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# $C^1$-HOMOGENEOUS COMPACTA IN $\mathbb{R}^n$ ARE $C^1$-SUBMANIFOLDS OF $\mathbb{R}^n$

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ABSTRACT. We give the characterization of $C^1$-homogeneous compacta in $\mathbb{R}^n$: Let $K$ be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then $K$ is $C^1$-homogeneous if and only if $K$ is a $C^1$-submanifold of $\mathbb{R}^n$.

## 1. INTRODUCTION

Several years ago, in his traditional course on ordinary differential equations at the Moscow State University, V. I. Arnol′d assigned to his students the following problem: given a one-parameter group $\mathcal{G} = \{h^t : \mathbb{R}^2 \to \mathbb{R}^2\}_{t\in[0,1]}$ of diffeomorphisms of $\mathbb{R}^2$, *continuously* depending on the parameter $t$, show that the group $\mathcal{G}$ actually depends on $t$ *smoothly* (cf. [1]). Arnol′d originally expected that the students would try to *generalize* the standard argument for the 1-dimensional case $\{h^t : \mathbb{R} \to \mathbb{R}\}_{t\in[0,1]}$ where continuity implies linearity. As it turned out, such a generalization didn't work and the exercise became an unsolved problem.

One can show that such an approach works only if the orbit consists of *smooth* curves. A proof for this special case can be found in [6] and it is based on a very simple geometric idea, a short description of which we quote from [20], where a generalization of the Arnol′d problem to arbitrary $C^1$-homogeneous *planar* compacta was treated (for related concepts see [4], [10], [11], [13]–[17]): given a smooth orbit $K \subset \mathbb{R}^2$ and a point $\omega \in K$, $K$ locally separates $\mathbb{R}^2$, for some neighborhood $U$ of $\omega$ in $\mathbb{R}^2$ we can get points $x, y \in U\backslash K$ from different components. Now construct tangent circles $C_x, C_y \subset U$ to $K$, centered at $x$ and $y$, and use $C^\infty$-homogeneity of the orbit $K$ to move $C_x$ and $C_y$ so that they become tangent at $\omega$, hence "wedging" $K$ at $\omega$, giving a tangent to $K$ at $\omega$. In order to get a solution of the Arnol′d problem, it essentially remains to observe that the tangents to $K$ change continuously at at least one point, hence by $C^\infty$-homogeneity at all points.

We begin by recalling from [20] that a subset $K \subset \mathbb{R}^n$ is said to be $C^1$-*homogeneous* if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset$

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$\mathbb{R}^n$ of $x$ and $y$, respectively, and a $C^1$-*diffeomorphism*

$$h\colon (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y),$$

i.e. $h$ and $h^{-1}$ have continuous first derivatives.

The purpose of this paper is to generalize the characterization of $C^1$-homogeneous compacta in $\mathbb{R}^2$ given in [20] to those which lie in $\mathbb{R}^n$, $n \geq 1$:

**Theorem 1.1.** *Let $K$ be a locally compact (possibly nonclosed) subset of $\mathbb{R}^n$. Then $K$ is $C^1$-homogeneous if and only if $K$ is a $C^1$-submanifold of $\mathbb{R}^n$.*

Notice that without the condition of *local compactness* Theorem 1.1 fails to be true—just consider the case when $K = \mathbb{Q}$ and $n = 1$. It is also clear that the $C^1$-homogeneity cannot be replaced by the *topological* homogeneity, as the example of the *Antoine* wild Cantor set in $\mathbb{R}^3$ demonstrates. On the other hand, observe that Theorem 1.1 remains valid if one replaces $\mathbb{R}^n$ by any $n$-dimensional $C^1$-manifold $M^n$. It will also be clear from the proof that one can weaken the $C^1$-homogeneity condition to the requirement that the map $h\colon (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y)$ be differentiable in $O_x$ and its derivative $h'$ be continuous at the point $x$. Finally, we remark that Theorem 1.1 remains valid if one replaces "$C^1$-homogeneity" and "$C^1$-submanifold of a $C^1$-manifold $M^n$" by "$C^n$-homogeneity" and "$C^n$-submanifold of a $C^n$-manifold $M^n$", respectively.

In conclusion we explain how our work is related to the classic *Hilbert-Smith* Conjecture, which asserts that *only* Lie groups can act freely on an $n$-manifold or, equivalently, that the $p$-adic integers $A_p$ can*not* act freely on any $n$-manifold $M^n$ (cf. [3], [7], [18]). Recall that S. Bochner and D. Montgomery [2] have proved that if a topological group $G$ acts on an $n$-manifold $M^n$ with *diffeomorphisms*, then $G$ must be a Lie group. Their result now follows immediately from our theorem. Indeed, in this case every orbit of the group $G$ is diffeomorphic to $G$ and is a smoothly homogeneous subset of the manifold $M^n$. Therefore, the orbits of $G$ are themselves smooth submanifolds of $M^n$ and thus, in particular, cannot be diffeomorphic to the $p$-adic integers $A_p$; hence neither can $G$.

The next possible step in an attack on the Hilbert-Smith Conjecture would be to answer the following question:

**Question 1.1.** *Can the $p$-adic integers $A_p$ act freely on any $n$-manifold $M^n$ by LIP-homeomorphisms?*

Namely, an LIP-homeomorphism $h\colon M^n \to M^n$ will always have a point of differentiability ([9], Theorem (3.1.6)). One should then try to get a dense collection of such $h$'s and apply our theorem above to get a contradiction. (For more on the relationship between the $C^1$-homogeneity and the Hilbert-Smith Conjecture, see the survey [21].)

## 2. A reduction of Theorem 1.1

**Definition 2.1.** Let $U \subset \mathbb{R}^k$ be an open subset and $K \subset \mathbb{R}^n$, $k \leq n$. An embedding $f\colon U \to K$ is called a chart in $K$ if $f(U)$ is open in $K$.

**Definition 2.2.** Let $X$ and $Y$ be metric spaces. A map $f\colon X \to Y$ is said to be *Lipschitz*, $f \in \mathrm{Lip}$, if there exists $M > 0$ such that for every pair of points $x, y \in X$, $\mathrm{dist}_Y(f(x), f(y)) \leq M \cdot \mathrm{dist}_X(x, y)$.

In the *if* direction, the proof of Theorem 1.1 is straightforward: Suppose that $K \subset \mathbb{R}^n$ is a $C^1$-submanifold of $\mathbb{R}^n$ and pick any two points $x, y \in K$. Then there exist open neighborhoods $O_x, O_y \subset \mathbb{R}^n$ of $x$ and $y$, respectively and $C^1$-diffeomorphisms $h_x\colon (O_x, O_x \cap K, x) \to (\mathbb{R}^n, \mathbb{R}^k \times 0, 0)$ and $h_y\colon (O_y, O_y \cap K, y) \to (\mathbb{R}^n, \mathbb{R}^k \times 0, 0)$, where $k = \dim K$. Consequently, the composition $h_y^{-1} h_x\colon (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y)$ is then also a $C^1$-diffeomorphism, and so $K$ is indeed $C^1$-homogeneous in $\mathbb{R}^n$.

In order to prove the *only if* direction of Theorem 1.1, it suffices to prove that $K$ has a Lipschitz chart, since then, by Proposition 2.1, $K$ has a point of differentiability.

**Propositon 2.1** (H. Rademacher)**.** *Every Lipschitz map $f\colon \mathbb{R}^m \to \mathbb{R}^m$ is almost everywhere differentiable.*

*Proof.* See for example [9], Theorem (3.1.6). □

By virtue of $C^1$-homogeneity, $K$ is then a differentiable $k$-manifold, and by Proposition 2.2 it has a point of continuous differentiability, so again by virtue of $C^1$-homogeneity, $K$ is continuously differentiable at every point and is thus a $C^1$-submanifold of $\mathbb{R}^n$.

**Propositon 2.2** (R. Baire)**.** *For every differentiable map $f\colon \mathbb{R}^m \to \mathbb{R}^n$, the set of points of continuity of $f'$ is an everywhere dense $G_\delta$-set.*

*Proof.* See for example [19], Theorem (15.3.3). □

## 3. The Lipschitz charts

We begin this section by a (very) nonrigorous description of an idea how to construct a Lipschitz chart. (The real argument is yet to come, in all details.) We shall illustrate this in the case when $K \subset \mathbb{R}^2$. Observe first, that if $K$ is not nowhere dense in $\mathbb{R}^2$, then it follows by the $C^1$-homogeneity that $K$ is an open subset of $\mathbb{R}^2$. So assume now that $K$ is nowhere dense in $\mathbb{R}^2$. Apply the local compactness of $K$ to get an open triangle in $\mathbb{R}^2 \backslash K$, one of whose vertices $x$ belongs to $K$.

Since $K$ is $C^1$-homogeneous, such a triangle can be found for all points $x \in K$. Invoking the Baire Category Theorem, one then concludes that there exists an open square in $\mathbb{R}^2$ whose intersection $K'$ with $K$ is nonempty and such that every point $x \in K'$ is a vertex of an isosceles triangle, the interior of which misses $K$, and all the triangles are parallel to one another (see Figure 1).

We may also assume that the length of the side of the square is smaller than the altitude of the isosceles triangles and that one of the sides of the square $AB$ is parallel to the basis of the triangles.

The triangles, symmetric to the ones above and denoted by a dotted line in Figure 1, also do not intersect $K'$, since the length of the side of the square was chosen to be less than the altitude of the triangles. Therefore, if the projection of $K'$ onto $AB$ is not nowhere dense, then its inverse is well defined and turns out to be a Lipschitz chart.

On the other hand, if the projection of $K'$ onto $AB$ is nowhere dense, then we apply the local compactness of find a point in $K'$ which is a vertex of an open nonconvex sector which does not intersect $K'$.

Since $K$ is $C^1$-homogeneous, every point of $K$ is a vertex of an open nonconvex sector not intersecting $K$. Applying the Baire Category Theorem, we can conclude

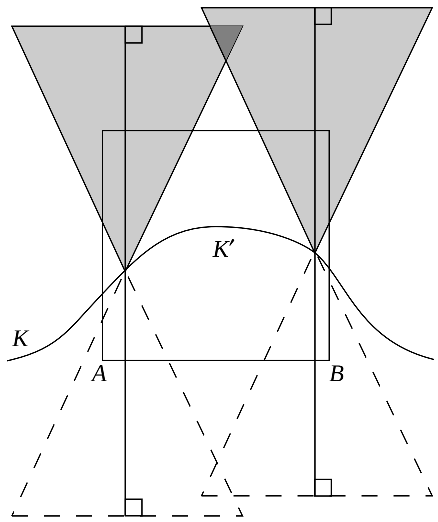


FIGURE 1

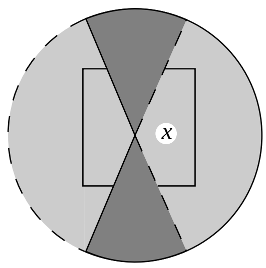


FIGURE 2

that there exists an open square whose intersection $K'$ with $K$ is nonempty, that every point of $K'$ is a vertex of a nonconvex sector which does not intersect $K$, and that all the sectors are parallel to one another. We may also assume that the radius of the sector is greater than the side of the square. Then the centrally symmetric sectors (denoted by dotted lines in Figure 2) also do not intersect $K'$.

Hence $K'$ has an isolated point, so $K$ must consist of isolated points only. (End of the nonrigorous introduction.)

For the purpose of a precise construction of a Lipschitz chart, we introduce some conventions and notations. Hereafter we shall denote by $K$ a locally compact subset of $\mathbb{R}^n$; by $S^{n-1} \subset \mathbb{R}^n$ the unit $(n-1)$-sphere, centered at the origin $0 \in \mathbb{R}^n$; by $\pi\colon \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ the radial projection of $\mathbb{R}^n \backslash \{0\}$ onto $S^{n-1}$; by $B^k = \{(x_1, \ldots, x_n) \in S^{n-1} | x_{k+1} \geq 0,\ x_i = 0 \text{ for } k+2 \leq i \leq n\}$ the $k$-dimensional closed semisphere in $S^{n-1}$, $k \in \{0, 1, \ldots, n-1\}$; and by $O_\varepsilon Y$ the closed $\varepsilon$-neighborhood of $Y$ in $S^{n-1}$ (or $\mathbb{R}^n$—this will be seen from the context). On $S^{n-1}$ we shall use the angular metrics. Finally, we shall assume that $A \in O(n)$, $0 < \varepsilon < \frac{\pi}{2}$, and $r > 0$, and we shall denote by $|\ |$ the norm, and by $+$ and $-$ the obvious linear operators in $\mathbb{R}^n$.

We define the *geometric derivative of $K$ at a point $x \in K$* to be the set

$$D_x K = \bigcap_{\delta>0} \mathrm{Cl}\ \pi\{y - x \mid y \in K, 0 < |y - x| < \delta\}.$$

The points of *$k$-dimensional Lipschitz property* are the members of the set

$$X^k = \{x \in K \mid \text{for some } A \text{ and } \varepsilon\colon D_x K \subseteq O_\varepsilon AB^k\}$$

where $k < n$ and $X^n = K$.

Define

$$T^k_\varepsilon(A, r) = \left\{ y \in \mathbb{R}^n \mid 0 < |y| < r, \frac{y}{|y|} \notin O_\varepsilon AB^k \right\},$$

and let

$$X^k_\varepsilon(A, r) = \{x \in K \mid K \cap (x + T^k_\varepsilon(A, r)) = \varnothing\}$$

where $k < n$, $X^n_\varepsilon(A, r) = K$.

**Lemma 3.1.** *$X^k_\varepsilon(A, r)$ is closed in $K$.*

*Proof.* For $k = n$ this is obvious. So let $k < n$, and choose any $x \in K$ such that $x = \lim_{n\to\infty} x_n$, where $x_n \in X^k_\varepsilon(A, r)$. If $y \in (x + T^k_\varepsilon(A, r)) \cap K$, then there exists $\delta > 0$ such that $O_\delta y \subset (x + T^k_\varepsilon(A, r))$ (since $T^k_\varepsilon(A, r)$ is open). Choose now $n$ such that $|x_n - x| < \delta$. For such $n$, we have that $y \in (x_n + T^k_\varepsilon(A, r))$. However, $y \in K$ and $(x_n + T^k_\varepsilon(A, r)) \cap K = \varnothing$, which yields a contradiction. Hence $x \in X^k_\varepsilon(A, r)$, and thus $X^k_\varepsilon(A, r)$ is indeed a closed subset of $K$. ☐

**Lemma 3.2.** *If $X^k = K$, then there exist $\varepsilon, A$, and $r$ such that $X^k_\varepsilon(A, r)$ is not nowhere dense in $K$.*

*Proof.* For $k = n$ this is obvious. So let $k < n$. Since by hypothesis $X^k = K$, it follows by the definition of $X^k$ that for any $x \in K$ there exist $A$ and $\varepsilon$ such that $D_x K \subset O_\varepsilon AB^k$. Choose $\delta \in (\varepsilon, \frac{\pi}{2})$. Since $D_x K \subset O_\varepsilon AB^k$, it follows from the definition of the geometric derivative that there exists $N \in \mathbb{N}$ such that $(x + T^k_\delta(A, \frac{1}{N})) \cap K = \varnothing$. Choose inside $O(n)$ an everywhere dense sequence $\{A_m\}_{m\in\mathbb{N}}$. Then there exists $M > N$ such that

$$T^k_{\pi/2-1/M}\left(A_M, \frac{1}{M}\right) \subset T^k_\delta\left(A, \frac{1}{N}\right);$$

therefore,

$$\left(x + T^k_{\pi/2-1/M}\left(A_M, \frac{1}{M}\right)\right) \cap K = \varnothing, \quad \text{i.e. } x \in X^k_{\pi/2-1/M}\left(A_M, \frac{1}{M}\right);$$

hence

$$\bigcup_{M=1}^{\infty} X^k_{\pi/2-1/M}\left(A_M, \frac{1}{M}\right) = K.$$

The assertion of the lemma now follows by an application of Lemma 3.1 and the Baire Category Theorem [8]. ☐

**Lemma 3.3.** *If $X^k = K$, then either $K$ has a $k$-dimensional Lipschitz chart or $k \geq 1$ and $X^{k-1} \neq \varnothing$.*

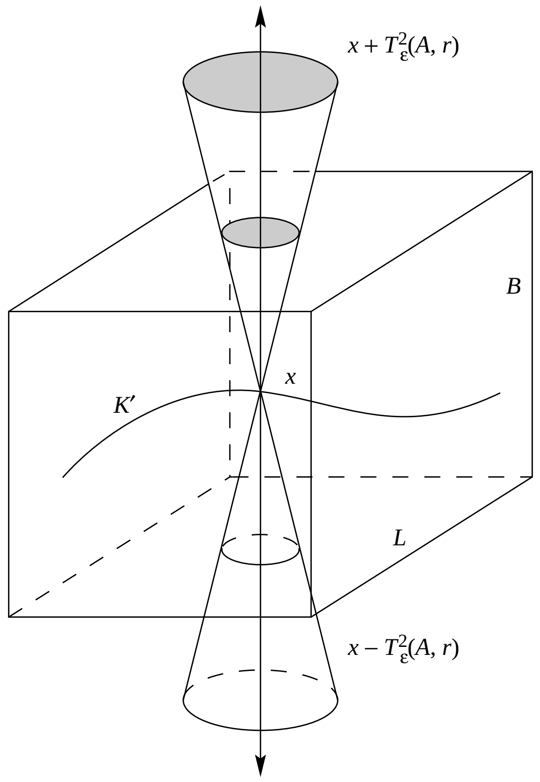


FIGURE 3

*Proof.* Since $X^k = K$, it follows by Lemma 3.2 that there exists $X^k_\varepsilon(A, r)$ which is not nowhere dense in $K$. Therefore there exist a point $x \in X^k_\varepsilon(A, r)$ and a neighborhood $O_x$ of $x$ such that $O_x \cap K \subset X^k_\varepsilon(A, r)$. Inside $O_x$ choose an open $n$-dimensional cube $B$ of diameter less than $r$ and centered at $x$, such that one of its $k$-dimensional sides $L$ is parallel to a hypersurface through $\partial AB^k$ and $O$ (for $k = n, L = B$) (see Figure 3 for $n = 3, k = 2$). Then $K' = B \cap K \subset X^k_\varepsilon(A, r)$.

For every point $y \in K'$,

$$[(y + T^k_\varepsilon(A, r)) \cup (y - T^k_\varepsilon(A, r))] \cap K' = \varnothing.$$

Indeed, since $y \in X^k_\varepsilon(A, r)$, we have that $(y + T^k_\varepsilon(A, r)) \cap K' = \varnothing$. On the other hand, if $z \in (y - T^k_\varepsilon(A, r)) \cap K'$, then $y \in (z + T^k_\varepsilon(A, r)) \cap K'$, which contradicts the fact that $z \in X^k_\varepsilon(A, r)$. Therefore (and since diam $B < r$), $p|_{K'} \colon K' \to L$ is one-to-one, where $p|_{K'}$ is the restriction onto $K'$ of the parallel projection $p \colon B \to L$. We must distinguish between two cases:

*Case* 1: *$p(K')$ is not nowhere dense in $L$.* (This is certainly true for $k = 0$, since $L$ is then a point.) Then $p(K')$ contains an open set $U \subset L$, and hence $q \colon U \to K$ is a $k$-dimensional Lipschitz chart, where $p(q(u)) = u$. Indeed, since $p$ is one-to-one, $q$ is well defined and is clearly an embedding; $q(U) = p^{-1}(U) \cap K$ is open in $K$. That $q$ is Lipschitz follows due to the existence of sets $y + T^k_\varepsilon(A, r)$ and $y - T^k_\varepsilon(A, r)$, in every point $y \in K'$, disjoint with $K'$.

*Case* 2: *$p(K')$ is nowhere dense in $L$.* Then $k \geq 1$. Since $K$ is locally compact, such must also be $K'$; hence $p(K')$ is also locally compact [8]. Therefore there exists a closed $k$-dimensional ball $V \subset L$, centered at $p(x)$, such that $p(K') \cap V$ is compact. Since $p(K')$ is nowhere dense, there exists $y \in L \backslash p(K')$ such that $\operatorname{dist}(y, p(x)) < \frac{\operatorname{diam} V}{4}$. Since $p(K') \cap V$ is compact, it follows that $\operatorname{dist}(y, p(K') \cap V) > 0$, and so there exists $p(z) \in p(K') \cap V$ such that $\operatorname{dist}(y, p(z)) = \operatorname{dist}(y, p(K') \cap V)$. Since $\operatorname{dist}(y, p(x)) < \frac{\operatorname{diam} V}{4}$, it follows that

$$\operatorname{dist}(p(z), p(x)) \leq \operatorname{dist}(p(z), y) + \operatorname{dist}(y, p(x)) \leq 2 \operatorname{dist}(y, p(x)) < \frac{\operatorname{diam} V}{2};$$

hence $p(z) \in V$. Let $C = \{u \in B \mid \operatorname{dist}(y, p(u)) < \operatorname{dist}(y, p(z))\}$. Since $\operatorname{dist}(y, p(x)) < \frac{\operatorname{diam} V}{4}$, it follows that for every $u \in C$

$$\operatorname{dist}(p(u), p(x)) \leq \operatorname{dist}(p(u), y) + \operatorname{dist}(y, p(x)) \leq 2 \operatorname{dist}(y, p(x)) < \frac{\operatorname{diam} V}{2};$$

hence $p(u) \in V$, and so $p(C) \subset V$.

Since $\operatorname{dist}(y, p(z)) = \operatorname{dist}(y, p(K') \cap V)$, it follows that the open ball in $L$, centered at $y$ and of radius $\operatorname{dist}(y, p(z))$, does not intersect $p(K') \cap V$. But, since this ball lies in $V$, it does not intersect $p(K')$ either. Consequently, $C \cap K' = \varnothing$.

Therefore,

$$D_z K \cap \pi[T^k_\varepsilon(A, r) \cup (-T^k_\varepsilon(A, r)) \cup (C - z)] = \varnothing,$$

i.e. $D_z K \subset O_\varepsilon AB^{k-1}$ and hence $X^{k-1} \neq \varnothing$. □

## 4. Proof of Theorem 1.1

By definition $X^n = K$. Therefore there is the minimal $k$ such that $X^k = K$. By Lemma 3.3, either there exists a $k$-dimensional Lipschitz chart (in which case we are done) or $k \geq 1$ and $X^{k-1} \neq \varnothing$, i.e. there exist $x \in K, A$, and $\varepsilon$ such that $D_x K \subset O_\varepsilon AB^{k-1}$. For every $y \in K$, there exists a diffeomorphism $h\colon (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y)$.

Therefore $D_y K = \pi h'_x D_x K \subset \pi h'_x O_\varepsilon AB^{k-1}$, where $h'_x$ denotes the derivative of $h$ at $x$, and $\pi h'_x AB^{k-1}$ is a $(k-1)$-semisphere in $S^{n-1}$ and $\pi h'_x O_\varepsilon AB^{k-1}$ is its closed neighborhood, contained in the $\delta$-neighborhood for some $\delta \in (0, \frac{\pi}{2})$. Hence $y \in X^{k-1}$, and thus $X^{k-1} = K$. This is a contradiction, so the latter case cannot occur.

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