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A Characterization of C^1-Homogeneous Subsets of the Plane.

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Sunto. - *Si dimostra che se K è un sottoinsieme k-dimensionale localmente compatto e C^1-omogeneo del piano $\mathbb{R}^2$, allora:* (i) *se $k = 0$, allora K è un insieme al più numerabile di punti isolati;* (ii) *se $k = 1$, allora K è una collezione al più numerabile di curve C^1 in $\mathbb{R}^2$ con intorni a due a due disgiunti;* (iii) *Se $k = 2$, allora K è un aperto di $\mathbb{R}^2$.*

Introduction.

In a well-known book by V. I. Arnol'd [1] there is the following interesting problem: given a one-parameter group $\mathcal{G} = \{h^t: \mathbb{R}^2 \to \mathbb{R}^2\}_{t \in [0,1]}$ of diffeomorphisms of $\mathbb{R}^2$ *continuously* depending on the parameter t, show that the group $\mathcal{G}$ actually depends on t *smoothly*. Arnol'd's idea was to *generalize* the standard argument used for the 1-dimensional case $\{h^t: \mathbb{R} \to \mathbb{R}\}_{t \in [0,1]}$ where continuity implies linearity. However, one of Arnol'd's students, I. Yaščenko, pointed out in late 1970's that such an approach doesn't work.

As it turns out, Arnol'd's idea works only if the orbits are *smooth* curves. A proof for this special case can be found in [4] and it is based on the following simple geometric idea: given a smooth orbit $K \subset \mathbb{R}^2$ and a point $\omega \in K$, K locally separates $\mathbb{R}^2$, for some neighborhood U of ω in $\mathbb{R}^2$ we can get points $x, y \in U \setminus K$ from different components. Now construct tangent circles $C_x, C_y \subset U$ to K, centered at x and y, and use C^∞-homogeneity of the orbit K to move C_x and C_y so that they become tangent at ω, hence «wedging» K at ω, giving a tangent to K at ω. In order to get a solution of the Arnol'd problem, it essentially remains to

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observe that the tangents to K change continuously at at least one point, hence by C^∞-homogeneity at all points.

The purpose of the present paper is to solve a generalization of the Arnol'd problem—to arbitrary C^1-homogeneous *planar* compacta. (For related concepts see [2], [6], [7], [8]-[12].) The concept of C^1-homogeneity for planar locally compact sets is an analogue of the earlier concept of C^∞-homogeneity for Jordan curves which was introduced in [4] and generalized in [3] to the notion of homogeneity of compact subsets of arbitrary topological manifolds. For related concepts and ideas see [2], [6]-[12].

DEFINITION. – *A locally compact subset K of $\mathbb{R}^2$ is said to be C^1-homogeneous if for every pair of points $x, y \in K$ there exist neighborhoods $O_x, O_y \subset \mathbb{R}^2$ of x and y, respectively and a C^1-diffeomorphism $h\colon (O_x, O_x \cap K, x) \to (O_y, O_y \cap K, y)$.*

The main result of this paper is the characterization of C^1-homogeneous subsets of the Euclidean plane $\mathbb{R}^2$.

THEOREM. – *Let K be a locally compact (possibly nonclosed) subset of $\mathbb{R}^2$. Then K is C^1-homogeneous if and only if K is a C^1-submanifold of $\mathbb{R}^2$, i.e.*

(i) *If* $\dim K = 0$, *then K is at most countable subset of isolated points in $\mathbb{R}^2$;*

(ii) *if* $\dim K = 1$, *then K is at most countable collection of C^1-curves with pairwise disjoint neighborhoods in $\mathbb{R}^2$; and*

(iii) *if* $\dim K = 2$, *then K is an open subset of $\mathbb{R}^2$.*

This result was first presented at the 1989 Pećs Colloquim on Topology and Applications. The preliminary version of this paper was written during the first author's visit to the Steklov Mathematical Institute in Moscow in 1990, on the basis of the long term agreement between the Slovenian Academy of Arts and Sciences and the Russian Academy of Sciences.

Preliminaries.

Let K be a C^1-homogeneous, locally compact subset of the plane. If $\dim K = 2$ then K must clearly contain an open subset of $\mathbb{R}^2$. Since K is C^1-homogeneous it follows that for each point $p \in K$, there is an open neighborhood $O_p \subset K$, hence K is an open subset of $\mathbb{R}^2$. Therefore we shall need to consider in the sequel only the cases when $\dim K \in \{0, 1\}$. Note, that this implies, in particular that K can not be dense in $\mathbb{R}^2$.

Fix an axis O_s in the plane $\mathbb{R}^2$. Suppose that $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi)$, $r \in \mathbb{R}^+$. Given an arbitrary point $V \in \mathbb{R}^2$, denote by $T(V, r, \alpha, \beta)$ the interior of the isosceles triangle ABV with the base AB, angle 2β at the top vertex V, of altitude r and with the angle between VD and O_s being equal to α, where D is the foot of the altitude. Let $X_{r,\alpha,\beta} = \{x \in T \mid T(x, r, \alpha, \beta) \cap K = \emptyset\}$, i.e. the triangle can intersect with K only at its boundary points (see Fig. 1).

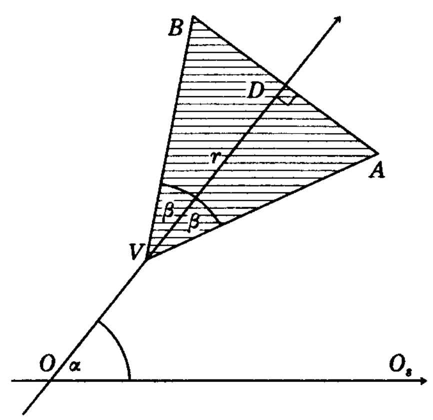

Fig. 1.

LEMMA. – *There exist r, α, β, such that $X_{r,\alpha,\beta}$ contains an open subset of K.*

PROOF. – We argue in three steps:

ASSERTION 1. – There exist r, α, β, such that $X_{r,\alpha,\beta} \neq \emptyset$.

PROOF. – Let $y \in K$ be any point and choose $U \subset \mathbb{R}^2$ to be an open disk containing y, such that $ClU \cap K$ is compact. Since K is not dense in $\mathbb{R}^2$, it follows that there is a point $z \in U \setminus K$.

Since $ClU \cap K$ is compact, it follows that there is a point $p \in ClU \cap K$, such that $\operatorname{dist}(z, p) = \operatorname{dist}(z, ClU \cap K) = d > 0$. Then $U \cap O_d(z) \cap K = \emptyset$, where $O_d(z)$ is the open ball of radius d, centered at z, and $U \cap O_d(z)$ contains an open equilateral triangle p as the top vertex and z as the foot of the altitude to the triangle's basis. So there exist r, α, β, such that $X_{r,\alpha,\beta} \neq '$. ■

ASSERTION 2. – For every $x \in K$, there exist $r \in \mathbb{Q}$, $\alpha \in \mathbb{Q}$, and $\beta \in \mathbb{Q}$, such that $T(x, r, \alpha, \beta) \cap K = \emptyset$.

PROOF. – Take the points $y, p \in K$ from the proof of Assertion 1. Let $x \in K$. By hypothesis, there exist neighborhoods $O_p, O_x \subset \mathbb{R}^2$ of p and x, respectively and a C^1-diffeomorphism h: $(O_p, O_p \cap K, p) \to (O_x, O_x \cap K, x)$. Since O_p is open, there is a small enough isosceles triangle T with p as the top vertex, such that $T \subset O_p$ and $T \cap K = \emptyset$.

Inside T we can find a triangle ΔpAB such that $h(pA)$ and $h(pB)$ are C^1-curves, meeting at x and having tangents at x which intersect at an angle greater than 0. There is therefore a triangle $T(x, r, \alpha, \beta)$ between these two tangents of $h(pA)$ and $h(pB)$ inside $h(\Delta pAB)$ with rational values for α, β, r. Since h is injective and $\Delta pAB \cap K = \emptyset$, it follows that $T(x, r, \alpha, \beta) \cap K = \emptyset$. So $T(x, r, \alpha, \beta)$ is the required triangle (see Fig. 2). ■

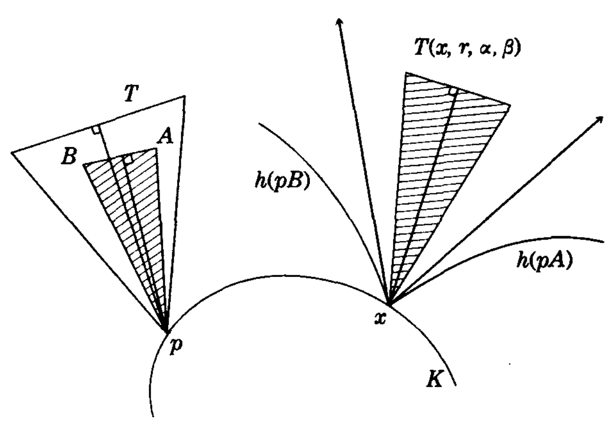

Fig. 2.

ASSERTION 3. – $X_{r,\alpha,\beta}$ is closed in K.

PROOF. – Let $p \in K$, such that $p = \lim_{n \to \infty} p_n$, where $p_n \in X_{r,\alpha,\beta}$. We need to show that $p \in X_{r,\alpha,\beta}$. There are triangles $\Delta p_n A_n B_n = T(p_n, r, \alpha, \beta)$, such that $\Delta p_n A_n B_n \cap K = \emptyset$. We may assume that the sequences $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ converge to some points A and B, respectively. Then $\Delta pAB \subseteq \bigcup_{n=1}^{\infty} \Delta p_n A_n B_n$. Hence $\Delta pAB \cap K = \emptyset$. This shows that $p \in X_{r,\alpha,\beta}$, i.e. that $X_{r,\alpha,\beta}$ is closed. ■

We now complete the proof of the lemma. By Assertion 2

$$\bigcup \{X_{r,\alpha,\beta} \mid r \in \mathbb{Q}, \alpha \in \mathbb{Q}, \beta \in \mathbb{Q}\} = K,$$

and by Assertion 3, $X_{r,\alpha,\beta}$ is closed. Since K is locally compact, using Baire category argument, we obtain that there are $r, \alpha, \beta \in \mathbb{Q}$, such that $X_{r,\alpha,\beta}$ contains an open subset of K. ■

Proof of the Theorem.

By lemma, we can find an open subset $V \subset \mathbb{R}^2$, and r, α, β such that $V \cap K \subset X_{r,\alpha,\beta}$. We can find an open square $U \subset V$, such that $U \cap K \neq \emptyset$, $Cl U \cap K$ is compact, diam $U < r$ and two sides of U are parallel to the altitude of $T(x, r, \alpha, \beta)$, where $x \in X_{r,\alpha,\beta}$. Let l be a straight line, parallel to the other two sides of U, situated below the square U. Let $\pi: U \to l$ be the orthogonal projection (see Fig. 3).

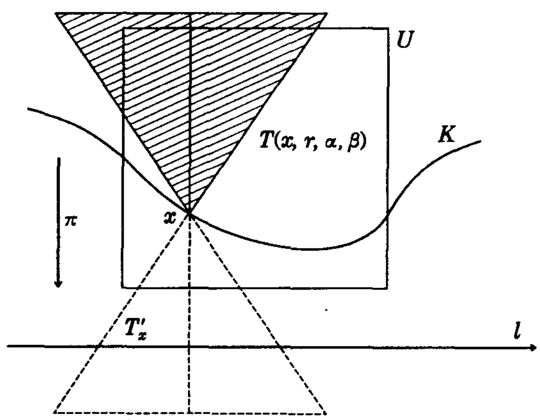

Fig. 3.

Denote $T(x, r, \alpha, \beta)$ by T_x and the reflection of T_x with respect to x by T'_x. If $y \in U \cap T'_x \cap K$, then $x \in T_y$, which is a contradiction. Hence $U \cap (T_x \cup T'_x) \cap K = \emptyset$. Since diam $U < r$ it follows that $ab\pi: U \cap K \to l$ is an injection. We must distinguish two cases.

Case 1. $\pi(U \cap K)$ *contains an open subset* W *of* l.

Let $\varphi: W \to \mathbb{R}$ be the map such that $\varphi(t) = \mathrm{dist}(t, \pi^{-1}(t) \cap K)$.

Since for each $x \in U \cap K$, $(T_x \cup T'_x) \cap U \cap K = \emptyset$ and since $\mathrm{diam}\, U < r$, it follows that $\varphi \in \mathrm{Lip}$, i.e. φ is a *Lipschitz* function. By [5], Theorem (3.1.6), there is a point $t_0 \in W$ such that $\varphi(t)$ is differentiable at t_0. Since K is C^1-homogeneous, φ is therefore differentiable at each point $t \in W$. For each $t \in W$, $\varphi'(t) = \lim_{n \to \infty} (\varphi(t + 1/n) - \varphi(t))/(1/n)$. So $\varphi'(t)$ is the pointwise limit of continuous functions. Therefore there is a point $t_1 \in W$, such that $\varphi'(t)$ is continuous at t_1. Since K is C^1-homogeneous it follows that $\varphi'(t)$ is continuous at each point $t \in W$, and K is a C^1-submanifold of $\mathbb{R}^2$.

Case 2. $\pi(U \cap K)$ *is nowhere dense in* l.

If $U \cap K = \{x\}$ then x is an isolated point of K. Since K is C^1-homogeneous, it follows that every point of K is then isolated so in this case K is at most countable set of isolated points.

Suppose now that there are at least two points $x, y \in U \cap K$. Since $\pi(U \cap K)$ is nowhere dense in l, it follows that there is a point $p \in l \setminus \pi(U \cap K)$ between $\pi(x)$ and $\pi(y)$. Since $\pi(U \cap K) \cap [\pi(x); \pi(y)]$ is compact (where $[\pi(x); \pi(y)]$ denotes the segment of l between $\pi(x)$ and $\pi(y)$), there is a point $\pi(z) \in \pi(U \cap K)$, such that $\mathrm{dist}(\pi(z), p) = \mathrm{dist}(\pi(U \cap K), p) = \rho$.

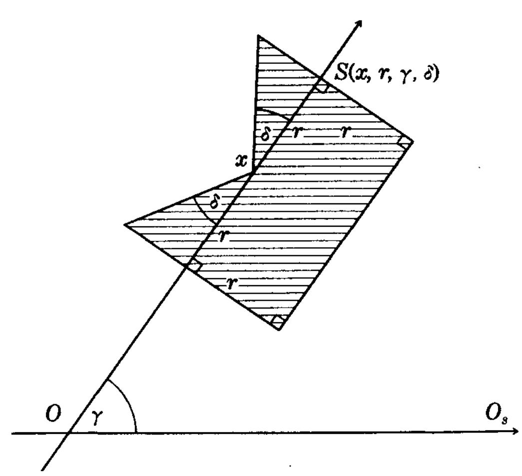

Fig. 4.

For an arbitrary $x \in \mathbb{R}^2$, $r > 0$, $\gamma \in [0, 2\pi)$, $\delta \in (-\pi/, \pi/2)$ denote by $S(x, r, \gamma, \delta)$ an open pentagon defined in the similar manner as $T(x, r, \alpha, \beta)$ (see Fig. 4). If $\delta < 0$ then $S(x, r, \gamma, \delta)$ is symmetric to $S(x, r, \gamma, -\delta)$ with respect to vertices.

Take z and ρ as above. Then there are $\gamma \in [0, 2\pi)$, $\delta \in (-\pi/2, \pi/2)$, such that $S(z, \rho, \gamma, \delta) \cap K = \emptyset$. There exist $r > 0$, $\gamma \in [0, 2\pi)$, $\delta \in (-\pi/2, \pi/2)$, such that $Y_{r,\gamma,\delta} = \{x \in K \mid S(x, r, \gamma, \delta) \cap K = \emptyset\}$ is dense in K (the proof of this fact is the same as the proof of Lemma). Hence we can find an open square $U \subset \mathbb{R}^2$, such that $U \cap K \neq \emptyset$, $U \cap K \subset Y_{r,\gamma,\delta}$ and $\operatorname{diam} U < r$. If $x, y \in U \cap K$, then $x \in S(y, r, \alpha, \beta)$ or $y \in S(x, r, \alpha, \beta)$, which is a contradiction. Hence, $U \cap K$ is an isolated point of K. This shows that also in this case K is at most countable set of isolated points. ■

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