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# GROUP ACTIONS ON MANIFOLDS AND SMOOTH AMBIENT HOMOGENEITY

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## 1. Introduction

*Hilbert's 5th problem* [16] was solved in 1952 by Gleason and Montgomery–Zippin [14, 23]. They proved that every locally Euclidean group is a Lie group. A more general version of Hilbert's 5th problem (also known as the *Hilbert–Smith conjecture*) asserts that among locally compact groups only Lie groups can act effectively on manifolds. It was proved except for the case where the acting group $G$ is a group of $p$-adic integers [4, 24, 32, 34]. In 1946, Bochner and Montgomery proved this conjecture for groups $G$ acting on a manifold $M$ by diffeomorphisms [3]. A simpler proof of this fact can be obtained with the use of the following concept of smooth ambient homogeneity [8].

A subset $K$ of a manifold $M$ is said to be *smoothly homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset M$ of $x$ and $y$ and a diffeomorphism $h: (O_x, O_xK, x) \to (O_y, O_yK, y)$. This is a property of the orbits for actions by diffeomorphisms of a topological group $G$ on a manifold $M$.

The main result of our paper is the following.

**Theorem 1.1.** *Let $K$ be a locally compact (possibly nonclosed) subset of a manifold $M$. Then $K$ is smoothly homogeneous if and only if $K$ is a smooth submanifold of $M$.*

The assumption of local compactness is important in this theorem: e.g., $\mathbb{Q}$ is a smoothly homogeneous subset of $\mathbb{R}$ which, obviously, is not a submanifold of $\mathbb{R}$.

If a topological group acts freely on a smooth manifold $M$, then every orbit of this action is diffeomorphic to $G$. If the group $G$ acts by diffeomorphisms, then every orbit of the action is a smoothly homogeneous subset of $M$. By our theorem, this subset is a smooth submanifold of $M$. Hence it itself is a manifold and so $G$ is a smooth manifold.

## 2. The Proof of Theorem 1.1

The main idea of the proof of Theorem 1.1 can be found in various forms in our earlier papers [8, 9, 26, 27]. It can be easily explained in the case where $K$ is a subset of $\mathbb{R}^2$. First, observe that if $K$ is not a nowhere dense subset of $\mathbb{R}^2$, then it follows from smooth homogeneity that $K$ is an open subset of $\mathbb{R}^2$. Assume now that $K$ is nowhere dense in $\mathbb{R}^2$. Applying the local compactness of $K$, we get an open triangle in $\mathbb{R}^2 \setminus K$, one of whose vertices belongs to $K$. Since $K$ is smoothly homogeneous, a triangle of this kind can be found for all points of $K$.

Recalling the Baire category theorem, we can infer that there exists an open square in $\mathbb{R}^2$ whose intersection $K'$ with $K$ is nonempty and such that every point $x \in K'$ is a vertex of an isosceles triangle, the interior of which misses $K$, and all the triangles are parallel to one another. We can also assume that the length of the sides of the triangles is less than the altitude of the isosceles triangles and that one of the sides $AB$ of the square is parallel to the base of the triangles. The triangles that are symmetric to the ones above do not intersect $K'$ either since the length of the side of the square was chosen to be less than the altitude

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of the triangles. Therefore, if the projection of $K'$ onto $AB$ is not nowhere dense, then it is well defined and turns out to be a Lipschitz chart. On the other hand, if the projection of $K'$ onto $AB$ is nowhere dense, then we apply the local compactness to find a point in $K'$ which is a vertex of the open nonconvex sector which does not intersect $K$. Exactly as above, we find that $K$ has an isolated point. Thus, $K$ must consist of only isolated points.

Applying the Baire category argument once again, we can infer that the constructed chart is actually a differentiable chart when $\dim K = n - 1$. But [12] gives us a tool which works in all cases. The Lipschitz charts, constructed in the first case, always have a point of differentiability [12]. By smooth homogeneity, $K$ is thus a smooth submanifold of $M$, as asserted.

Our proof reveals a close relationship between the concept of homogeneity and the idea of taming the subsets of manifolds, pinched by tangent balls. The latter problem was extensively investigated in the past by various authors [7, 13, 15, 18–22].

The proof of Theorem 1.1 can be used to prove its analog, stating that only $C^n$-submanifolds of the $C^n$-manifold $M$ are $C^n$-homogeneous, where $C^n$-homogeneity is also defined as smooth homogeneity, only we must replace the term *smooth* by *$C^n$-differentiable*. The *$C^n$-case* can also be obtained from the $C^1$-case by applying it to the tangent bundle of $M$.

## 3. Conclusion

It would be interesting to obtain an analytical version of Theorem 1.1.

**Conjecture 3.1.** *If a locally compact subset $K$ of an analytical manifold $M$ is analytically homogeneous, then $K$ is an analytical submanifold of $M$.*

The idea of our proof can also be used to provide the following theorem.

**Theorem 3.2.** *If a locally compact subset $K$ of a given smooth manifold $M$ in uniformly quasiconformally homogeneous, then $K$ is a Lipschitz submanifold of $M$.*

To take the next step in the attack on the Hilbert–Smith conjecture, it would be interesting to prove an analog of our theorem for Lipschitz homogeneous subsets.

**Conjecture 3.3.** *If a locally compact subset $K$ of a Lipschitz manifold $M$ is Lipschitz homogeneous, then $K$ is a Lipschitz submanifold of $M$.*

To prove the Hilbert–Smith conjecture for actions by Lipschitz maps, it suffices to establish that $K$ cannot be a Cantor set in this conjecture. A Lipschitz homeomorphism will always have a point of differentiability [12]. One could also try to get a dense collection of such diffeomorphisms and apply Theorem 1.1 to get a contradiction.

**Conjecture 3.4.** *Suppose that $A^p$ acts freely on $\mathbb{R}^n$ by Lipschitz homeomorphisms. Then there exist $h \in A^p$ and $x$ in $\mathbb{R}^n$ such that $h'(x) \neq 0$.*

The analog of Theorem 1.1 for arbitrary homeomorphisms is not true: the Antoine necklace [1] is an ambiently homogeneous Cantor set in $\mathbb{R}^3$. For further examples in this direction, see [6, 30, 31, 33]. However, these examples cannot be extended to free actions of $A^p$ on $\mathbb{R}^n$. Thus, the Hilbert–Smith conjecture for action by homeomorphisms remains an open question.

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