Obstructions for Seifert fibrations and an extension of the Bolsinov–Fomenko theorem on integrable Hamiltonian systems

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In 1994, Bolsinov and Fomenko [1] proved a theorem on the topological orbital classification of non-degenerate integrable Hamiltonian systems with two degrees of freedom on 3-dimensional constant-energy manifolds. For the motivations and for a short survey, see [2], §1 and [1], §1. It was shown that two such systems are equivalent if a particular invariant is the same for each. This invariant is a graph with some additional labels on its vertices and edges. A necessary condition was that the Hamiltonian system under consideration does not have unstable periodic orbits with a non-orientable separatrix. Since orbits of this type occur in examples, for instance, in the Kovalevskaya top, it is of interest to remove the above condition. We show that the Bolsinov-Fomenko theorem holds without this condition.

Theorem (cf. [1], Theorem 4.1). *Let X be the set of non-degenerate integrable Hamiltonian systems with two degrees of freedom on constant-energy orientable 3-manifolds, up to an orientation-preserving topological orbital equivalence. Then there is an injection of X into the set of t-labelled graphs W regarded up to t-equivalence.*

The definitions of a t-labelled graph and of t-equivalence are as in [1]; see also [2], [4]. In fact, the more general situation needs no additions or corrections with respect to [1], except for the new condition that the P-labels can be atoms with stars. We note that the image of the injection in the theorem and the dependence of t-labels on the orientation of the constant-energy 3-manifold are described in [1], §12.3 and §13.5. Moreover, in §13 of [1] another labeling on W was constructed, the so-called t-molecule, which is simpler in a sense. These phenomena can also be extended to our more general situation; we recall that the P-labels can now be atoms with stars.

Our proof is based upon the following general observation, which could possibly be applied to other problems. A bifurcation of Liouville tori in a Bott integrable Hamiltonian system can be described by a neighbourhood of $F^{-1}(c)$, where F is an additional integral and c is a critical value of it. If the critical submanifold of F corresponding to c is a circle, then this neighbourhood is a Seifert fibration Q over a (non-closed) 2-surface P [3]. More precisely, by a *double* P^* we mean a 2-surface with boundary and with an involution χ on P^* that has finitely many fixed points, which are called *stars*. We set $P = P^*/\chi$ (P^* is called the double of the *surface* P). Let $p\colon P^* \to P$ be the projection. By N we denote the p-image of the set of fixed points of χ (that is, of the stars). By $\tilde{P}$ we denote the closed surface obtained by attaching discs to the boundary circles of P. A 3-*atom* is a fibre bundle over S^1 with fibre P^* and sewing map χ, that is, $Q(P^*) \cong P^* \times I/\{(a,0) \sim (\chi a, 1)\}$ (cf. [2], Definition 2.2). By this definition, $Q(P^*)$ depends only on P and not on P^*. Therefore, in what follows we write $Q(P)$ or simply Q instead of $Q(P^*)$. Let us define a map $\pi\colon Q \to P$ by $\pi[(a,t)] = p(a)$ (a Seifert fibration having singular fibres only over stars and only of type $(2,1)$).

To study the bifurcation of Liouville tori, we construct a Poincaré section of the flow on Q [1]. If the critical circle has an *orientable* separatrix diagram (or, equivalently, P has no stars), then $Q \cong P \times S^1$, and the Poincaré section can be chosen to be a cross-section. Therefore, Poincaré sections can be classified by the methods of classical obstruction theory. If the critical circle has a *non-orientable* separatrix diagram (or, equivalently, P has stars), then the Seifert fibration is not locally trivial. Nevertheless, a Poincaré section is a Seifert analogue of a cross-section. An embedding $f\colon P^* \to Q$ is called a *Seifert section* if $\pi \circ f = p$. In the smooth category, we must assume in addition that f is transversal to the fibres of the map π. In [1] the Seifert sections were called transversal platforms. The main part of our proof is the classification of the *Seifert* sections

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of a Seifert fibration. The proof of the theorem modulo the classification theorem stated below is similar to that in [1]; for a detailed proof, see [4]. We omit the $\mathbb{Z}$-coefficients in the notation for cohomology groups. For a space with involution, the symmetric (co)homology groups are denoted by adding the subscript S to the standard notation.

Classification Theorem. *For a fixed double P^*, the set X of Seifert sections regarded up to isotopy over π is in one-to-one correspondence with $H^1(P)$.*

Proof. Let us define a map

$$q\colon P^* \times S^1 \cong P^* \times I/\{(a,0) \sim (a,1)\} \to P^* \times I/\{(a,0) \sim (\chi a,1)\} \cong Q$$

by the formula

$$q[(a,t)] = \begin{cases} [(a,2t)], & 0 \le t \le \frac{1}{2}, \\ [(\chi a, 2t-1)], & \frac{1}{2} \le t \le 1\,. \end{cases}$$

Since χ is an involution, it follows that q is well defined and continuous.

Let $f\colon P^* \to Q$ be a Seifert section. For each $x \in P^*\backslash N$, there is a *unique* point $f'(x) \in P^* \times S^1$ such that $qf'(x) = f(x)$. For each $x \in N$ there are *two* points $s,t \in S^1$ such that $q(x,s) = q(x,t) = f(x)$. Since a small punctured disc neighbourhood of x in P^* is connected, we can choose $f'(x)$ to be either (x,s) or (x,t) so that the map $f'\colon P^* \to P^* \times S^1$ becomes continuous. This map f' is a classical section of the trivial bundle $P^* \times S^1 \to P^*$. Since f is an embedding, it follows that $p_2 f'(x)$ and $p_2 f'(\chi x)$ are not antipodes for any point $x \in P^*$. Here $p_2\colon P^* \times S^1 \to S^1$ stands for the projection. Therefore, there is a canonical homotopy between f' and a *symmetric* section f'' (that is, a section f'' such that $p_2 f''(x) = p_2 f''(\chi x)$ for any $x \in P^*$). Moreover, the map $q \circ F$ is a Seifert section and $(q \circ F)'' = F$ for any symmetric section $F\colon P^* \to P^* \times S^1$. Obviously, Seifert sections f and g are isotopic over π if and only if the corresponding symmetric sections f'' and g'' are symmetrically homotopic (or, equivalently, isotopic). Then X is in one-to-one correspondence with the set X'' of symmetric sections of the trivial bundle $P^* \times S^1 \to P^*$ regarded up to a symmetric homotopy. In turn, the latter set is in one-to-one correspondence with $H^1_S(P^*;\mathbb{Z})$. We can readily see that $H^1_S(P^*;\mathbb{Z}) \cong H^1(P;\mathbb{Z})$. □

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