Bull. Aust. Math. Soc. **94** (2016), 326–336
doi:10.1017/S0004972716000228

ON GEHRING–MARTIN–TAN GROUPS WITH AN ELLIPTIC GENERATOR

DUŠAN REPOVŠ✉ and ANDREI VESNIN

(Received 30 October 2015; accepted 27 February 2016; first published online 12 May 2016)

Abstract

The Gehring–Martin–Tan inequality for two-generator subgroups of PSL(2, $\mathbb{C}$) is one of the best known discreteness conditions. A Kleinian group G is called a Gehring–Martin–Tan group if the equality holds for the group G. We give a method for constructing Gehring–Martin–Tan groups with a generator of order four and present some examples. These groups arise as groups of finite-volume hyperbolic 3-orbifolds.

2010 *Mathematics subject classification*: primary 57M50; secondary 20H15.

Keywords and phrases: hyperbolic orbifold, discreteness condition, two-generated group.

1. Introduction

In this paper we are interested in discreteness conditions for groups of isometries of a hyperbolic 3-space $\mathbb{H}^3$. It was shown by Jørgensen that it suffices to understand the discreteness problem for a class of two-generated groups. The most famous necessary discreteness conditions are the Shimizu lemma and the Jørgensen inequality [2]. There are some generalisations of these conditions for complex and quaternionic hyperbolic spaces (see, for example, [5, 8, 15]).

A description of the set of all two-generated discrete nonelementary groups of isometries $\mathbb{H}^3$ for which the equality holds in the Jørgensen inequality is an open problem of special interest. For many elegant results concerning this problem, see [4, 7, 17] and references therein.

We shall consider a different discreteness condition for two-generated groups which was independently proved by Gehring and Martin [7] and Tan [18]. Like the Jørgensen inequality, this condition takes the form of an inequality involving the trace of one of the generators and the trace of the commutator of generators. We shall say that a group

This research was supported in part by the Slovenian Research Agency (ARRS) grant BI-RU/14-15-001. The first author was supported in part by the ARRS grants P1-0292-0101, J1-7025-0101 and J1-6721-0101. The second author was supported in part by the RFBR grant 16-01-00414 and the Laboratory of Quantum Topology, Chelyabinsk State University contract no. 14.Z50.31.0020 with the Ministry of Education and Science of the Russian Federation.

is *a GMT-group* if it can be generated by two elements for which the equality holds. The following problem arises naturally.

PROBLEM 1.1. *Find all GMT-groups.*

The problem is still open. In this paper we shall present a method to construct new examples of GMT-groups from known examples.

The most interesting example of a GMT-group is a group related to the well-known figure-eight knot. Denote by $\mathcal{F}(n)$ the orbifold with the underlying space S^3 and singular set the figure-eight knot $\mathcal{F}$ with singularity index n, $n \geq 4$. According to [1], the orbifolds $\mathcal{F}(n)$ are extreme in the following sense: let L_n denote the set of all orientable hyperbolic 3-orbifolds with nonempty singular set and with all torsion orders bounded below by n. Therefore,

$$L_2 \supset L_3 \supset L_4 \supset \cdots \quad \text{and} \quad \cap L_n = \emptyset.$$

Then, for all $n \geq 4$, the unique lowest-volume element of L_n is the orbifold $\mathcal{F}(n)$. A formula for $\operatorname{vol} \mathcal{F}(n)$ was given in [21]. It was shown in [19] that $\mathcal{F}(4)$ is extreme in the sense of discreteness conditions: the orbifold group of $\mathcal{F}(4)$ is a GMT-group. Below we shall use the orbifold group of $\mathcal{F}(4)$ as a starting point for constructions of new examples of GMT-groups.

In Section 2 we shall give basic definitions and describe some properties of GMT-groups. In particular, we shall prove Lemma 2.4, which gives a method for constructing new GMT-groups. Next, we shall apply this method. In Section 3 we shall prove that some 3-orbifold hyperbolic groups related to the figure-eight knot are GMT-groups. In Section 4 we shall give examples of GMT-groups which are subgroups of the Picard group.

2. Gehring–Martin–Tan discreteness condition

Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space presented by the Poincaré model in the upper halfspace. Then the boundary $\partial\mathbb{H}^3$ can be identified with $\overline{\mathbb{C}}$. It is well known that the group $\mathrm{Iso}(\mathbb{H}^3)$ of all orientation-preserving isometries of $\mathbb{H}^3$ is isomorphic to

$$\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{I}\},$$

where I denotes the unit matrix. In the sequel we shall not distinguish between a matrix $M \in \mathrm{SL}(2, \mathbb{C})$ and its equivalence class $\{\pm M\} \in \mathrm{PSL}(2, \mathbb{C})$. An action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

on

$$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C}, t \in \mathbb{R}_+\}$$

is defined by the following rule:

$$g(z, t) = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2 t^2}, \frac{t}{|cz + d|^2 + |c|^2 t^2} \right).$$

Recall that a matrix $M \in \mathrm{SL}(2, \mathbb{C}) \backslash \{\pm \mathrm{I}\}$ is said to be:

- *elliptic* if $\mathrm{tr}^2(M) \in [0,4)$;
- *parabolic* if $\mathrm{tr}^2(M) = 4$; and
- *loxodromic* if $\mathrm{tr}^2(M) \in \mathbb{C}\setminus[0,4]$.

In particular, a loxodromic element is said to be:

- *hyperbolic* if $\mathrm{tr}(M) \in (-\infty,0) \cup (2,+\infty)$.

We shall say that an element of the group $\mathrm{PSL}(2,\mathbb{C})$ is *elliptic*, *parabolic* or *loxodromic* if its representative in $\mathrm{SL}(2,\mathbb{C})$ is of such type.

A group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be *discrete* if it is a discrete set in the matrix quotient topology. A group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be *elementary* if there exists a finite G-orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$, and *nonelementary* otherwise.

In 1977, Jørgensen proved in [9] that a nonelementary group $G < \mathrm{PSL}(2,\mathbb{C})$ is discrete if and only if any two elements $f, g \in G$ generate a discrete group. His result motivated many other investigations of discreteness conditions for two-generated groups. In the present paper we shall discuss the necessary discreteness condition obtained in 1989 by Gehring and Martin [7] and independently by Tan [18]. We formulate their result as follows.

Theorem 2.1 [7, 18]. *Suppose that $f, g \in \mathrm{PSL}(2,\mathbb{C})$ generate a discrete group. If $\mathrm{tr}[f,g] \neq 1$, then the following inequality holds:*

$$|\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f,g] - 1| \geq 1. \tag{2.1}$$

This result makes the following definitions natural. For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ such that $\mathrm{tr}[f,g] \neq 1$, define

$$\mathcal{G}(f,g) = |\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f,g] - 1|.$$

Let $G < \mathrm{PSL}(2,\mathbb{C})$ be a two-generated group. The value

$$\mathcal{G}(G) = \inf_{\langle f,g\rangle = G} \mathcal{G}(f,g)$$

is referred to as the *Gehring–Martin–Tan number* (or, shortly, the *GMT-number*) of G. A two-generated discrete group $G < \mathrm{PSL}(2,\mathbb{C})$ is said to be a *GMT-group* if it can be generated by f and g such that $\mathcal{G}(f,g) = 1$.

The following statement shows (see also [20]) that the property $\mathcal{G}(f,g) = 1$ implies many restrictions on f.

Lemma 2.2. *Suppose that $f, g \in \mathrm{PSL}(2,\mathbb{C})$ generate a discrete group and $\mathrm{tr}[f,g] \neq 1$. Assume that f is one of the following transformations:*

(i) *parabolic;*
(ii) *hyperbolic;*
(iii) *elliptic of order 2 or 3; or*
(iv) *elliptic with trace $\mathrm{tr}^2(f) = 4\cos^2(\pi k/n)$, where $(n,k) = 1$, $n/k \geq 6$.*

Then, for any g, $\mathcal{G}(f,g) > 1$.

PROOF. The result follows immediately from the classification of elements of $\mathrm{PSL}(2,\mathbb{C})$ and from the fact that $\mathcal{G}(f,g)$ is defined for pairs f,g such that $\mathrm{tr}[f,g] \neq 1$.

(i) If f is parabolic, then $\mathrm{tr}^2(f) = 4$ and therefore $|\mathrm{tr}^2(f) - 2| = 2 > 1$.
(ii) If f is hyperbolic, then $\mathrm{tr}(f) \in (-\infty, 0) \cup (2, \infty)$, so $|\mathrm{tr}^2(f) - 2| > 2$.
(iii) If f is elliptic of order 2, then $\mathrm{tr}^2(f) = 0$, so $|\mathrm{tr}^2(f) - 2| = 2$. If f is elliptic of order 3, then $\mathrm{tr}^2(f) = 1$, so $|\mathrm{tr}^2(f) - 2| = 1$. Since $\mathrm{tr}[f,g] \neq 1$, we get $\mathcal{G}(f,g) > 1$.
(iv) If f is elliptic with trace $\mathrm{tr}^2(f) = 4\cos^2(\pi k/n)$, where $(n,k) = 1$ and $n/k \geq 6$, then $\mathrm{tr}^2(f) \geq 4\cos^2(\pi/6) = 3$, so $|\mathrm{tr}^2(f) - 2| \geq 1$. Since $\mathrm{tr}[f,g] \neq 1$, $\mathcal{G}(f,g) > 1$. □

The following statement gives a way to find GMT-subgroups of GMT-groups with a generator of order four.

LEMMA 2.3 [20]. *Let $\langle f,g\rangle$ be a GMT-group with $\mathcal{G}(f,g) = 1$, where f is elliptic of order four. Then a group generated by f and $h = gfg^{-1}$ is a GMT-group with $\mathcal{G}(f,h) = 1$.*

The following statement gives a method for constructing GMT-groups as extensions of GMT-groups with a generator of order four.

LEMMA 2.4. *Let $\langle f,g\rangle$ be a GMT-group with $\mathcal{G}(f,g) = 1$, where f is elliptic of order four. Assume that $h \in \mathrm{PSL}(2,\mathbb{C})$ is an involution of $\langle f,g\rangle$ with one of the following conjugation actions:*

(i) $hfh^{-1} = g$;
(ii) $hfh^{-1} = f^{-1}$; *or*
(iii) $hfh^{-1} = fg^{-1}f^{-1}$.

Then $\langle f,h\rangle$ is a GMT-group.

PROOF. Since f is elliptic of order four, we have $\mathrm{tr}^2(f) = 2$ and hence $\mathcal{G}(f,g) = |\mathrm{tr}[f,g] - 1|$ and $\mathcal{G}(f,h) = |\mathrm{tr}[f,h] - 1|$. Since h is an involution, it follows that either $\langle f,g\rangle$ is a subgroup of index 2 in $\langle f,h\rangle$, or $\langle f,g\rangle$ and $\langle f,h\rangle$ coincide. Therefore, $\langle f,h\rangle$ is discrete. Recall that by [2] the identify

$$\mathrm{tr}[f,hfh^{-1}] = (\mathrm{tr}[f,h] - 2)\left(\mathrm{tr}[f,h] - \mathrm{tr}^2(f) + 2\right) + 2 \tag{2.2}$$

holds for any $f,h \in \mathrm{PSL}(2,\mathbb{C})$. From (2.2), using $\mathrm{tr}^2(f) = 2$,

$$\mathrm{tr}[f,hfh^{-1}] = (\mathrm{tr}[f,h] - 2)\mathrm{tr}[f,h] + 2.$$

Hence,

$$|\mathrm{tr}[f,hfh^{-1}] - 1| = |\mathrm{tr}[f,h] - 1|^2. \tag{2.3}$$

Consider the case (i) with $hfh^{-1} = g$. Then

$$|\mathrm{tr}[f,hfh^{-1}] - 1| = |\mathrm{tr}[f,g] - 1| = \mathcal{G}(f,g) = 1.$$

Therefore, $\mathcal{G}(f,h) = 1$.

Consider the case (ii) with $hfh^{-1} = f^{-1}$. Then

$$|\mathrm{tr}[f,hfh^{-1}] - 1| = |\mathrm{tr}[f,f^{-1}] - 1| = 1.$$

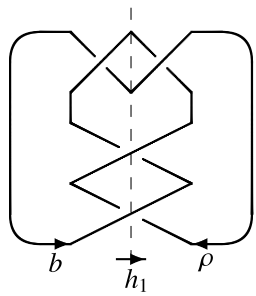

Figure 1. Generators of the group $\pi_1(S^3 \backslash \mathcal{F})$.

Therefore, $\mathcal{G}(f,h) = 1$.

Consider the case (iii) with $hfh^{-1} = fg^{-1}f^{-1}$. Then

$$\begin{aligned}|\mathrm{tr}[f, hfh^{-1}] - 1| &= |\mathrm{tr}[f, fg^{-1}f^{-1}] - 1| = |\mathrm{tr}(fg^{-1}f^{-1}g) - 1| \\ &= |\mathrm{tr}([f,g]^{-1}) - 1| = |\mathrm{tr}[f,g] - 1| = 1.\end{aligned}$$

Here we used the relation $\mathrm{tr}\,(\alpha^{-1}) = \mathrm{tr}\,\alpha$ for $\alpha \in \mathrm{SL}(2,\mathbb{C})$. Therefore, $\mathcal{G}(f,h) = 1$. □

In the next section we shall realise a method based on Lemma 2.4.

3. The figure-eight knot and related orbifolds

Let us denote by $\mathcal{F}$ the figure-eight knot in the 3-sphere S^3 presented by its diagram in Figure 1. The knot group $\pi_1(S^3 \backslash \mathcal{F})$ can be easily found by the Wirtinger algorithm. Taking generators b and ρ, the corresponding loops are marked in Figure 1. Thus,

$$\pi_1(S^3 \backslash \mathcal{F}) = \langle \rho, b \mid \rho^{-1}\,[b,\rho] = [b,\rho]\,b \rangle,$$

where $[b,\rho] = b\rho b^{-1}\rho^{-1}$. It is well known [16] that the group $\pi_1(S^3 \backslash \mathcal{F})$ has a faithful representation in $\mathrm{PSL}(2,\mathbb{C})$.

Denote by $\mathcal{F}(n)$ the orbifold with the underlying space S^3 and singular set $\mathcal{F}$ with singularity index n, where $n \geq 4$. Cyclic n-fold coverings of $\mathcal{F}(n)$ are known as *Fibonacci manifolds* (see [13, 22] for their interesting properties). We call $\mathcal{F}(n)$ *the figure-eight orbifold*. Denote its orbifold group by $\Gamma_n = \pi^{\mathrm{orb}}\mathcal{F}(n)$. The group Γ_n has the following presentation:

$$\Gamma_n = \langle \rho_n, b_n \mid \rho_n^n = b_n^n = 1,\ \rho_n^{-1}\,[b_n,\rho_n] = [b_n,\rho_n]\,b_n \rangle,$$

where generators ρ_n and b_n correspond to loops $\rho, b \in \pi_1(S^3 \backslash \mathcal{F})$. It is well known that for $n \geq 4$ the group Γ_n has a faithful representation in $\mathrm{PSL}(2,\mathbb{C})$. According to [14], this representation is defined on the generators by

$$\rho_n = \begin{pmatrix} \cos(\pi/n) & ie^{d_n/2}\sin(\pi/n) \\ ie^{-d_n/2}\sin(\pi/n) & \cos(\pi/n) \end{pmatrix}, \quad b_n = \begin{pmatrix} \cos(\pi/n) & ie^{-d_n/2}\sin(\pi/n) \\ ie^{d_n/2}\sin(\pi/n) & \cos(\pi/n) \end{pmatrix}. \quad (3.1)$$

The quantity d_n, defined as the complex distance between the axis of f_n and the axis of g_n, is given by

$$\cosh d_n = \frac{1}{4}\Big(1 + \cot^2(\pi/n) - i\sqrt{3\cot^4(\pi/n) + 14\cot^2(\pi/n) - 5}\Big).$$

The image of Γ_n under this representation is a nonelementary discrete group. In what follows, we shall not distinguish between the group Γ_n and its image under the faithful representation.

GMT-numbers of the figure-eight knot group and of the figure-eight orbifold groups were studied in [19]. It was shown that $\mathcal{G}(\pi_1(S^3\setminus\mathcal{F})) = 3$ and the following result was obtained.

THEOREM 3.1 (See [19]). *Let $n \geq 4$. Then the following inequalities hold for the figure-eight orbifold groups:*

$$1 \leq \mathcal{G}(\Gamma_n) \leq 3 - 4\sin^2\frac{\pi}{n}.$$

By writing the above inequalities for $n = 4$, we immediately get the following result.

COROLLARY 3.2 (See [19]). *The figure-eight orbifold group Γ_4 is a GMT-group.*

This result can be checked directly. Indeed, by (3.1), we have $\mathrm{tr}^2(\rho_n) = 4\cos^2(\pi/n)$, so $|\mathrm{tr}^2(\rho_4) - 2| = 0$. Also, by [19, Lemma 1], for any $\lambda \in \mathbb{R}$,

$$|\mathrm{tr}[\rho_n, b_n] - \lambda| = \sqrt{(\lambda^2 - 3\lambda + 3) + 4(\lambda - 1)\sin^2(\pi/n)}$$

and this gives $|\mathrm{tr}[\rho_n, b_n] - 1| = 1$ for any n. Hence, $\mathcal{G}(\rho_n, b_n) = 1$.

4. Quotient orbifolds of the figure-eight orbifold

LEMMA 4.1. *The figure-eight orbifold group Γ_n, $n \geq 4$, has involutions of types (i), (ii) and (iii) from Lemma 2.4.*

PROOF. Being one of the simplest knots, the figure-eight knot has been intensively studied. In 1914, Dehn [6] demonstrated that the figure-eight group Γ has eight outer automorphisms forming the dihedral group

$$\langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1\rangle,$$

where

$$\begin{cases} \sigma(\rho) = b, \\ \sigma(b) = \rho \end{cases} \quad \text{and} \quad \begin{cases} \tau(\rho) = \rho b \rho^{-1}, \\ \tau(b) = b^{-1}\rho b. \end{cases}$$

Later, in 1931, Magnus [11] proved that Γ has no other outer automorphisms. It is obvious that the actions

$$\begin{cases} \sigma(\rho_n) = b_n, \\ \sigma(b_n) = \rho_n \end{cases} \quad \text{and} \quad \begin{cases} \tau(\rho_n) = \rho_n b_n \rho_n^{-1}, \\ \tau(b_n) = b_n^{-1}\rho_n b_n \end{cases}$$

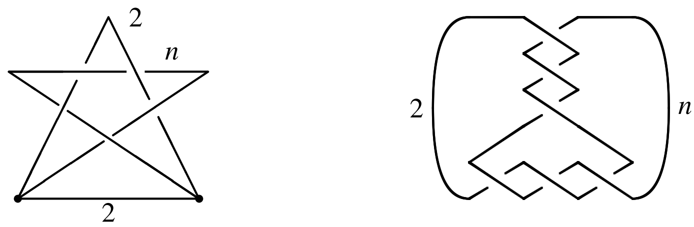

FIGURE 2. Singular sets of $O_1(n)$ and $O_2(n)$.

TABLE 1. Automorphisms of Γ_n.

	σ	τ	τ^2	τ^3	$\sigma\tau$	$\sigma\tau^2$	$\sigma\tau^3$
ρ_n	b_n	$\rho_n b_n \rho_n^{-1}$	b_n^{-1}	$b_n^{-1}\rho_n^{-1}b_n$	$b_n^{-1}\rho_n b_n$	ρ_n^{-1}	$\rho_n b_n^{-1}\rho_n^{-1}$
b_n	ρ_n	$b_n^{-1}\rho_n b_n$	ρ_n^{-1}	$\rho_n b_n^{-1}\rho_n^{-1}$	$\rho_n b_n \rho_n^{-1}$	b_n^{-1}	$b_n^{-1}\rho_n^{-1}b_n$

are outer automorphisms of Γ_n for any n. The action of the group $\langle\sigma, \tau\rangle$ by automorphisms on Γ_n is presented in Table 1 (see also [23]).

Note that $\mathcal{F}(n)$, $n \geq 4$, is a hyperbolic 3-orbifold of finite volume (see, for example, [3]). There are two involutions acting as described in the statement of Lemma 2.4. For any n, there is $h_1 \in \mathrm{Iso}(\mathbb{H}^3)$ such that

$$\sigma(\rho_n) = h_1\rho_n h_1^{-1} = b_n,$$

where the involution σ is of type (i) in Lemma 2.4, and also $h_2 \in \mathrm{Iso}(\mathbb{H}^3)$ such that

$$\sigma\tau^2(\rho_n) = h_2\rho_n h_2^{-1} = \rho_n^{-1},$$

where the involution $\sigma\tau^2$ is of type (ii). For any n, there exists $h_3 \in \mathrm{Iso}(\mathbb{H}^3)$ which realises τ^2 with conjugation by b_n:

$$b_n(\tau^2(b_n))b_n^{-1} = h_3 b_n h_3^{-1} = b_n\rho_n^{-1}b_n^{-1}$$

and the involution h_3 is of type (iii). □

Let us define two orbifolds with a 3-sphere S^3 as the underlying space. Denote by $O_1(n)$ the orbifold with singular set the spatial theta-graph presented in Figure 2 with singularities 2, 2 and n at its edges, as indicated in the figure. Denote by $O_2(n)$ the orbifold with singular set the two-component link 6_2^2 with singularities 2 and n at its components, as presented in Figure 2.

THEOREM 4.2. *The orbifold group $\pi^{\mathrm{orb}}O_1(4)$ is a GMT-group.*

PROOF. For a fixed n, consider the involution $h_1 \in \mathrm{Iso}(\mathbb{H}^3)$ from the proof of Lemma 4.1 such that $h_1\rho_n h_1^{-1} = b_n$ and $h_1 b_n h_1^{-1} = \rho_n$. For ρ_n and b_n given by (3.1), we have $h_1 = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. An extension of Γ_n by h_1 has the following presentation:

$$\Delta_n = \langle \rho_n, b_n, h_1 \mid \rho_n^n = b_n^n = h_1^2 = 1,\ \rho_n^{-1}[b_n, \rho_n] = [b_n, \rho_n]\, b_n,\ h_1\rho_n h_1^{-1} = b_n\rangle. \quad (4.1)$$

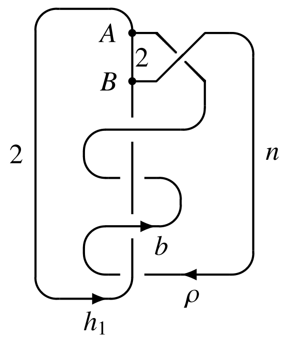

FIGURE 3. Singular set of $\mathcal{F}(n)/h_1$.

It is easy to see that the conjugation by h_1 is induced by an involution of S^3 whose axis corresponds to the dotted line in Figure 1 and intersects the figure-eight knot $\mathcal{F}$ in two points. This symmetry induces an isometry, also denoted by h_1, of the orbifold $\mathcal{F}(n)$.

The quotient space $\mathcal{F}(n)/h_1$ has S^3 as its underlying space. Its singular set is a spatial graph with two vertices presented by a diagram in Figure 3. This graph can be described as the torus knot 5_1 with a tunnel AB. Points A and B are the images of intersection points of the singular set of $\mathcal{F}(n)$ with the axis of the involution h_1. Two edges of this graph, which are images of the axis of h_1, have singularity index 2, and the third edge, which is the image of the singular set of $\mathcal{F}(n)$, has singularity index n.

It can be checked directly (see, for example, [23]) that the orbifold group of $\mathcal{F}(n)/h_1$ is isomorphic to Δ_n with generators ρ, b and h_1, as pictured in Figure 3. Indeed, the relations (4.1) hold by the Wirtinger algorithm. In particular, the relation $\rho_n^{-1}\,[b_n,\rho_n] = [b_n,\rho_n]\,b_n$ is a consequence of the fact that the loop around its unknotting tunnel AB (see [12] about unknotting tunnels) is an element of order two in the orbifold group. Obviously, the spatial theta-graphs presented diagrammatically in Figure 2(left) and Figure 3 are equivalent, so $\pi^{\text{orb}}\mathcal{O}_1(n) = \Delta(n)$. Eliminating b_n from (4.1), we see that Δ_n is a two-generated group with generators ρ_n and h_1.

Suppose that $n = 4$. By Corollary 3.2, Γ_4 is a GMT-group and the pair Γ_4 and h_1 satisfies case (i) of Lemma 2.4. Hence, $\pi^{\text{orb}}\mathcal{O}_1(4)$ is a GMT-group. □

THEOREM 4.3. *The orbifold group $\pi^{\text{orb}}\mathcal{O}_2(4)$ is a GMT-group.*

PROOF. To see the symmetry of h_3, we shall redraw the singular set of the orbifold $\mathcal{F}(n)$ as in Figure 4. Define $\lambda = b\rho b^{-1}$ (see Figure 4). It is easy to see that h_3 corresponds to a rotational symmetry of order two such that b goes to λ^{-1} and λ goes to b^{-1}. Therefore, $h_3bh_3^{-1} = b\rho^{-1}b^{-1}$, corresponding to the case (iii) of Lemma 2.4. Using $\rho = b^{-1}h_3b^{-1}h_3b$ from the defining relation $\rho^{-1}[b,\rho] = [b,\rho]b$, we get the relation

$$bh_3bh_3b^{-1}h_3b^{-1}h_3bh_3 = h_3bh_3b^{-1}h_3b^{-1}h_3bh_3b,$$

which corresponds to the canonical defining relation of the two-generated fundamental group of the two-bridge link 10/3 pictured on the right in Figure 2 (see also [23]).

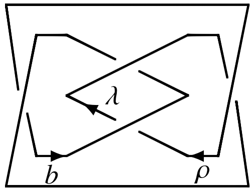

FIGURE 4. Singular set of the orbifold $\mathcal{F}(n)$.

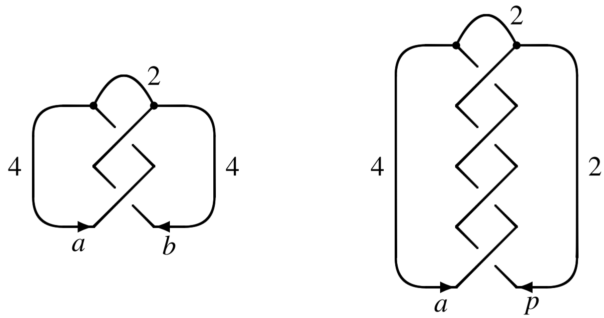

FIGURE 5. Singular sets of orbifolds O_3 and O_4.

Thus, the group generated by ρ_n, b_n and h_3 is the orbifold group $\pi^{\mathrm{orb}}O_2(n)$. Suppose that $n = 4$. By Corollary 3.2, Γ_4 is a GMT-group and the pair Γ_4 and h_3 satisfies the case (iii) of Lemma 2.4. Hence, $\pi^{\mathrm{orb}}O_2(4)$ is a GMT-group. □

5. More examples of GMT-groups

In this section we shall give two more examples of GMT-groups. Let us denote the orbifolds with the singular set presented in Figure 5 by O_3 and O_4. The singular set of O_3 is a spatial graph with two vertices that can be described as a Hopf link with an unknotting tunnel. The singular set of O_4 can be described as a double link with an unknotting tunnel. Singularity indices are presented in Figure 5. Both singular vertices of O_3 belong to $\partial\mathbb{H}^3$. One of the singular vertices of O_4 belongs to $\partial\mathbb{H}^3$, whereas the other one lies in $\mathbb{H}^3$.

THEOREM 5.1. *The orbifold groups of* $\pi^{\mathrm{orb}}O_3$ *and* $\pi^{\mathrm{orb}}O_4$ *are GMT-groups.*

PROOF. The orbifold group of $\pi^{\mathrm{orb}}O_3$ has the following presentation:

$$\pi^{\mathrm{orb}}O_3 = \langle a, b \mid a^4 = b^4 = 1,\ [a, b]^2 = 1\rangle,$$

where a relation $[a, b]^2 = 1$ corresponds to a loop around a tunnel. Concerning the hyperbolicity of this orbifold, see, for example, [10]. Let us use the letters a and b also for images of generators in the group $\mathrm{Iso}(\mathbb{H}^3)$, corresponding to a faithful representation. Then $\mathrm{tr}^2(a) = 2$ and $\mathrm{tr}[a, b] = 0$. Hence, $\pi^{\mathrm{orb}}O_3$ is a GMT-group.

It can be seen from Figure 5 that the singular set of O_3 has a symmetry of order two that exchanges a and b. This symmetry induces an involution τ of $\pi^{\text{orb}}O_3$ defined by

$$\tau(a) = pap^{-1} = b \quad \text{and} \quad \tau(b) = pbp^{-1} = a$$

for some $p \in \text{Iso}(\mathbb{H}^3)$. It is easy to verify that the quotient orbifold O_3/p is isometric to O_4. Moreover, $\pi^{\text{orb}}O_4$ is two-generated with generators a and p, and p satisfies case (i) of Lemma 2.4. Hence, $\pi^{\text{orb}}O_4$ is a GMT-group and has the presentation

$$\pi^{\text{orb}}O_4 = \langle a, p \mid a^4 = p^2 = 1, \quad (apapa^{-1}pa^{-1}p)^2 = 1\rangle. \qquad \square$$

References

[1] C. K. Atkinson and D. Futer, 'The lowest volume 3-orbifolds with high torsion', *Trans. Amer. Math. Soc.* (to appear) doi:10.1090/tran/6920.

[2] A. F. Beardon, *The Geometry of Discrete Groups* (Springer, New York, 1983).

[3] M. Boileau, S. Maillot and J. Porti, *Three-Dimensional Orbifolds and their Geometric Structures*, Panoramas et Synthéses, 15 (Société Mathématique de France, Paris, 2003).

[4] J. Callahan, 'Jørgensen number and arithmeticity', *Conform. Geom. Dyn.* **13** (2009), 160–186.

[5] W. Cao and H. Tan, 'Jørgensen inequality for quaternionic hyperbolic space with elliptic elements', *Bull. Aust. Math. Soc.* **81** (2010), 121–131.

[6] M. Dehn, 'Dei beiden Kleeblattschlingen', *Math. Ann.* **75** (1914), 402–413.

[7] F. W. Gehring and G. J. Martin, 'Stability and extremality in Jørgensen's inequality', *Complex Var. Theory Appl.* **12** (1989), 277–282.

[8] K. Gongopadhyay and A. Mukherjee, 'Extremality of quaternionic Jørgensen inequality'. arXiv:math1503.08802.

[9] T. Jørgensen, 'A note on subgroups of SL(2, ℂ)', *Q. J. Math. Oxford Ser. (2)* **28**(110) (1977), 209–211.

[10] E. Klimenko and N. Kopteva, 'Two-generated Kleinian orbifolds'. arXiv:math/0606066.

[11] W. Magnus, 'Untersuchungen fiber einige unendliche diskontinuierliche Gruppen', *Math. Ann.* **105** (1931), 52–74.

[12] K. Marimoto, M. Sakuma and Y. Yokota, 'Identifying tunnel number one knots', *J. Math. Soc. Japan* **48**(4) (1996), 667–688.

[13] S. Matveev, C. Petronio and A. Vesnin, 'Two-sided asymptotic bounds for the complexity of some closed hyperbolic three-manifolds', *J. Aust. Math. Soc.* **86** (2009), 205–219.

[14] A. Mednykh and A. Rasskazov, 'On the structure of the canonical fundamental set for the 2-bridge link orbifolds', Universität Bielefeld, Preprint, 1998.

[15] J. R. Parker, 'Shimizu's lemma for complex hyperbolic space', *Internat. J. Math.* **3** (1992), 291–308.

[16] R. Riley, 'A quadratic parabolic group', *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 281–288.

[17] H. Sato, 'The Jørgensen number of the Whitehead link group', *Bol. Soc. Mat. Mexicana* (3) **10** (2004), 495–502.

[18] D. Tan, 'On two-generator discrete groups of Mobius transformations', *Proc. Amer. Math. Soc.* **106** (1989), 763–770.

[19] A. Vesnin and A. Masley, 'On Jørgensen numbers and their analogues for groups of figure-eight orbifolds', *Sib. Math. J.* **55** (2014), 807–816.

[20] A. Vesnin and A. Masley, 'Two-generated subgroups of PSL(2, ℂ) which are extreme for Jørgensen inequality and its analogues', *Proc. Semin. Vector and Tensor Analysis with their Applications to Geometry, Mechanics and Physics*, 30 (Moscow State University, 2015), 1–54.

[21] A. Vesnin and A. Mednykh, 'Hyperbolic volumes of the Fibonacci manifolds', *Sib. Math. J.* **36** (1995), 235–245.

[22] A. Vesnin and A. Mednykh, 'Fibonacci manifolds as two-fold coverings over the three-dimensional sphere and the Meyerhoff–Neumann conjecture', *Sib. Math. J.* **37** (1996), 461–467.

[23] A. Vesnin and A. Rasskazov, 'Isometries of hyperbolic Fibonacci manifolds', *Sib. Math. J.* **40** (1999), 9–22.

DUŠAN REPOVŠ, Faculty of Education and Faculty of Mathematics and Physics,
University of Ljubljana, Kardeljeva pl. 16, 1000 Ljubljana, Slovenia
e-mail: dusan.repovs@guest.arnes.si

ANDREI VESNIN, Laboratory of Quantum Topology,
Chelyabinsk State University, Chalyabinsk, Russia
and
Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
e-mail: vesnin@math.nsc.ru