_{\lambda,\mu} \tag{17}$$

and $d_{\lambda,\mu} = \deg \chi_{\lambda,\mu} = \deg \chi_\lambda \cdot \deg \chi_\mu = d_\lambda d_\mu$. Moreover, $\lambda \in H(k,0)$, $\mu \in H(0,l)$ by Lemma 4. Applying Lemma 6.2.5 from [10], we obtain

$$\sum_{\substack{\lambda \in H(k,0) \\ \lambda \vdash q}} d_\lambda \leq C n^r k^q, \qquad \sum_{\substack{\mu \in H(0,l) \\ \mu \vdash n-q}} d_\mu \leq C n^r l^{n-q}$$

for some constants C, r, and hence

$$\sum_{\substack{\lambda \in H(k,0), \lambda \vdash q \\ \mu \in H(0,l), \mu \vdash n-q}} d_\lambda d_\mu \leq C^2 n^{2r} k^q l^{n-q}. \tag{18}$$

On the other hand, the graded colength

$$l_{q,n-q}(\widetilde{L}) = \sum_{\substack{\lambda \vdash q \\ \mu \vdash n-q}} m_{\lambda,\mu}$$

is not greater than the non-graded colength $l_n(\widetilde{L})$. Since $l_n(\widetilde{L})$ is polynomially bounded by Lemma 6, one can find a polynomial $\varphi(n)$ such that

$$m_{\lambda,\mu} \leq \varphi(n) \tag{19}$$

for any $m_{\lambda,\mu}$ in (17). It now follows from (17), (18) and (19) that for $\psi(n) = C^2 n^{2r} \varphi(n)$ we have

$$c_n^{gr}(\widetilde{L}) \leq \psi(n) \sum_{q=1}^{n} \binom{n}{q} k^q l^{n-q} = \psi(n)(k+l)^n, \tag{20}$$

and we have obtained an upper bound for $c_n^{gr}((\widetilde{L}))$.

On the other hand, it was proved in [2, Lemma 3.1] that for any associative G-graded algebra A, where G is a finite group, an ordinary n-th codimension is less than or equal to the graded n-th codimension, for any n. Proof of this lemma does not use associativity. Hence

$$c_n^{gr}(\widetilde{L}) \geq c_n(\widetilde{L}), \tag{21}$$

and Theorem 2 now follows from (20), (21) and Proposition 2. $\square$

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