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A Nash type solution for hemivariational inequality systems

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ABSTRACT

In this paper, we prove an existence result for a general class of hemivariational inequality systems using the Ky Fan version of the KKM theorem Fan (1984) [10] or Tarafdar fixed points Tarafdar (1987) [11]. As application, we give an infinite-dimensional version for the existence result of Nash generalized derivative points introduced recently by Kristály (2010) [5]. We also give an application to a general hemivariational inequality system.

1. Introduction

In the last few years, many papers have been dedicated to the study of the existence and multiplicity of solutions for hemivariational inequality systems or differential inclusion systems defined on bounded or unbounded domains; see [1–6]. In these papers, the authors use the critical point theory for locally Lipschitz functions, combined with the *Principle of Symmetric Criticality* and different topological methods. For a comprehensive treatment of hemivariational inequality and hemivariational inequality systems on bounded domains using the critical point theory for non-smooth functionals, we refer the reader to the monographs of Motreanu and Rădulescu [7] and Motreanu and Panagiotopoulos [8]. For very recent results concerning variational inequalities and elliptic systems using the critical point theory and different variational methods, see also the book by Kristály et al. [9].

The aim of this paper is to prove the existence of at least one solution for a general class of hemivariational inequality systems on a closed and convex set (either bounded or unbounded), without using the critical point theory. We apply a version of the well-known theorem of Knaster–Kuratowski–Mazurkiewicz due to Ky Fan [10] or the Tarafdar fixed point theorem [11]. We start the paper by giving in Section 2 the assumptions and by formulating the hemivariational inequality system problem that we study. The main results concerning the existence of at least one solution for the hemivariational inequality systems that we study are given in Section 3. Section 4 contains applications to Nash and Nash generalized derivative points and existence results for some abstract class of hemivariational inequality systems.

2. Assumptions and formulation of the problem

Let $X_1, X_2, \ldots, X_n$ be reflexive Banach spaces and $Y_1, Y_2, \ldots, Y_n, Z_1, \ldots, Z_n$ Banach spaces, such that there exist linear operators $T_i : X_i \to Y_i, T_i : X_i \to Z_i$ for $i \in \{1, \ldots, n\}$. We suppose that the following condition holds:

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(TS) $T_i : X_i \to Y_i$ and $S_i : X_i \to Z_i$ are compact for $i = \overline{1, n}$.

We denote by X_i^* the topological dual of X_i and $\langle \cdot, \cdot \rangle_i$ denotes the duality pairing between X_i^*, whereas X_i for $i = \overline{1, n}$. Also, let $K_i \subset X_i$ be closed, convex sets for $i = \overline{1, n}$ and we consider $A_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ the continuous functions which are locally Lipschitz in the ith variable and we denote by $A_i^\circ(u_1, \ldots, u_i, \ldots, u_n; v_i)$ the partial Clarke derivative in the directional derivative in the ith variable, i.e., the Clarke derivative of the locally Lipschitz function $A_i(u_1, \ldots, u_i, \ldots, u_n)$ at the point $u_i \in Y_i$ in the direction $v_i \in Y_i$, that is

$$A_i^\circ(u_1, \ldots, u_i, \ldots u_n; v_i) = \limsup_{\substack{w \to u_i \\ \tau \searrow 0}} \frac{A_i(u_1, \ldots, w + \tau v_i, \ldots u_n) - A_i(u_1, \ldots, w, \ldots, u_n)}{\tau}.$$

We suppose that for every $i = \overline{1, n}$ the following condition holds:
(A) the functions $A_i^\circ : Y_1 \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ are upper semi-continuous.

We also consider the following nonlinear operators $F_i : K_1 \times \cdots \times K_i \times \cdots \times K_n \to X_i^\star$, $i = \overline{1, n}$. We suppose that the operators F_i satisfy the following condition:
(F) the functions $(u_1, \ldots, u_n) \mapsto \langle F_i(u_1, \ldots, u_n), v_i \rangle_i$ are weakly upper semi-continuous for every $v_i \in X_i$ and $i = \overline{1, n}$.

Definition 2.1 (*See [12]*)**.** Let Z be a Banach space and $j : Z \to \mathbb{R}$ a locally Lipschitz function. We say that j is regular at $u \in Z$ if for all $v \in Z$ the one-sided directional derivative $j'(u; v)$ exists and $j'(u; v) = j^\circ(u; v)$. If j is regular at every point $u \in Z$ we say that j is regular.

We have the following elementary result.

Proposition 2.1. *Let $J : Z_1 \times \cdots \times Z_n \to \mathbb{R}$ be a regular, locally Lipschitz function. Then the following assertions hold:*
(a) $\partial J(u_1, \ldots, u_n) \subseteq \partial_1 J(u_1, \ldots, u_n) \times \cdots \times \partial_n J(u_1, \ldots, u_n)$ *(see [12, Proposition 2.3.15]), where $\partial_i J$, $i = \overline{1, n}$ denotes the Clarke subdifferential in the ith variable;*
(b) $J^\circ(u_1, \ldots, u_n; v_1, \ldots, v_n) \le \sum_{i=1}^n J_i^\circ(u_1, \ldots, u_n; v_i)$*, where J_i° denotes the Clarke derivative in the ith variable; and*
(c) $J^\circ(u_1, \ldots, u_n; 0, \ldots, v_i, \ldots, 0) \le J_i^\circ(u_1, \ldots, u_n; v_i)$.

We introduce the following notations:

- $K = K_1 \times \cdots \times K_n$,
- $u = (u_1, \ldots, u_n)$
- $Tu = (T_1 u_1, \ldots, T_n u_n)$
- $Su = (S_1 u_1, \ldots, S_n u_n)$
- $A(Tu, Tv - Tu) = \sum_{i=1}^n A_i^\circ(Tu, T_i v_i - T_i u_i)$
- $F(u, v - u) = \sum_{i=1}^n \langle F_i u, v_i - u_i \rangle_i$.

In this paper we study the following problem:
Find $u = (u_1, \ldots, u_n) \in K_1 \times \cdots \times K_n$ such that for all $v = (v_1, \ldots, v_n) \in K_1 \times \cdots \times K_n$ and $i \in \{1, \ldots, n\}$ we have:

$$A_i^\circ(Tu; Tv_i - Tu_i) + \langle F_i(u), v_i - u_i \rangle_i + J_i^\circ(Su; S_i v_i - S_i u_i) \ge 0. \tag{QHS}$$

In this case we say that $u = (u_1, \ldots, u_n)$ is a Nash equilibrium point for the system (QHS).

To prove our main result we use the FKKM theorem due to Ky Fan [10] and the Tarafdar fixed point theorem [11].

Definition 2.2. Suppose that X is a vector space and $E \subset X$. A set-valued mapping $G : E \to 2^X$ is called a KKM mapping, if for any $x_1, \ldots, x_n \in E$ the following holds

$$\operatorname{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n G(x_i).$$

The following version of the KKM theorem is due to Ky Fan [10].

Theorem 2.1. *Suppose that X is a locally convex Hausdorff space, $E \subset X$ and that $G : E \to 2^X$ is a closed-valued KKM map. If there exists $x_0 \in E$ such that $G(x_0)$ is compact, then $\bigcap_{x \in E} G(x) \ne \emptyset$.*

Theorem 2.2. *Let K be a non-empty, convex subset of a Hausdorff topological vector space X. Let $G : K \hookrightarrow 2^K$ be a set-valued map such that*
(i) *for each $u \in K$, $G(u)$ is a non-empty convex subset of K;*
(ii) *for each $v \in K$, $G^{-1}(v) = \{u \in K : v \in G(u)\}$ contains an open set O_v which may be empty;*
(iii) $\cup_{v \in K} O_v = K$*; and*
(iv) *there exists a non-empty set K_0 contained in a compact convex subset K_1 of K such that $D = \cap_{v \in K_0} O_v^c$ is either empty or compact (where O_v^c is the complement of O_v in K).*

Then there exists a point $u_0 \in K$ such that $u_0 \in G(u_0)$.

Tarafdar in [11] proved the equivalence of Theorems 2.1 and 2.2.

3. Main results

Theorem 3.1. *Let $K_i \subset X_i$, $i = \overline{1, n}$ be non-empty, bounded, closed and convex sets. Let $A_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ be a locally Lipschitz function in the ith variable for all $i \in \{1, \ldots, n\}$ satisfying condition **(A)**. We suppose that the operators $T_i : X_i \to Y_i$, $S_i : X_i \to Z_i$ and $F_i : K_1 \times \cdots \times K_n \to X_i^\star$ $(i = \overline{1, n})$ satisfy the condition **(TS)** respectively **(F)**. Final we consider the regular locally Lipschitz function $J : Z_1 \times \cdots \times Z_n \to \mathbb{R}$. Under these conditions the problem (QHS) admits at least one solution.*

Before proving Theorem 3.1, we make two remarks.

Remark 3.1. We observe that for every $v \in K$ the function

$$u \mapsto A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su)$$

is weakly upper semi-continuous. Indeed, from the condition **(A)** and from the fact that the operators T_i are compact follows that $A(Tu, Tv - Tu)$ is weakly upper semi-continuous. From **(F)** it follows that $F(u, v - u)$ is weakly upper semi-continuous. The third term, i.e., $J^\circ(Su; Sv - Su)$ is weakly upper semi-continuous, because $J^\circ(\cdot; \cdot)$ is upper semi-continuous and the operators $S_i : X_i \to Z_i$ are compact.

Remark 3.2. If there exists $u \in K$, such that for every $v \in K$ we have:

$$A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) \geq 0, \tag{3.1}$$

then $u \in K$ is a solution of the problem (QHS). Indeed, if we fix an $i = \{1, \ldots, n\}$ and put $v_j := u_j, j \neq i$ in the above inequality and using (iii) Proposition 2.1 we get that

$$A_i^\circ(Tu; Tv_i - Tu_i) + \langle F_i(u_i), v_i - u_i \rangle_i + J_i^\circ(Su; S_i v_i - S_i v_i) \geq 0. \tag{QHS}$$

for all $i \in \{1, \ldots, n\}$.

In what follows we give two proofs, using Theorems 2.1 and 2.2.

First proof: Let $G : K \hookrightarrow 2^K$ be the set-valued map defined by

$$G(v) = \{u \in K : A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) \geq 0\}.$$

For every $v \in K$, we have $G(v) \neq \emptyset$ because $v \in G(v)$ and taking into account that the function

$$u \mapsto A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su)$$

is weakly upper semi-continuous, it follows that the set $G(v)$ is weakly closed. Now we prove that G is a KKM mapping. We argue by contradiction, let $v_1, \ldots, v_k \in K$ and $w \in \text{conv}\{v_1, \ldots, v_k\}$ such that $w \notin \cup_{i=1}^k G(v_i)$. From this it follows that

$$A(Tw, Tv_i - Tw) + F(w, v_i - w) + J^\circ(Sw; Sv_i - Sw) < 0, \tag{3.2}$$

for all $i = \{1, \ldots, k\}$. Because of $w \in \text{conv}\{v_1, \ldots, v_k\}$ the existence of $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^k \lambda_i = 1$ such that $w = \sum_{i=1}^k \lambda_i v_i$ follows. If we multiply the inequalities (3.2) with λ_i and adding for $i = \{1, \ldots, k\}$ we obtain

$$A(Tw, Tw - Tw) + F(w, w - w) + J^\circ(Sw; Sw - Sw) < 0 \tag{3.3}$$

because the functions $A(\cdot, \cdot)$, $F(\cdot, \cdot)$ and $J^\circ(\cdot, \cdot)$ are positive homogeneous and convex in the second variable. From inequality (3.3) it follows that $0 = A(Tw, Tw - Tw) + F(w, w - w) + J^\circ(Sw; Sw - Sw) < 0$, which is a contradiction. Because the set K is bounded, convex and closed, it follows that it is weakly closed and by the Eberlein–Smulian theorem we have is weakly compact. Because $G(v) \subset K$ is weakly closed, we have that $G(v)$ is weakly compact and from Theorem 2.1 it follows that $\cap_{v \in K} G(v) \neq \emptyset$, therefore from Remark 3.2 it follows that the problem (QHS) has a solution.

Second proof: Using Remark 3.2 we prove the existence of an element $u \in K$ such that for every $v \in K$ we have

$$A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) \geq 0.$$

In this case $u \in K$ will be the solution of systems (QHS).

We argue by contradiction. Let us assume that for each $u \in K$, there exists $v \in K$ such that

$$A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) < 0. \tag{3.4}$$

Now, we define the set-valued mapping $G : K \hookrightarrow 2^K$ by

$$G(u) = \{v \in K : A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) < 0\}. \tag{3.5}$$

From (3.4) it follows that the set $G(u) \neq \emptyset$ for every $u \in K$. Because the function $A(\cdot, \cdot) + F(\cdot, \cdot) + J^\circ(\cdot; \cdot)$ is convex in the second variable, we get that $G(u)$ is a convex set. Now, we prove that for every $v \in K$, the set $G^{-1}(v) = \{u \in K : v \in G(u)\}$ is weakly open. Indeed, from weakly upper semi-continuity of the function

$$u \mapsto A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su)$$

it follows that

$$[G^{-1}(v)]^c = \{u \in K : A(Tu, Tv - Tu) + F(u, v - u) + J^\circ(Su; Sv - Su) \geq 0\}$$

is weakly closed, therefore $G^{-1}(v)$ is weakly open.
Now we verify (iii) from Theorem 2.2, i.e., $\bigcup_{v \in K} G^{-1}(v) = K$. Because for every $v \in K$ we have $G^{-1}(v) \subset K$, it follows that $\bigcup_{v \in K} G^{-1}(v) \subset K$. Conversely, let $u \in K$ be fixed. Since $G(u) \neq \emptyset$ there exists $v_0 \in K$ such that $v_0 \in G(u)$. In the next step we verify (iv) Theorem 2.2. We assert that $D = \bigcap_{v \in K}[G^{-1}(v)]^c$ is empty or weakly compact. Indeed, if $D \neq \emptyset$, then D is a weakly closed set of K since it is the intersection of weakly closed sets. But K is weakly compact hence we get that D is weakly compact. Taking $O_v = G^{-1}(v)$ and $K_0 = K_1 = K$ we can apply Theorem 2.2 to conclude that there exists $u_0 \in K$ such that $u_0 \in G(u_0)$. This give

$$0 = A(Tu_0, Tu_0 - Tu_0) + F(u_0, u_0 - u_0) + J^0(Su_0; Su_0 - Su_0) < 0,$$

which is a contradiction. Therefore the system (QHS) has a solution.

Remark 3.3. If in Theorem 3.1 the sets $K_i,\ i = \overline{1, n}$ are only convex and closed but not bounded we impose the following coercivity condition.
(CC) there exist $K_i^0 \subset K_i$ compact sets and $v_i^0 \in K_i^0$ such that for all $v = (v_1, \dots, v_n) \in K_1 \times \cdots \times K_n \setminus K_1^0 \times \cdots \times K_n^0$ we have

$$A(Tv, Tv^0 - Tv) + F(v, v^0 - v) + \sum_{i=1}^{n} J_i^0(Sv, S_i v_i^0 - S_i v_i) < 0,$$

where $v^0 = (v_1^0, \dots, v_n^0)$. In this case the problem (**QHS**) has a solution.

4. Applications

In this section we are concerned with two applications. In the first application we study the relation between Nash equilibrium and Nash generalized derivative equilibrium points for a hemivariational inequality system and in the second application we give an existence result for an abstract class of hemivariational inequality systems.

Let $X_1, \dots, X_n$ be Banach spaces and $K_i \subset X_i$ and the functions $f_i : K_1 \times \cdots \times K_i \times \cdots \times K_n \to \mathbb{R}$ for $i \in \{1, \dots, n\}$. The following notion was introduced by Nash [13,14]:

Definition 4.1. An element $(u_1^0, \dots, u_n^0) \in K_1 \times \cdots \times K_n$ is Nash equilibrium point of functions $f_1, \dots, f_n$ if for each $i \in \{1, \dots, n\}$ and $(u_1, \dots, u_n) \in K_1 \times \cdots K_n$ we have

$$f_i(u_1^0, \dots, u_i, \dots, u_n^0) \geq f_i(u_1^0, \dots, u_i^0, \dots, u_n^0).$$

Now let $D_i \subset X_i$ be open sets such that $K_i \subset D_i$ for all $i \in \{1, \dots, n\}$. We consider the function $f_i : K_1 \times \cdots \times D_i \times \cdots K_n \to \mathbb{R}$ which are continuous and locally Lipschitz in the ith variable. The next notion was introduced recently by Kristály [5] and is a little bit different form for functions defined on Riemannian manifolds.

Definition 4.2. If $(u_1^0, \dots, u_n^0) \in K_1 \times \cdots \times K_n$ is an element such that

$$f_i^0(u_1^0, \dots, u_n^0; u_i - u_i^0) \geq 0,$$

for every $i = \{1, \dots, n\}$ and $(u_1, \dots, u_n) \in K_1 \times \cdots \times K_n$ we say that $(u_1^0, \dots, u_n^0)$ is a Nash generalized derivative points for the functions $f_1, \dots, f_n$.

Remark 4.1. If the functions f_i, $i \in \{1, \dots, n\}$ are differentiable in the ith variable, then the above notion coincides with the Nash stationary point introduced in [15].

Remark 4.2. It is easy to observe that any Nash equilibrium point is a Nash generalized derivative point.

The following result is an existence result for Nash generalized derivative points and is an infinite-dimensional version of a result from the paper [5]. Therefore, if in Theorem 3.1 we choose $F_i = 0$, $i \in \{1, \dots, n\}$ and $J = 0$ we obtain the following result.

Theorem 4.1. *(i) Let $Y_1, Y_2, \dots, Y_n$ and $X_1, X_2, \dots, X_n$, be a reflexive Banach spaces and $T_i : X_i \to Y_i$ compact, linear operators. We consider the closed, convex, bounded sets $K_i \subset X_i$ and the functions $A_i : Y_1 \times \cdots \times Y_n \to \mathbb{R}, i = 1, \dots, n$ which are locally Lipschitz in the ith variable and satisfies the condition* (A). *In these conditions, there exists $(u_1^0, \dots, u_i^0, \dots, u_n^0) \in K_1 \times \cdots \times K_i \times \cdots \times K_n$ such that for all $i \in \{1, \dots, n\}$ and $(u_1, \dots, u_i, \dots, u_n) \in K_1 \times \cdots \times K_i \times \cdots \times K_n$ we have*

$$A_i^0((T_1 u_1^0, \dots, T_i u_i^0, \dots, T_n u_n^0); T_i u_i - T_i u_i^0) \geq 0,$$

i.e., $(u_1^0, \dots, u_i^0, \dots, u_n^0)$ is a Nash generalized derivative points for the function A_i, $i \in \{1, \dots, n\}$.

(ii) If the sets K_i, $i = \{1, \dots, n\}$ are only closed and convex we suppose that there exists the bounded, closed sets $K_i^0 \subset K_i$ and $v_i^0 \in K_i^0$, $i = \{1, \dots, n\}$ such that for every $(u_1, \dots, u_n) \in K_1 \times \cdots \times K_n \setminus K_1^0 \times \cdots \times K_n^0$ we have

$$A(Tu, Tv^0 - Tu) < 0.$$

Then there exist $u^0 = (u_1^0, \dots, u_i^0, \dots, u_n^0) \in K_1 \times \cdots \times K_i \times \cdots \times K_n$ such that for all $i \in \{1, \dots, n\}$ and $u = (u_1, \dots, u_i, \dots, u_n) \in K_1 \times \cdots \times K_i \times \cdots \times K_n$ we have

$$A_i^0(Tu_0; T_iu_i - T_iu_i^0) \geq 0,$$

i.e., $u^0 = (u_1^0, \dots, u_i^0, \dots, u_n^0)$ is a Nash generalized derivative points for the functions A_i, $i \in \{1, \dots, n\}$.

In the next step we give an existence result for a general system of hemivariational inequalities. In this case in Theorem 3.1 we choose $Y_i = Z_i$, $i \in \{1, \dots, n\}$ and we suppose that the functions $A_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ are differentiable in the ith variable for $i \in \{1, \dots, n\}$. In this case we suppose that the functions $A_i' : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ are continuous for $i \in \{1, \dots, n\}$. Let also $J : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ a locally Lipschitz regular function.

Under these conditions we have the following result.

Corollary 4.1. *Let $J, A_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ be the function as above and suppose that the condition **(TS)** holds and let $K_i \subset X_i$, $i = \{1, \dots, n\}$ be bounded, closed and convex sets. Under these conditions there exist an element $u^0 = (u_1^0, \dots, u_n^0) \in K_1 \times \cdots \times K_n$ such that for every $u = (u_1, \dots, u_n) \in K_1 \times \cdots \times K_n$ and $i \in \{1, \dots, n\}$ we have:*

$$A_i'(Tu^0; T_iu_i - T_iu_i^0) + J_i^0(Tu^0; T_iu_i - T_iu_i^0) \geq 0.$$

If in Theorem 3.1 we take $A_i = 0$ then we obtain the following existence result for a general class of hemivariational inequality systems.

Corollary 4.2. *Let $K_i \subset X_i$ bounded, closed and convex subsets of the reflexive Banach spaces X_i for $i \in \{1, \dots, n\}$. We suppose that $F_i : K_1 \times \cdots \times K_n \to X_i^\star$ satisfies the condition **(F)** and $J : Z_1 \times \cdots \times Z_n \to \mathbb{R}$ is a regular locally Lipschitz function and the condition **(TS)** holds. Then there exists $u^0 = (u_1^0, \dots, u_i^0, \dots, u_n^0) \in K_1 \times \cdots \times K_i \times \cdots K_n$ such that for every $u = (u_1, \dots, u_i, \dots, u_n) \in K_1 \times \cdots \times K_i \times \cdots K_n$ and $i \in \{1, \dots, n\}$ we have*

$$\langle F_i(u^0); u_i - u_i^0 \rangle_i + J_i^0(Su^0; S_iu_i - S_iu_i^0) \geq 0.$$

The above result generalizes the main result from the paper of Kristály [16].

Indeed, let $\Omega \subset \mathbb{R}^N$ be a bounded, open subset. Let $j : \Omega \times \underbrace{\mathbb{R}^k \times \cdots \mathbb{R}^k}_{n} \to \mathbb{R}$ a Carathéodory function such that $j(x, \cdot, \dots, \cdot)$ is locally Lipschitz for every $x \in \Omega$ and satisfies the following assumptions for all $i \in \{1, \dots, n\}$:

(j_i) there exists $h_1^i \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}_+)$ and $h_2^i \in L^\infty(\Omega, \mathbb{R}_+)$ such that

$$|z_i| \leq h_1^i(x) + h_2^i(x)|y|_{\mathbb{R}^{kn}}^{p-1}$$

for almost $x \in \Omega$ and every $y = (y_1, \dots, y_n) \in \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{n}$ and $z_i \in \partial_i j(x, y_1, \dots, y_n)$.

In this case let $S = (S_1, \dots, S_n) : X_1 \times \cdots \times X_n \to L^p(\Omega, \mathbb{R}^k) \times \cdots L^p(\Omega, \mathbb{R}^k)$ and $J \circ S : K_1 \times \cdots \times K_n \to \mathbb{R}$ is defined by

$$J(Su) = \int_\Omega j(x, S_1u_1(x), \dots S_nu_n(x))\mathrm{d}x.$$

Using a result from Clarke [12] we have:

$$J_i^0(Su; S_iv_i) \leq \int_\Omega j_i^0(x, S_1u_1(x), \dots S_nu_x; S_iv_i(x))\mathrm{d}x, \tag{I}$$

for every $i \in \{1, \dots, n\}$ and $v_i \in X_i$.

Therefore we have the following existence result obtained by Kristály [16].

Corollary 4.3. *Let $K_i \subset X_i$ bounded, closed and convex subsets of the reflexive Banach spaces X_i for $i \in \{1, \dots, n\}$. We suppose that $F_i : K_1 \times \cdots \times K_n \to X_i^\star$ satisfies the condition **(F)** and $j : \Omega \times \underbrace{\mathbb{R}^k \times \cdots \mathbb{R}^k}_{n} \to \mathbb{R}$ a Carathéodory function such that $j(x, \cdot, \dots, \cdot)$ is a regular, locally Lipschitz function satisfying condition (j_i) and the condition **(TS)** holds. Then there exists $u^0 = (u_1^0, \dots, u_i^0, \dots, u_n^0) \in K_1 \times \cdots \times K_i \times \cdots K_n$ such that for every $u = (u_1, \dots, u_i, \dots, u_n) \in K_1 \times \cdots \times K_i \times \cdots K_n$ and $i \in \{1, \dots, n\}$ we have*

$$\langle F_i(u^0); u_i - u_i^0 \rangle_i + \int_\Omega j_i^0(x, S_1u_1^0(x), \dots, S_nu_n^0(x); S_iu_i(x) - S_iu_i^0(x))\mathrm{d}x \geq 0.$$

Remark 4.3. If $n = 1$ we obtain a similar result from the paper of Panagiotopoulos, Fundo and Rădulescu [17].

Remark 4.4. If the Banach spaces X_i, $i \in \{1, \ldots, n\}$ are separable and the domain $\Omega \subset \mathbb{R}^N$ is unbounded then a similar inequality to (I) was proved in the paper Dályai and Varga [18]. Therefore, we can state a similar result as Corollary 4.3 in the case when $\Omega \subset \mathbb{R}^N$ is an unbounded domain.

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