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Two-Sided Bounds for the Volume of Right-Angled Hyperbolic Polyhedra

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Abstract—For a compact right-angled polyhedron R in Lobachevskii space $\mathbb{H}^3$, let $\mathrm{vol}(R)$ denote its volume and $\mathrm{vert}(R)$, the number of its vertices. Upper and lower bounds for $\mathrm{vol}(R)$ were recently obtained by Atkinson in terms of $\mathrm{vert}(R)$. In constructing a two-parameter family of polyhedra, we show that the asymptotic upper bound $5v_3/8$, where v_3 is the volume of the ideal regular tetrahedron in $\mathbb{H}^3$, is a double limit point for the ratios $\mathrm{vol}(R)/\mathrm{vert}(R)$. Moreover, we improve the lower bound in the case $\mathrm{vert}(R) \le 56$.

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1. RIGHT-ANGLED POLYHEDRA IN HYPERBOLIC 3-SPACE

In each of the possible geometric spaces, right-angled polyhedra are very convenient "building blocks" for various geometric constructions. We consider polyhedra in hyperbolic 3-space $\mathbb{H}^3$. Necessary and sufficient conditions under which a polyhedron of given combinatorial type can be realized as a compact convex right-angled polyhedron in $\mathbb{H}^3$ were described by Pogorelov in 1967 in the very first issue of "Matematicheskie Zametki" [1]. The simplest compact right-angled hyperbolic polyhedron is the dodecahedron with all dihedral angles $\pi/2$ (and, therefore, all plane angles $\pi/2$). Naturally, such polyhedra are very useful in constructing hyperbolic 3-manifolds. Thus, we can try to construct a hyperbolic 3-manifold by using a right-angled polyhedron as its fundamental polyhedron. Alternatively, we can construct a hyperbolic 3-manifold whose fundamental group is the torsion-free subgroup in the Coxeter group generated by reflections in the faces of a right-angled polyhedron [2]. In what follows, we consider only compact polyhedra, i.e., polyhedra without ideal vertices.

We begin by recalling two recent results. In [3], Inoue introduced two operations on right-angled polyhedra, which he called *decomposition* and *edge surgery*, and proved that Löbell polyhedra (which will be discussed later) are universal in the following sense.

Theorem 1 ([3, Theorem 9.1]). *Suppose that P_0 is a compact right-angled hyperbolic polyhedron. Then there exists a sequence of disjoint unions of right-angled hyperbolic polyhedra $P_1, \dots, P_k$ such that, for $i = 1, \dots, k$, P_i is obtained from P_{i-1} by either a decomposition or an edge surgery and P_k is a set of Löbell polyhedra. Furthermore,*

$$\mathrm{vol}(P_0) \ge \mathrm{vol}(P_1) \ge \mathrm{vol}(P_2) \ge \cdots \ge \mathrm{vol}(P_k).$$

Atkinson [4] estimated the volume of a right-angled polyhedron in terms of the number of its vertices as follows.

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Theorem 2 ([4, Theorem 2.3]). *If P is a compact right-angled hyperbolic polyhedron with V vertices, then*

$$(V-2)\cdot\frac{v_8}{32}\le \mathrm{vol}(P) < (V-10)\cdot\frac{5v_3}{8},$$

where v_8 is the volume of the ideal regular octahedron and v_3 is the volume of the ideal regular tetrahedron. There exists a sequence of compact polyhedra P_i, where P_i has V_i vertices, such that $\mathrm{vol}(P_i)/V_i$ tends to $5v_3/8$ as i goes to infinity.

The family of polyhedra P_i proposed by Atkinson was described in the proof of Proposition 6.4 in [4].

In this paper, we show that Löbell polyhedra can serve as a suitable family for which the upper bound is realized. Thus, Löbell polyhedra play an important role not only in Theorem 1, but also in Theorem 2.

Let $\mathrm{vert}(R)$ denote the number of vertices of a right-angled polyhedron R. The following result shows that the value $5v_3/8$ is a double limit point in the sense that it is a limit point for the limit points of the sequence of ratios $\mathrm{vol}(R)/\mathrm{vert}(R)$.

Theorem 3. *For any integer $k\ge 1$, there exists a sequence of compact right-angled hyperbolic polyhedra $R_k(n)$ such that*

$$\lim_{n\to\infty}\frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))}=\frac{k}{k+1}\cdot\frac{5v_3}{8}.$$

As will be seen from the proof, the polyhedra $R_1(n)$ are Löbell polyhedra, and the polyhedra $R_k(n)$, $k>1$, are towers composed of them.

In addition, in Corollary 4, we improve the lower bounds from Theorem 2 for the case $\mathrm{vert}(R)\le 56$.

2. LÖBELL POLYHEDRA AND MANIFOLDS

Löbell polyhedra were introduced in [2] as a generalization of a right-angled 14-hedron used in [5].

Recall that, in 1931, to prove the existence of Clifford–Klein space forms (i.e., of closed manifolds) of constant negative curvature, Löbell [5] constructed the first example of a closed orientable hyperbolic 3-manifold. This manifold was obtained by gluing together eight copies of the right-angled 14-hedron (denoted below by $R(6)$ and shown in Fig. 1). Its upper and lower bases are regular hexagons, and the lateral surface is formed by 12 pentagons whose arrangement is similar to that of the pentagons on the lateral surface of the dodecahedron. Obviously, $R(6)$ can be regarded as the generalization of the right-angled dodecahedron obtained by replacing the pentagonal upper and lower bases by hexagonal ones.

As shown in [2], the dodecahedron and the polyhedron $R(6)$ are elements of the following family of polyhedra. For each $n\ge 5$, consider the right-angled polyhedron $R(n)$ in $\mathbb{H}^3$ with $2n+2$ faces, two of which (regarded as its upper and lower bases) are regular n-gons. Its lateral surface is formed by $2n$ pentagons whose arrangement is similar to that of the pentagons on the lateral surface of the dodecahedron. In particular, $R(5)$ is a right-angled dodecahedron (see Fig. 1). The existence of such polyhedra $R(n)$ in $\mathbb{H}^3$ can easily be proved by using Pogorelov's theorem [1] or Andreev's theorem [6].

The approach proposed in [2] allows constructing both orientable and nonorientable closed hyperbolic 3-manifolds from eight copies of any compact right-angled hyperbolic polyhedron. Namely, each four-coloring of the faces of a right-angled polyhedron such that no two faces of the same color have a common edge determines the torsion-free subgroup composed of orientation-preserving isometries; this subgroup has index eight in the Coxeter group of the original polyhedron.

Thus, each four-coloring determines an orientable hyperbolic 3-manifold obtained from eight copies of any right-angled polyhedron. Using the same approach, we can also construct nonorientable hyperbolic 3-manifolds, but this requires five to seven colors.

In [2], it was noted that the manifold constructed by Löbell in 1931 can be obtained by using a four-coloring of the polyhedron $R(6)$. In addition, for each $n\ge 5$, examples of both orientable and nonorientable manifolds constructed from eight copies of $R(n)$ were given explicitly. Closed orientable hyperbolic 3-manifolds encoded by four-colorings of the polyhedra $R(n)$, $n\ge 5$, are called *Löbell*

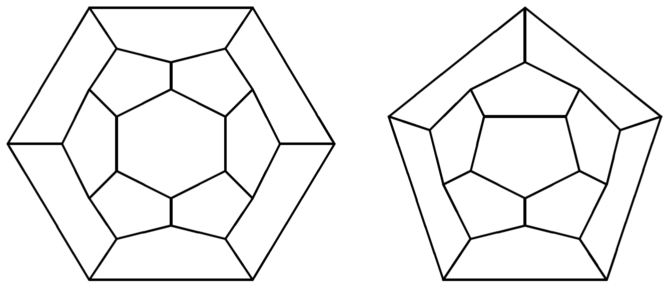

Fig. 1. The polyhedra $R(6)$ and $R(5)$.

manifolds. (Note that, for given n, there may be more than one such manifold.) It is natural to call such polyhedra $R(n)$ *Löbell polyhedra*.

The properties of Löbell manifolds have been studied in many works: exact formulas for volumes were obtained in [7] and [8]; invariant trace fields of fundamental groups of manifolds and their arithmetical properties were studied by numerical methods in [9]; in [10], many manifolds were described as double branched coverings of the 3-sphere; and two-sided bounds for the complexity of Löbell manifolds were obtained in [11], [12].

Volume formulas for hyperbolic polyhedra traditionally use the *Lobachevskii function*

$$\Lambda(x) = -\int_0^x \log|2\sin(t)|\,dt,$$

which dates back to Lobachevskii's 1832 paper.

Since each Löbell manifold with parameter n can be obtained by isometrically gluing together eight copies of the polyhedron $R(n)$, the formula for the volume of Löbell manifolds proved in [7] implies the following formula for $\operatorname{vol}(R(n))$.

Theorem 4. *For all $n \geq 5$, the following formula holds*:

$$\operatorname{vol}(R(n)) = \frac{n}{2}\left(2\Lambda(\theta_n) + \Lambda\left(\theta_n + \frac{\pi}{n}\right) + \Lambda\left(\theta_n - \frac{\pi}{n}\right) + \Lambda\left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

where

$$\theta_n = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

It is easy to verify that $\theta_n \to \pi/6$ and

$$\frac{\operatorname{vol}(R(n))}{n} \to \frac{5v_3}{4}$$

as $n \to \infty$. Here we have used the equalities $v_3 = 3\Lambda(\pi/3) = 2\Lambda(\pi/6)$. Moreover, in [12, Proposition 2.10], the asymptotic behavior of the volumes of Löbell manifolds was established. This readily implies the following description of the asymptotic behavior of $\operatorname{vol}(R(n))$ as n approaches infinity.

Proposition 1. *For sufficiently large n, the following inequalities hold*:

$$\frac{5v_3}{4} \cdot n - \frac{17v_3}{2n} < \operatorname{vol}(R(n)) < \frac{5v_3}{4} \cdot n.$$

Using the relation $\operatorname{vert}(R(n)) = 4n$, we obtain the following statement.

Corollary 1. *For sufficiently large n, the following inequalities hold*:

$$\frac{5v_3}{16} - \frac{17v_3}{8n^2} < \frac{\operatorname{vol}(R(n))}{\operatorname{vert}(R(n))} < \frac{5v_3}{16}.$$

3. PROOF OF THEOREM 3

We shall use the Löbell polyhedra $R(n)$ as building blocks for constructing right-angled polyhedra with the required properties. Let us represent each polyhedron $R(n)$ by its lateral surface, as shown in Fig. 2 for the polyhedra $R(6)$ and $R(5)$, assuming that the left- and the right-hand edge must be identified.

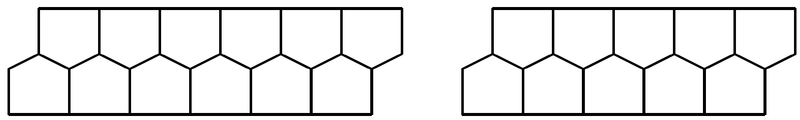

Fig. 2. The polyhedra $R(6)$ and $R(5)$.

For an integer $k \geq 1$, let $R_k(n)$ denote the polyhedron constructed from k copies of the polyhedron $R(n)$ by gluing them together along their n-gonal faces as a tower. In particular, $R_1(n) = R(n)$. The polyhedron $R_3(6)$ is shown in Fig. 3.

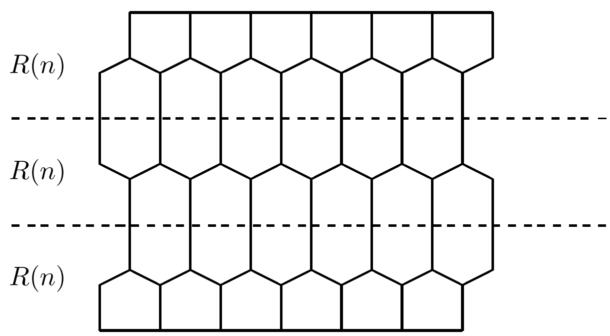

Fig. 3. The polyhedron $R_3(6)$.

Obviously, $R_k(n)$ is a right-angled polyhedron with n-gonal upper and lower bases and lateral surface formed by $2n$ pentagons and $(k-1)\cdot n$ hexagons.

Since $\mathrm{vol}(R_k(n)) = k \cdot \mathrm{vol}(R(n))$, Proposition 1 implies that, for sufficiently large n, we have

$$k \cdot \frac{5v_3}{4} \cdot n - k \cdot \frac{17v_3}{2n} < \mathrm{vol}(R_k(n)) < k \cdot \frac{5v_3}{4} \cdot n.$$

Since $\mathrm{vert}\, R_k(n) = (2k+2)n$, we obtain

$$\frac{k}{k+1} \cdot \frac{5v_3}{8} - \frac{k}{k+1} \cdot \frac{17v_3}{4n^2} < \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} < \frac{k}{k+1} \cdot \frac{5v_3}{8}.$$

Thus, the family of right-angled polyhedra $R_k(n)$ is such that, for each integer $k \geq 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} = \frac{k}{k+1} \cdot \frac{5v_3}{8},$$

and the upper bound $5v_3/8$ is a double limit point in the sense that it is the limit point as $k \to \infty$ for the limit points mentioned above:

$$\lim_{k,n\to\infty} \frac{\mathrm{vol}(R_k(n))}{\mathrm{vert}(R_k(n))} = \frac{5v_3}{8}.$$

This proves the theorem.

4. OTHER VOLUME ESTIMATES

Since the 1-skeleton of a right-angled compact hyperbolic polyhedron P is a 3-valent planar graph, it follows from Euler's formula for the polyhedron that

$$V = 2F - 4,$$

where V is the number of vertices of the polyhedron P and F is the number of its faces. Moreover, Euler's formula implies that P has at least 12 faces (this number of faces corresponds to the dodecahedron). Thus, Theorem 2 implies the following result.

Corollary 2. *Suppose that P is a compact right-angled hyperbolic polyhedron with F faces. Then*

$$(F-3)\cdot\frac{v_8}{16} \le \operatorname{vol}(P) < (F-7)\cdot\frac{5v_3}{4}.$$

Recall that the constants v_3 and v_8 are

$$v_3 = 3\Lambda\left(\frac{\pi}{3}\right) = 1.0149416064096535\dots,$$

$$v_8 = 8\Lambda\left(\frac{\pi}{4}\right) = 3.663862376708876\dots.$$

Since the area of a right-angled n-gon is $\pi/2\cdot(n-4)$, the area of the lateral surface of a compact right-angled hyperbolic polyhedron P with F faces is $\pi\cdot(F-6)$. Thus, Corollary 2 implies the following result.

Corollary 3. *Suppose that P is a compact right-angled hyperbolic polyhedron, the area of whose lateral surface is equal to S. Then*

$$\left(\frac{S}{\pi}+3\right)\cdot\frac{v_8}{16} \le \operatorname{vol}(P) < \left(\frac{S}{\pi}-1\right)\cdot\frac{5v_3}{4}.$$

Note that Theorem 4 can be used to prove that the volume $\operatorname{vol}(R(n))$ is a monotone increasing function of n (see the corresponding proofs in [2] or [12]) and calculate the volumes of Löbell polyhedra. In particular,

$$\operatorname{vol}(R(5)) = 4.306\dots, \qquad \operatorname{vol}(R(6)) = 6.203\dots, \qquad \operatorname{vol}(R(7)) = 7.563\dots.$$

Together with Theorem 1, this implies that the right-angled hyperbolic polyhedron of smallest volume is $R(5)$ (a dodecahedron) and the second smallest (in volume) is the polyhedron $R(6)$. Thus, if P is a compact right-angled hyperbolic polyhedron distinct from the dodecahedron, then

$$\operatorname{vol}(P) \ge 6.203\dots.$$

Thus, we obtain the following statement.

Corollary 4. *Suppose that P is a compact right-angled hyperbolic polyhedron with V vertices and F faces distinct from the dodecahedron. Then*

$$\operatorname{vol}(P) \ge \max\left\{(V-2)\cdot\frac{v_8}{32},\, 6.203\dots\right\}, \qquad \operatorname{vol}(P) \ge \max\left\{(F-3)\cdot\frac{v_8}{16},\, 6.203\dots\right\}.$$

The bounds from Corollary 4 improve the lower bounds from Theorem 2 for $V \le 56$ and the lower bounds from Corollary 2 for $F \le 30$.

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