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New results on embeddings of polyhedra and manifolds in Euclidean spaces

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Abstract. The aim of this survey is to present several classical results on embeddings and isotopies of polyhedra and manifolds in $\mathbb{R}^m$. We also describe the revival of interest in this beautiful branch of topology and give an account of new results, including an improvement of the Haefliger–Weber theorem on the completeness of the deleted product obstruction to embeddability and isotopy of highly connected manifolds in $\mathbb{R}^m$ (Skopenkov) as well as the unimprovability of this theorem for polyhedra (Freedman, Krushkal, Teichner, Segal, Skopenkov, and Spież) and for manifolds without the necessary connectedness assumption (Skopenkov). We show how algebraic obstructions (in terms of cohomology, characteristic classes, and equivariant maps) arise from geometric problems of embeddability in Euclidean spaces. Several classical and modern results on completeness or incompleteness of these obstructions are stated and proved. By these proofs we illustrate classical and modern tools of geometric topology (engulfing, the Whitney trick, van Kampen and Casson finger moves, and their generalizations).

Contents

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Introduction

Many theorems in mathematics state that an arbitrary space of a given abstractly defined class is a subspace of a certain 'standard' space of this class. Examples include the Cayley theorem on embeddings of finite groups in symmetric groups, the theorem on the representation of compact Lie groups as virtual subgroups of $GL(V)$ for a certain linear space V, the Urysohn theorem on embeddings of normal spaces with countable base in Hilbert space, the general position theorem for embeddings of finite polyhedra in $\mathbb{R}^m$, the Menger–Nöbeling–Pontryagin theorem on embeddings of finite-dimensional compact spaces in $\mathbb{R}^m$, the Whitney theorem on embeddings of smooth manifolds in $\mathbb{R}^m$, the Nash theorem on embeddings of Riemannian manifolds in $\mathbb{R}^m$, the Gromov theorem on embeddings of symplectic manifolds in $\mathbb{R}^{2n}$, and so on. The Kolmogorov–Arnold solution of Hilbert's 13th problem can also be stated in terms of embeddings (§ 5). Being of interest in themselves, these embeddability theorems also prove to be powerful tools for solving other problems. Subtler problems of embeddability and classification (up to isotopy) of embeddings of a given space in $\mathbb{R}^m$ for a *given* m are among the most important classical problems in topology.

In this survey, from the vast variety of methods and results concerning embeddability and isotopy problems, we have chosen the deleted product method. (We also consider the van Kampen and Whitney methods, which are earlier-known special cases of it.) The deleted product method is a demonstration of the general mathematical idea of 'complements of diagonals' (Borsuk, Lefschetz, Shapiro, and Wu). The classical Haefliger–Weber theorem (1963–1967) states that the deleted product condition is sufficient for embeddability of polyhedra and manifolds (respectively, isotopy of embeddings of polyhedra and manifolds) under the 'metastable' dimension restriction $m \geqslant \frac{3(n+1)}{2}$ (respectively, $m \geqslant \frac{3n}{2}+2$). In 1995–1998, the second author completely solved the well-known problem as to whether it is possible to weaken the metastable restriction in these theorems for piecewise linear manifolds and polyhedra (for details, see §§ 2–4). This result was obtained by combining and sharpening methods and results from various branches of topology: theory of immersions, homotopy theory, engulfing, the generalized Whitney trick, and generalized van Kampen and Casson finger moves. In this survey, we present the Haefliger–Weber theorem, the above-mentioned results of the second author together with some corollaries, and the connections between the embedding problem and other fields of mathematics (in particular, algebraic topology and functional analysis).

The first part opens with the statement and discussion of these problems (§ 1). The van Kampen and Whitney methods and related results are presented in §§ 2 and 3. We state in § 4 the classical Haefliger–Weber result on the sufficiency of the deleted product criterion for embeddability of n-dimensional spaces in $\mathbb{R}^m$

for $m \geqslant \frac{3(n+1)}{2}$ (and for the isotopy of embeddings for $m > \frac{3(n+1)}{2}$) without additional high-connectedness assumptions. Here we also present the sufficiency theorem for this criterion under the condition $m < \frac{3(n+1)}{2}$ and additional high-connectedness assumptions. A variety of corollaries are given in §§ 2–4. We also state the Haefliger–Hirsch theorem on immersions of manifolds in $\mathbb{R}^m$ and a piecewise linear analogue of it due to Harris and the second author. In § 5, we discuss recent results in the theory of *basic embeddings*, which emerged at the junction of functional analysis and topology in the course of investigation of Hilbert's 13th problem. This section is independent of the other parts of the survey.

The second part deals with proofs of completeness theorems (Haefliger, Weber, and Skopenkov) and incompleteness theorems (Freedman, Krushkal, Teichner, Segal, Skopenkov, and Spież) for the deleted product obstruction. These proofs are used to illustrate classical and modern methods of geometric topology: the Whitney trick, engulfing, and van Kampen finger moves (as well as their generalizations). The theory of embeddings is not the only area where these methods apply. In § 6, we construct examples (announced in §§ 2 and 4) of incompleteness of the van Kampen and deleted product obstructions for $4 \leqslant m \leqslant \frac{3n}{2} + 1$. The construction of these examples involves higher-dimensional Casson finger moves. The proofs of completeness of the deleted product criterion are carried out in the piecewise linear case (studied by Weber and the second author). For the reader's convenience, we use the historical approach in the exposition of proofs: each method is first illustrated in a simple particular case and then applied in its full strength. The Whitney trick and van Kampen finger moves are exemplified in § 7 by the proofs of Theorem 2.1.a (van Kampen–Shapiro–Wu) for $n \geqslant 3$ and Theorem 4.1.e (Weber) for $m = 2n + 1$. The engulfing method is illustrated in § 8 by the proof of Theorem 3.2.a (Penrose–Whitehead–Zeeman–Irwin). In §§ 9 and 10, the methods of §§ 7 and 8 are applied to the proof of Theorem 4.1.e in the general case $m \geqslant \frac{3(n+1)}{2}$. The exposition of the second part of the proof in § 10 is based on a method from [136]. It is substantially simpler than the original proof [153] and does not resort to the Freudenthal suspension theorem. In § 11, we outline the proof of a sharpening of Theorem 4.1.e in the piecewise linear case (Theorem 4.2.e) and a piecewise linear version of the Haefliger–Hirsch Theorem 4.1.i (Theorem 4.2.i). In § 12, the idea is given for proofs of isotopic versions of the above-mentioned results.

I. Obstructions to embeddability and isotopy

§1. Problems of embeddability and isotopy

A compactum, a polyhedron, or a smooth manifold K is said to be topologically, piecewise-linearly, or smoothly *embeddable* in $\mathbb{R}^m$ if there is a continuous, piecewise-linear, or smooth injective map $f\colon K \to \mathbb{R}^m$. (In the smooth case, it is additionally required that df be a monomorphism at each point.) Such a map f is called an *embedding* of K in $\mathbb{R}^m$. The first main problem dealt with in this survey is to find conditions under which a given polyhedron K is embeddable in $\mathbb{R}^m$ for a given m. In most of this survey (except for § 5), the piecewise linear case is considered; in particular, all maps are assumed to be piecewise linear, unless stated otherwise. We recall that a map $f\colon K \to \mathbb{R}^m$ is said to be *piecewise linear* if it is linear on each simplex of some triangulation of K. The results and references for the smooth case in §§ 3 and 4 are given in parentheses. By a polyhedron we always understand a *finite* polyhedron. For a beautiful survey on smooth embeddings, see [35]. For $m \geqslant 2\dim K + 1$, a polyhedron K can always be embedded in $\mathbb{R}^m$. This follows by a general position argument. (Moreover, even $K \times I$ can be embedded in $\mathbb{R}^m$ [122].) Thus, the problem is of interest only for $m \leqslant 2\dim K$.

The similar notions of immersions and quasi-embeddings are of interest in themselves and prove useful in the study of embeddings. Their definitions can be obtained by weakening the injectivity condition in the definition of an embedding in two dual ways. A polyhedron K is said to be *immersible* (respectively, *quasi-embeddable*) in $\mathbb{R}^m$ if for each $\varepsilon > 0$ there is a map $f\colon K \to \mathbb{R}^m$ such that $f(x) \neq f(y)$ whenever $\operatorname{dist}(x, y) < \varepsilon$ (respectively, $\operatorname{dist}(x, y) > \varepsilon$). For $m - n \geqslant 3$, any piecewise linear embedding of a piecewise linear n-manifold in $\mathbb{R}^m$ is locally flat; hence, our definition coincides with the usual one for piecewise linear manifolds [54]. Many results of this survey have their counterparts for immersions and quasi-embeddings.

Another interesting problem is how to determine whether two given embeddings $f, g\colon K \to \mathbb{R}^m$ are 'of the same type'. The best-known example is the classification of knots in $\mathbb{R}^3$. More precisely, two topological, piecewise linear, or smooth embeddings $f, g\colon K \to \mathbb{R}^m$ are said to be (*ambient*-) *isotopic* if there is a topological, piecewise linear, or smooth surjective homeomorphism $F\colon \mathbb{R}^m \times I \to \mathbb{R}^m \times I$ such that $F(y, 0) = (y, 0)$ for each $y \in \mathbb{R}^m$, $F(f(x), 1) = g(x)$ for each $x \in K$, and $F(\mathbb{R}^m \times \{t\}) = \mathbb{R}^m \times \{t\}$ for each $t \in I$. Such a homeomorphism F is called an (*ambient*) *isotopy*. The same term is used for the homotopy $\mathbb{R}^m \times I \to \mathbb{R}^m$ or the family of maps $F_t\colon \mathbb{R}^m \to \mathbb{R}^m$ generated by F in an obvious manner. The second classical topological problem considered in this survey is to find conditions under which two embeddings $f, g\colon K \to \mathbb{R}^m$ are isotopic. (Just as for the embedding problem, we mostly consider the piecewise linear case.) If $m \geqslant 2\dim K + 2$, then any two embeddings of a polyhedron K in $\mathbb{R}^m$ are isotopic by a general position argument. Likewise, two topological, piecewise linear, or smooth immersions $f, g\colon K \to \mathbb{R}^m$ are said to be *regularly homotopic* if there is a topological, piecewise linear, or smooth immersion $F\colon K \times I \to \mathbb{R}^m \times I$ such that $F(x, 0) = (f(x), 0)$, $F(x, 1) = (g(x), 1)$ for each $x \in K$, and $F(K \times \{t\}) \subset \mathbb{R}^m \times \{t\}$ for each $t \in I$.

Obviously, (ambient) isotopy is an equivalence relation on the set of embeddings of K in $\mathbb{R}^m$. It is a stronger equivalence relation than non-ambient isotopy, isoposition, concordance, bordism, and so on. Two embeddings $f, g\colon K \to \mathbb{R}^m$ are said to be (*non-ambient*-) *isotopic* if there is an embedding $F\colon K \times I \to \mathbb{R}^m \times I$ such that $F(x, 0) = (f(x), 0)$, $F(x, 1) = (g(x), 0)$ for each $x \in K$, and $F(K \times \{t\}) \subset \mathbb{R}^m \times \{t\}$ for each $t \in I$. In the smooth category, or, for $m - \dim K \geqslant 3$, in the topological and piecewise linear categories, isotopy implies ambient isotopy ([4], [26], § 7, [66], and [74]). For $m - \dim K \leqslant 2$, this is not the case. (For example, any knot $S^1 \to S^3$ is piecewise-linearly isotopic to the trivial knot but need not be ambient-isotopic to it.) Two embeddings $f, g\colon K \to \mathbb{R}^m$ are said to be (orientation-preserving) *isopositioned* if there is an (orientation-preserving) homeomorphism $h\colon \mathbb{R}^m \to \mathbb{R}^m$ such that $h \circ f = g$. The Alexander–Guggenheim theorem asserts that orientation-preserving isoposition is equivalent to isotopy [125]. Two embeddings $f, g\colon K \to \mathbb{R}^m$ are said to be *ambient-concordant* if there is a surjective homeomorphism $F\colon \mathbb{R}^m \times I \to \mathbb{R}^m \times I$ (which is called a *concordance*) such that

$F(y,0) = (y,0)$ for each $y \in \mathbb{R}^m$ and $F(f(x),1) = (g(x),1)$ for each $x \in K$. The definition of *non-ambient concordance* can be obtained from that of non-ambient isotopy in a similar way by dropping the last condition of level preservation. In the smooth category, or, for $m - \dim K \geqslant 3$, in the topological and piecewise linear categories, concordance implies ambient concordance and isotopy [70], [73], [165]. (This is not the case in codimensions 1 and 2.) This result allows one to reduce the isotopy problem to the relative embeddability problem (see § 12 for details).

We give a list (by no means complete) of references on closely related questions of geometric topology. In embeddability and isotopy problems, $\mathbb{R}^m$ can be replaced by an *arbitrary* space Y. The cases in which Y is a manifold or a product of trees have been studied most widely ([145], Theorem 4.6 and Remark, [33], [34], and [167]). For embeddings of *products*, *twisted products*, and *Lie groups*, see [2], [116], [117], [121], and references therein. For embeddings *up to cobordism*, see [11]. For embeddings *up to homotopy*, see [152], § 11. For the classification of *link maps*, see [40], [138], and references therein. For embeddings of polyhedra in *certain* manifolds see [10], [97], [108], [133], [150], [151], and references therein. For the problem of *embeddability of compacta* and the close problem of *approximability by embeddings*, see [3], [16], [17], [23], [104], [105], [118], § 9, [119], [129], and references therein. For the problem of *intersection of compacta*, see [25], [140], and references therein. For *basic embeddings*, see § 5 of the present paper.

§2. The van Kampen obstruction

As shown in § 1, the first non-trivial case of the embeddability problem is the investigation of embeddability of n-polyhedra in the Euclidean space $\mathbb{R}^{2n}$. For $n = 1$, the solution is given by the Kuratowski criterion [89] (see also [118], § 2 and references therein). However, for $n > 1$ there is no simple criterion. (For example, there are infinitely many closed non-orientable 2-surfaces that cannot be embedded in $\mathbb{R}^3$, none containing any other; see also [128].) In [80], an obstruction to embeddability of n-polyhedra in $\mathbb{R}^{2n}$ was constructed for arbitrary n, and some steps were made towards proving that the obstruction is complete. (See Theorem 2.1.a below and the historical remark in the end of this section.)

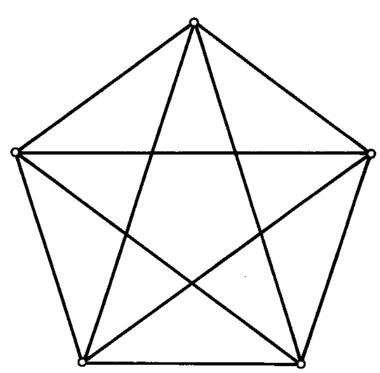

Figure 2.1

To explain the idea of the construction of the van Kampen obstruction, we sketch the proof of the fact that the complete graph K_5 with five vertices is non-planar (Fig. 2.1). We take an arbitrary generic map $f\colon K_5 \to \mathbb{R}^2$. For any two edges σ and τ, the intersection $f\sigma \cap f\tau$ consists of finitely many points. Let v_f be the sum mod 2 of the numbers $|f\sigma \cap f\tau|$ over all unordered pairs $\{\sigma, \tau\}$ of disjoint edges of K_5. For the map f shown in Fig. 2.1, $v_f = 1$. Every generic map $f\colon K_5 \to \mathbb{R}^2$ can be transformed to any other such map through isotopies of $\mathbb{R}^2$ and Reidemeister moves for graphs on the plane (Fig. 2.2). The proof of this assertion, which is similar to that of the Reidemeister theorem for knots, is omitted here, since the assertion itself is needed only for sketching the idea but not for the rigorous proof. For each edge of K_5 with vertices a and b, the graph $K_5 - \{a, b\}$ obtained from K_5 by deleting the vertices a and b and the interiors of edges adjacent to a and b is a circle. (It is this property of K_5 that is needed in the proof.) Therefore, v_f is invariant under the Reidemeister moves, and hence $v_f = 1$ for *each* generic map $f\colon K_5 \to \mathbb{R}^2$. It follows that K_5 is non-planar.

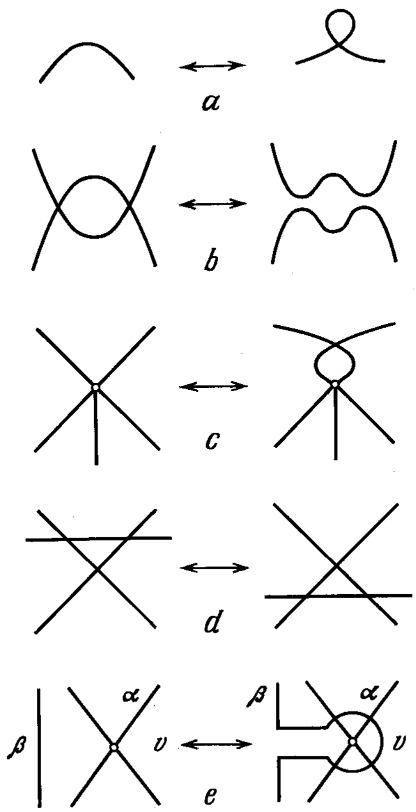

Figure 2.2

Now let us discuss some generalizations of this proof; they will be used later. This proof actually implies a stronger assertion. Let e be an edge of K_5 and Σ the

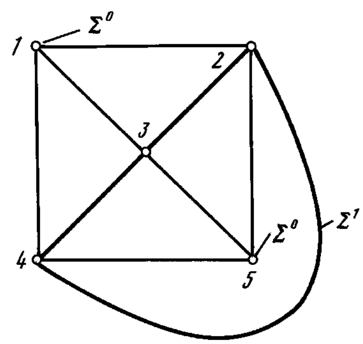

Figure 2.3

cycle in K_5 formed by the edges non-adjacent to e. Let $\overset{\circ}{e} = \operatorname{Int} e$. Then $K_5 - \overset{\circ}{e}$ is embeddable in $\mathbb{R}^2$ (Fig. 2.3), and for each embedding $g\colon K_5 - \overset{\circ}{e} \to \mathbb{R}^2$, the g-images of the ends of e (the 0-sphere) lie on different sides of $g\Sigma$. Likewise, one can prove that the graph K_{33} (three houses and three wells) is not embeddable in $\mathbb{R}^2$ and the 2-skeleton K of the 6-simplex is not embeddable in $\mathbb{R}^4$. Moreover, let e be a 2-simplex of K, and let $P = K - \overset{\circ}{e}$. Then P is embeddable in $\mathbb{R}^4$ and contains two disjoint spheres Σ^2 and $\Sigma^1 = \partial e$ such that for each embedding $P \to \mathbb{R}^4$ the images of these spheres link with a non-zero linking coefficient [28]. (More precisely, the linking coefficient is ± 1.)

Let us introduce the main definition of this section. We choose a triangulation T of a polyhedron K. The space

$$\widetilde{T} = \bigcup \{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \varnothing\}$$

is called the *simplicial deleted product* of K. Since the equivariant homotopy type of $\widetilde{T}$ depends only on K ([132], Lemma 2.1 and [63], § 4), we replace $\widetilde{T}$ by $\widetilde{K}$ in this section. The group $\mathbb{Z}_2$ acts on $\widetilde{K}$ by the transposition of factors. Let $K^* = \widetilde{K}/\mathbb{Z}_2$.

Now we are in a position to define the van Kampen obstruction $v(K)$ for $n = 1$. Throughout this section we omit $\mathbb{Z}_2$ coefficients from the notation of (co)chain and (co)homology groups. For any generic piecewise linear map $f\colon K \to \mathbb{R}^2$ and disjoint edges σ and τ of T, the intersection $f(\sigma) \cap f(\tau)$ consists of finitely many points. Let $v_f(\sigma, \tau) = |f(\sigma) \cap f(\tau)| \mod 2$. Then $v_f \in C^2(K^*)$. The cochain v_f is invariant under isotopy of $\mathbb{R}^2$ and the first four Reidemeister moves (Fig. 2.2.a–d). The fifth Reidemeister move (Fig. 2.2.e) alters v_f by adding the cochain that is equal to 1 on the class of the 2-simplex $\alpha \times \beta$ for $v \in \alpha$ and is zero on the other 2-simplices. This cochain is the coboundary $\delta[v \times \beta]$ of the elementary cochain in $B^2(K^*)$ that is equal to 1 on the class of the 1-simplex $v \times \beta$ and is zero on the other 2-simplices. Then the equivalence class

$$v_2(K) = [v_f] \in H^2(K^*) = C^2(K^*)/B^2(K^*)$$

is independent of f. (Since $\dim K^* = 2$, it follows that $C^2(K^*) = Z^2(K^*)$.) The resulting cohomology class $v_2(K)$ is the *van Kampen obstruction* to embeddability of K in $\mathbb{R}^2$ (more precisely, the mod 2 analogue of this obstruction). It is clear that $v_2(K)=0$ for all planar graphs K.

Following [132], Lemma 3.5, let us prove that $[v_f]$ is independent of f without using the fact concerning the Reidemeister moves (not proved here). For given generic maps $f_0,f_1\colon K\to\mathbb{R}^2$, we consider an arbitrary generic homotopy $f\colon K\times I\to\mathbb{R}^2$ and define a cochain $v_f\in C^{2n-1}(K^*)$ by the formula $v_f[\sigma^n\times\tau^{n-1}]=|f(\sigma\times I)\cap f(\tau\times I)|\mod 2$. Then the fact that v_f is independent of f follows from the relation $v_{f_0}-v_{f_1}=\delta v_f$, which is easy to check.

Likewise, one defines the mod 2 van Kampen obstruction $v_2(K)\in H^{2n}(K^*)$ to embeddability of an n-polyhedron K in $\mathbb{R}^{2n}$. The genuine van Kampen obstruction $v(K)$ (with integer coefficients) is constructed as follows. We choose a triangulation of K and define $\widetilde{K}$ and K^* as above. Next, we choose an orientation of $\mathbb{R}^{2n}$ and of n-simplices of K. For any generic map $f\colon K\to\mathbb{R}^{2n}$ and any two disjoint oriented n-simplices $\sigma,\tau\in K$, the intersection $f(\sigma)\cap f(\tau)$ consists of finitely many points. We define a cochain $v_f\in C^{2n}(\widetilde{K},\mathbb{Z})$ (the *intersection cochain*) by the formula $v_f(\sigma,\tau)=f\sigma\cdot f\tau=\sum_{P\in f\sigma\cap f\tau}\operatorname{sign}P$, where $\operatorname{sign}P=+1$ if the positive n-bases of $f\sigma$ and $f\tau$ (in this order) constitute a positive $2n$-basis in $\mathbb{R}^{2n}$ and $\operatorname{sign}P=-1$ otherwise. Clearly, $v_f(\sigma\times\tau)=(-1)^n v_f(\tau\times\sigma)$. Here v_f lies in the subgroup $C^{2n}_s(\widetilde{K},\mathbb{Z})\subset C^{2n}(\widetilde{K},\mathbb{Z})$ of cochains assuming equal values on $2n$-cells $\sigma\times\tau$ and $\tau\times\sigma$ (for even n) or opposite values (for odd n). The cohomology class

$$v(K)=[v_f]\in H^{2n}_s(\widetilde{K},\mathbb{Z})=C^{2n}_s(\widetilde{K},\mathbb{Z})/B^{2n}_s(\widetilde{K},\mathbb{Z})$$

(like $v_2(K)$, it is independent of f) is the *van Kampen obstruction* to embeddability of K in $\mathbb{R}^{2n}$. We note that $H^{2n}_s(\widetilde{K},\mathbb{Z})\cong H^{2n}(K^*,\mathbb{Z})$ for even n [29]. One can readily show that $v(K)$ is independent of the choice of the orientations in $\mathbb{R}^{2n}$ and on the n-simplices of K, up to an automorphism of $H^{2n}_s(\widetilde{K},\mathbb{Z})$.

The above constructions can be generalized in several ways. For a subpolyhedron A of a polyhedron K, one can define the obstruction to extending a given embedding $A\subset\partial B^m$ to an embedding $K\to B^m$ in a similar way [29]. Likewise, one can construct the difference class $u(f)\in H^{2n}_s(\widetilde{K},\mathbb{Z})$ of an embedding $f\colon K\to\mathbb{R}^{2n+1}$ [147]. For the van Kampen obstruction to *approximation by embeddings*, see [15], § 4, [119], [3], and [121].

Theorem 2.1. a) ([80], [132], [159], [127], [29]) *For the embeddability of a finite n-polyhedron K in $\mathbb{R}^{2n}$, it is necessary that $v(K)=0$. For $n\neq 2$, this condition is sufficient, whereas for $n=2$ it is not.*

b) [160]. *If two embeddings $f,g\colon K\to\mathbb{R}^{2n+1}$ of a finite n-polyhedron K are isotopic, then $u(f)=u(g)$. For $n\geqslant 2$, this condition is sufficient for isotopy, whereas for $n=1$ it is not. (However, for $n=1$, embeddings f,g such that $u(f)=u(g)$ are homologous* [147].)

The relative van Kampen obstruction is complete for $n\neq 2$ (for $n\geqslant 3$, see [160], while for $n=1$ this follows from the relative version of the Kuratowski criterion) and is incomplete for $n=2$ (see § 6). Theorem 2.1.a implies the piecewise linear case of Theorem 3.1 below (which was proved earlier than Theorem 2.1.a). For $n\geqslant 3$, Theorem 2.1.a has an interesting corollary: every acyclic n-polyhedron is piecewise-linearly embeddable in $\mathbb{R}^{2n}$ ([159]; see also [60]). For $n = 2$, it was proved independently of Theorem 2.1.a that any acyclic 2-polyhedron is topologically embeddable in $\mathbb{R}^4$ [85]. The van Kampen obstruction can possibly be applied to find the minimal m such that a polyhedron that is a product of graphs is embeddable in $\mathbb{R}^m$ ([2], [32]).

In this paper, we do not present the proof of Theorem 2.1.b. (For $n \geqslant 2$, it can be proved by analogy with Theorem 2.1.a with the use of ideas from § 12; for $n = 1$, see [147].) The necessity part in Theorem 2.1.a was actually proved in the construction of the van Kampen obstruction. The sufficiency in Theorem 2.1.a for $n \geqslant 3$ is proved in § 7, while for $n = 1$ it is a consequence of the Kuratowski graph planarity criterion. The incompleteness of the van Kampen obstruction for $n = 2$ is proved in § 6.

§3. The Whitney obstruction

Theorem 3.1 ([80], [158]). *Every piecewise linear* (*smooth*) *n-manifold is piecewise-linearly* (*smoothly*) *embeddable in $\mathbb{R}^{2n}$.*

The proof of Theorem 3.1 for the piecewise linear (smooth) case is given in § 8 (§ 7). The dimension $2n$ in Theorem 3.1 is the best possible for $n = 2^k$ (since $\mathbb{R}P^n$ cannot be embedded in $\mathbb{R}^{2n-1}$) and is not the best possible for other n (see Corollary 3.4). There is a celebrated and difficult conjecture saying that every closed n-manifold can be embedded in $\mathbb{R}^{2n+1-\alpha(n)}$, where $n = 2^{k_1} + \cdots + 2^{k_{\alpha(n)}}$ and $k_1 < \cdots < k_{\alpha(n)}$. (For the solution of a similar conjecture for immersions, see [21] and [93].) We note that the n-manifold $N = \mathbb{R}P^{2^{k_1}} \times \cdots \times \mathbb{R}P^{2^{k_{\alpha(n)}}}$ is not embeddable in $\mathbb{R}^{2n-\alpha(n)}$.

Theorem 3.2. a) ([42], [112], [78]) *Every closed $(2n - m)$-connected piecewise linear* (*smooth*) *n-manifold N is piecewise-linearly* (*smoothly*) *embeddable in $\mathbb{R}^m$ provided that $m \geqslant n + 3$* ($m \geqslant \frac{3(n+1)}{2}$).

b) ([42], [78]) *Suppose that N and M are closed $(2n - m)$- and $(2n - m + 1)$-connected piecewise linear* (*smooth*) *n-manifolds* (*with or without boundary*) *and $f\colon N \to M$ is a map such that $f|_{\partial N}$ is an embedding in ∂M. Then f is piecewise-linearly* (*smoothly*) *homotopic* rel ∂N *to an embedding provided that $m \geqslant n + 3$* ($m \geqslant \frac{3(n+1)}{2}$).

The proof of the piecewise linear case of Theorem 3.2 is given in § 8. We note that the piecewise linear case of Theorem 3.2.a is not interesting for $m < \frac{3n}{2} + 1$. Indeed, then $2n - m > \frac{n}{2} - 1$ and every $(2n - m)$-connected manifold is a homotopy sphere. Since $\frac{3n}{2} + 1 > m \geqslant n + 3$, it follows that $n \geqslant 6$. Thus $N \cong S^n$, and Theorem 3.2.a obviously holds. We note that the smooth case of Theorem 3.1 is false for $m < \frac{3n}{2} + 1$, although N is a homotopy sphere [61]. For generalizations of Theorem 3.2, see [57], [65], [67], [41], [72], [38], [81], [13], [71], [95], [163], and also below in this section.

Let us introduce some notation, which will be used in §§ 3 and 4. For a group G, let $G_{(l)}$ (respectively, $G_{[l]}$) be G for even l and $G/2G$ (respectively, the subgroup of G formed by elements of order 2) for odd l. To any homomorphism $\varphi : G \to H$, we assign homomorphisms $\varphi_{(l)}\colon G_{(l)} \to H_{(l)}$ and $\varphi_{[l]} : G_{[l]} \to H_{[l]}$ defined in an obvious way. If G is a finite Abelian group, then, clearly, $G_{(l)} \cong G_{[l]}$.

We recall the following definitions. A closed manifold N (respectively, a pair $(N, \partial N)$) is said to be *homologically k-connected* if N is connected and $H_i(N) = 0$ for each $i = 1, \dots, k$ (respectively, $H_i(N, \partial N) = 0$ for each $i = 0, \dots, k$). In both cases, this condition is equivalent to $\widetilde{H}_i(X) = 0$ for each $i = 0, \dots, k$, where X is either N or $(N, \partial N)$ and the $\widetilde{H}_i$ are the reduced homology groups. We note that if $H_0(N, \partial N) = 0$, then the manifold N has no closed connected components. We adopt the following conventions: 0-connectedness is equivalent to homological 0-connectedness and to connectedness, and k-connectedness for $k \leqslant 0$ is 0-connectedness.

In 1934, Whitney proved that for any orientable manifold N and any generic immersion $f\colon N \to \mathbb{R}^m$, the homology class

$$\overline{W}_{m-n}(N) \in H_{2n-m}(N, \mathbb{Z}_{(m-n)}) \cong H^{m-n}(N, \mathbb{Z}_{(m-n)})$$

of the projection of the singular submanifold

$$\widetilde{\Delta}(f) = \bigl\{(x, y) \in N \times N \mid x \neq y, fx = fy\bigr\}$$

on N is independent of f. This class is called the *Whitney obstruction* to embeddability of N in $\mathbb{R}^m$: if N can be embedded in $\mathbb{R}^m$, then $\overline{W}^k(N) = 0$ for $k \geqslant m - n$ ([156]; see also [107], [114], [124], and [157]).

Theorem 3.3 ([53], [153]; for the case $m < \frac{3(n+1)}{2}$, see [69], §11, [135], and [137])**.** *Suppose that N is a closed $(2n-m-1)$-connected piecewise linear (smooth) n-manifold. For $m \geqslant n+3$ ($m \geqslant \frac{3(n+1)}{2}$ or $(m,n) = (12l-1, 8l-1)$), N is piecewise-linearly (smoothly) embeddable in $\mathbb{R}^m$ if and only if $\overline{W}_{m-n}(N) = 0$.*

For $m \geqslant \frac{3n+1}{2}$, the condition of *homotopy* $(2n-m-1)$-connectedness in Theorem 3.3 can be weakened to *homology* $(2n-m-1)$-connectedness. Theorem 3.3 follows from Theorems 4.1.e and 4.2.e and Corollary 4.3. Just as for Theorem 3.2, the piecewise linear case of Theorem 3.2 is not interesting for $m < \frac{3n}{2}$, since it follows from [83] that every piecewise linear homology n-sphere can be embedded in $\mathbb{R}^{n+1}$ (piecewise-linearly if $n \neq 3$, and only topologically if $n = 3$). The following assertion is a consequence of Theorem 3.3 and [100], [101] for $n \neq 3, 4$. (For $n = 3, 4$, this was proved separately.)

Corollary 3.4. *The following assertions hold.*

1) ([109], [53], [58], [150], [153], [8], [24], [27], [123]) *Every closed, orientable (for $n = 2^k$), piecewise linear (smooth) n-manifold is piecewise-linearly (smoothly) embeddable in $\mathbb{R}^{2n-1}$.*

2) ([46], [153]; for $n = 6$, see [135]) *If $n \geqslant 6$ ($n \geqslant 8$) is even and $n \neq 2^k(2^h+1)$ for any integers $k, h \geq 2$, then every piecewise linear (smooth) n-manifold N such that $H_1(N) = 0$ is piecewise-linearly (smoothly) embeddable in $\mathbb{R}^{2n-2}$.*

3) ([135], [137]) *Every closed homologically 2-connected smooth 7-manifold is smoothly embeddable in $\mathbb{R}^{11}$.*

Let us complete this section by stating the isotopy analogues of the above results.

Theorem 3.5 ([163], [42], [142], [36]). *Every piecewise linear* (*respectively, smooth, topological locally flat*) *embedding* $S^n \to S^m$ *is piecewise-linearly* (*respectively, smoothly, topologically*) *unknotted if* $m - n \geqslant 3$ (*respectively,* $m \geqslant \frac{3(n+1)}{2}$, $m \geqslant n+3$).

In Theorem 3.5, the dimension restrictions are sharp, and the local flatness assumption in the topological case is necessary ([43], [47], [126]).

Theorem 3.6. *Suppose that* N *is a closed connected piecewise linear* (*smooth*) n*-manifold and* $m - n \geqslant 3$ ($m \geqslant \frac{3n}{2} + 2$).

1) ([42], [166]) *If* N *is* $(2n - m + 1)$*-connected, then any two piecewise linear* (*smooth*) *embeddings* $N \to \mathbb{R}^m$ *are piecewise-linearly* (*smoothly*) *isotopic.*

2) ([53], [153], [69], [7], [8]; for $m \leqslant \frac{3(n+1)}{2}$, see [135]) *If* N *is orientable and* $(2n-m)$*-connected, then the piecewise linear* (*smooth*) *isotopy classes of embeddings* $N \to \mathbb{R}^m$ *are in a one-to-one correspondence with* $H^{m-n-1}(M, \mathbb{Z}_{(m-n)})$.

3) ([53], [153]; for $n = 3$, see [135]) *If* $n \neq 2$ *and* N *is non-orientable, then the piecewise linear* (*smooth*) *isotopy classes of embeddings* $N \to \mathbb{R}^{2n}$ *are in a one-to-one correspondence with* $H^{n-1}(M, \mathbb{Z}_{(n)}) \otimes \mathbb{Z}_{(n-1)}$.

For $m \geqslant \frac{3n+1}{2}$, the condition of *homotopy* $(2n-m)$-connectedness in Theorem 3.6 can be replaced by *homology* $(2n - m)$-connectedness ([135], [137]).

§4. The deleted product condition

The 'complements of the diagonal' idea plays an important role in various branches of mathematics ([31], [37], [149]). The deleted product necessary condition for embeddability and isotopy is a manifestation of this idea in the theory of embeddings. In fact, it originates from two celebrated theorems: the Lefschetz fixed point theorem and the Borsuk antipodes theorem [9].

Before stating the above-mentioned necessary condition, we consider the following example. Let us prove that S^n cannot be embedded in $\mathbb{R}^n$. Although this fact is obvious, the method of the proof given here admits a wide generalization. Suppose the contrary: there is an embedding $f\colon S^n \to \mathbb{R}^n$. We define a map $\widetilde{f}\colon S^n \to S^{n-1}$ by setting $\widetilde{f}(x) = \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$. Here by $-x$ we denote the antipode of the point $x \in S^n$. Since f is an embedding, it follows that $\widetilde{f}$ is well defined. Obviously, $\widetilde{f}$ is equivariant with respect to the antipodal involutions on S^n and S^{n-1}. The restriction $\widetilde{f}|_{S^{n-1}}$ extends to S^n and hence is homotopic to zero. However, by the Borsuk–Ulam theorem, every equivariant map $S^{n-1} \to S^{n-1}$ is not null-homotopic. This contradiction proves the non-embeddability of S^n in $\mathbb{R}^n$. We have actually proved that there is a pair of points x and $-x$ such that $f(x) = f(-x)$.

To state the deleted product necessary condition, we need the following definition. The *deleted product* $\widetilde{N}$ of a topological space N is the product of N by itself minus the diagonal:

$$\widetilde{N} = \{(x, y) \in N \times N \mid x \neq y\}.$$

Now suppose that $f\colon N \to \mathbb{R}^m$ is an embedding of a polyhedron N in the Euclidean space $\mathbb{R}^m$. Then the map $\widetilde{f}\colon \widetilde{N} \to S^{m-1}$ given by the formula $\widetilde{f}(x, y) = \dfrac{f(x) - f(y)}{\|f(x) - f(y)\|}$

is well-defined. This map is equivariant with respect to the involution $t(x,y) = (y,x)$ on N and the antipodal involution on S^{m-1}. Thus the *deleted product necessary condition for embeddability of N in $\mathbb{R}^m$* is the existence of at least one equivariant map $\widetilde{N} \to S^{m-1}$. The existence of an equivariant map $\widetilde{N} \to S^{m-1}$ is equivalent to the existence of a section of the bundle $\widetilde{N} \times S^{m-1}/(t \times a) \overset{S^{m-1}}{\underset{g}{\to}} \widetilde{N}/t$, where t is the involution $(x,y) \leftrightarrow (y,x)$ on $\widetilde{N}$, a is the antipodal involution on S^{m-1}, and the map g is given by the formula $g[(x,y),\alpha] = [(x,y)]$. If X is a polyhedron or a smooth manifold, then one can check the deleted product necessary condition by using methods of obstruction theory ([22], [160], [44]). In particular, the van Kampen and Whitney obstructions defined above are the first obstructions to the existence of a section of g, that is, can be derived from the deleted product necessary condition in a purely algebraic way. Since the simplicial deleted product $\widetilde{T}$ (§ 2) is an equivariant retract of $\widetilde{N}$ [63], it follows that the deleted product condition is equivalent to the existence of an equivariant map $\widetilde{T} \to S^{m-1}$.

The deleted product necessary condition for isotopy is constructed as follows. For isotopic embeddings $f,g\colon N \to \mathbb{R}^m$ and an isotopy $F\colon N \times I \to \mathbb{R}^m$ between them, we define a map $\Phi\colon \widetilde{N} \times I \to S^{m-1}$ by the formula

$$\Phi(x,y,t) = \frac{F(x,t) - F(y,t)}{\|F(x,t) - F(y,t)\|}\,.$$

This map is an equivariant homotopy between $\widetilde{f}$ and $\widetilde{g}$. Thus the *deleted product necessary condition for isotopy of embeddings $f,g\colon N \to \mathbb{R}^m$* is an equivariant homotopy of $\widetilde{f}$ and $\widetilde{g}$. This condition is equivalent to the equivalence of sections of the bundle $\widetilde{N} \times S^{m-1}/(t \times a) \overset{S^{m-1}}{\underset{g}{\to}} \widetilde{N}/t$ ([160], [147]). We note that this condition (and its generalizations in terms of isovariant maps or deleted p-fold products [50], [88]) detects neither the ambience of isotopy nor the distinction between the smooth, piecewise linear, and topological categories. Therefore, this condition (and its generalizations) cannot be used to distinguish knots in $\mathbb{R}^3$. Nevertheless, the deleted product condition works well in codimension $\geqslant 3$ (see below).

The deleted product necessary condition to immersibility is constructed as follows. For a sufficiently small neighbourhood $O\Delta$ of the diagonal $\Delta \subset N \times N$, let $SN = O\Delta - \Delta$. If N is a polyhedron, then the equivariant homotopy type of SN is independent of $O\Delta$. For an immersion $h : N \to \mathbb{R}^m$, the map $\widetilde{h}$ is well defined on SN. Thus the *deleted product necessary condition for immersibility of N in $\mathbb{R}^m$* is the existence of an equivariant map $SN \to S^{m-1}$. The deleted product necessary condition for regular homotopy of immersions f and g is the equivariant homotopy of $\widetilde{f}$ and $\widetilde{g}$ on SN.

Thus, let us consider the following assertions for a polyhedron (a smooth manifold) N. The converses of these assertions have just been proved.

(EXI.ε) If there is an equivariant map $\Phi\colon \widetilde{N} \to S^{m-1}$, then N is piecewise-linearly (smoothly) embeddable in $\mathbb{R}^m$.

(EXI.e) If there is an equivariant map $\Phi\colon \widetilde{N} \to S^{m-1}$, then there is a piecewise linear (smooth) embedding $f\colon N \to \mathbb{R}^m$ such that $\widetilde{f} \simeq_{\mathrm{eq}} \Phi$.

(EXI.i) If there is an equivariant map $\Phi\colon SN \to S^{m-1}$, then there is a piecewise linear (smooth) immersion $h\colon N \to \mathbb{R}^m$ such that $\widetilde{h} \simeq_{\mathrm{eq}} \Phi$ on SN.

(CLA.e) If $f_0, f_1\colon N \to \mathbb{R}^m$ are two piecewise linear (smooth) embeddings and $\widetilde{f_0} \simeq_{\mathrm{eq}} \widetilde{f_1}$, then f_0 and f_1 are piecewise-linearly (smoothly) isotopic.

(CLA.i) If $h_0, h_1\colon N \to \mathbb{R}^m$ are piecewise linear (smooth) immersions and $\widetilde{h_0} \simeq_{\mathrm{eq}} \widetilde{h_1}$ on SN, then h_0 and h_1 are piecewise-linearly (smoothly) regularly homotopic.

To state the sufficiency theorems for the deleted product conditions, let us introduce the following notation. First, we set $d = 3n - 2m + 2$. Further, everywhere in this section we omit $\mathbb{Z}$-coefficients in the notation of (co)homology groups. For $n \leqslant k+2$, the stable suspension mapping is denoted by $\Sigma^\infty\colon \pi_{n+k}(S^n) \to \pi_{2k+2}(S^{k+2}) = \pi_k^S$.

Theorem 4.1. *For an n-polyhedron (respectively, a smooth n-manifold) N, assertions EXI are true under the following conditions*:

ε) $m - n \geqslant 3$ (respectively, $m \geqslant \frac{3(n+1)}{2}$), and the piecewise linear n-manifold N is such that the pair $(N, \partial N)$ is $[\frac{d}{3}]$-connected and $\pi_1 \partial N = 0$;

e) *$m \geqslant \frac{3(n+1)}{2}$ (respectively, either $m \geqslant \frac{3(n+1)}{2}$, or $(N, \partial N)$ is $(d-2)$-connected, $\pi_1 \partial N = 0$, and $m \geqslant 6$);*

i) *$m \geqslant \frac{3(n+1)}{2}$ (respectively, either $m \geqslant \frac{3n+1}{2}$, or $(N, \partial N)$ is homologically $(d-2)$-connected).*

Theorem 4.2. *For an n-polyhedron N, the piecewise linear cases of assertions EXI are true under the following conditions*:

e) *N is a piecewise linear manifold and either $m \geqslant n+3$ and N is closed and d-connected, or $(N, \partial N)$ is $(d-1)$-connected, $\pi_1 \partial N = 0$, and $m \geqslant 6$;*

i) *either $m \geqslant \frac{3n}{2} + 1$, or N is a piecewise linear manifold, $(N, \partial N)$ is $(d-1)$-connected, $\pi_1 \partial N = 0$, and $n \geqslant 6$.*

For $d \leqslant 1$ it suffices in the closed case of Theorem 4.2.e to require only *homological* simple connectedness [137]. Theorem 4.1.ε is a folklore result (see the proof in [137]). Theorem 4.1.e was proved in [46], Theorems 1′ and 6.4, and [153], Theorem 1; see also [55], [136], and [146]. Theorem 4.1.i was proved in [52] and [55], Theorem 2. Theorem 4.2 was proved in [135] and [137]. In this paper we only prove Theorem 4.1.e in the piecewise linear case for $m \geqslant \frac{3(n+1)}{2}$, sketch the proof of Theorem 4.2.e assuming that Theorem 4.2.i is valid, and present the idea of a possible proof of Theorem 4.2.i. We shall also construct Example 4.5.ε (see below).

Theorem 4.2.e is most interesting for closed N, since for this case assertion (EXI.i) is not proved (we conjecture that it is false) and the smooth case of Theorem 4.2.e is false (by Examples 4.4.ε and e′).

An interesting corollary of [18] and [19] was derived in [136]: a Peano continuum K is embeddable in $\mathbb{R}^2$ if and only if there is an equivariant map $\widetilde{K} \to S^1$.

It follows from the smoothing theory ([51], §1.6 and [50], §11.1) that if N is a smooth manifold, then smoothing of the embedding in the closed case of Theorem 4.2.e encounters a single obstruction in $H^n(N, C_{n-1}^{m-n})$. Since $C_{8k-2}^{4k} = 0$ (see [47], §8.15, where this relation is, however, misprinted as $C_{4k-2}^{3k} = 0$), we see that the following assertion holds.

Corollary 4.3. *The smooth case of* (EXI.e) (*and its corollaries*) *is true for any homologically simply connected closed smooth* $(8l-1)$-*manifold* N *and* $m = 12l-1$.

All results of § 3 except for Theorem 3.2.b are also corollaries of Theorems 4.1.e, 4.2.e, 4.6.e, and 4.7.e (although most of them can be proved independently) [42]. It follows from Theorems 4.1.e and 4.2.e that under the hypotheses of these theorems

1) the piecewise linear (smooth) embeddability of N in $\mathbb{R}^m$ is independent of the piecewise linear (smooth) structures on N;

2) If N is topologically embeddable in $\mathbb{R}^m$, then N is piecewise-linearly embeddable in $\mathbb{R}^m$;

3) If N is quasi-embeddable in $\mathbb{R}^m$, then N is embeddable in $\mathbb{R}^m$.

Theorems 4.1.i, 4.2.i, 4.6, and 4.7 have similar corollaries (see [137] for details).

The proof of Theorem 4.2.e [137] does not give its relative and approximation versions (which are true for Theorem 4.1.e; see [153], Theorems 7 and 3). In contrast with this, the proof of the case $m \geqslant \frac{3n}{2}+1$ of Theorem 4.2.i [137] does give the approximation version (in which the immersion obtained is arbitrarily close to a given piecewise linear map g) and the relative version (if $g\colon N \to \mathbb{R}^m$ is a piecewise linear map, $A \subset N$ a subpolyhedron, and $\Phi\colon SN \to S^{m-1}$ an equivariant map such that $g|_A$ is an immersion and $\widetilde{g} \simeq_{\mathrm{eq}} \Phi$ on SA, then there is an immersion $h\colon N \to \mathbb{R}^m$ such that $h = g$ on A and $\widetilde{h} \simeq_{\mathrm{eq}} \Phi$ on SN). For the smooth case, the approximation version is true even without the assumption $m \geqslant \frac{3n}{2}+1$, in the following form: if N is immersible in $\mathbb{R}^m$, then every map $N \to \mathbb{R}^m$ can be approximated by immersions [39].

Obviously, the restriction on the connectedness of N can be eliminated from Theorem 4.2.e for $d=0$ if we require only (EXI.ε) but not (EXI.e). For $m < \frac{5n+6}{4}$, Theorem 4.2.e is not interesting: we have $d > \frac{n}{2}-1$ and $n \geqslant 6$, and hence N is a homotopy sphere, $N \cong S^n$, and Theorem 4.2.e is true (cf. the remark after Theorem 3.2). But the proof cannot be simplified for $m \geqslant \frac{5n+6}{4}$, and it also can be considered as a step towards the proof of an analogue of Theorem 4.2.e for embeddings in *manifolds*.

Example 4.4. ε) The smooth case of assertion (EXI.ε) is false for $(m,n) = (19,16)$ and a homotopy sphere N.

e) If $l \geqslant 3$ and $\Sigma^\infty\colon \pi_q(S^l) \to \pi^S_{q-l}$ is not epimorphic, then neither the piecewise linear nor the smooth case of assertion (EXI.e) is true for the disconnected manifold $N = S^q \sqcup S^q$ and $m = q+l+1$.

e$'$) The smooth case of assertion (EXI.e) (even in Theorem 3.6.2) is false for the closed (and even $(2n-m)$-connected) manifold $N = S^{2k} \times S^{2k}$ and $m = \frac{3n}{2}+1 = 6k+1$.

Example 4.5. ε) The polyhedral case of assertion (EXI.ε) is false for each pair (m,n) such that $4 \leqslant m \leqslant \frac{3n}{2}+1$.

e) If $l \geqslant 3$ and $\Sigma^\infty_{[l-1]}\colon \pi_q(S^l)_{[l-1]} \to \pi^S_{q-l,[l-1]}$ is not epimorphic, then neither the piecewise linear nor the smooth case of assertion (EXI.e) is true for $N = S^1 \times S^q$ and $m = q+l+1$.

e$'$) (conjecture) If $l \geqslant 2$, $\Sigma^\infty_{(l)}\colon \pi_q(S^l)_{(l)} \to \pi^S_{q-l,(l)}$ is not epimorphic, and $\Sigma^\infty\colon \pi_q(S^{l+1}) \to \pi^S_{q-l-1}$ is monomorphic, then neither the piecewise linear nor the smooth case of assertion (EXI.e) is true for $N = S^1 \times S^q$ and $m = q+l+2$.

e$''$) The 3-adic solenoid Σ (that is, the intersection of an infinite sequence of solid tori each of which is inscribed in the previous one with degree 3) is not embeddable in $\mathbb{R}^2$, although there is an equivariant map $\widetilde{\Sigma} \to S^1$.

e$'''$) (conjecture) There is a non-planar tree-like continuum K for which there are no equivariant maps $\widetilde{K} \to S^1$.

i) The piecewise linear case of (EXI.i) is false for each m such that $5 \leqslant m \leqslant \frac{3n+1}{2}$ and for $m = n = 4$.

Example 4.4.ε was constructed in [61] (see also [109] and [117]). The assertion of Example 4.4.e follows from [47]. The assertion of Example 4.4.e$'$ follows from [8], Theorems 1.5 and 1.6 and [7], Theorem 4.2; see the proof in [137]. Example 4.5.ε was constructed in [29], [77], [99], [130], and [131]. Examples 4.5.e, e$'$, and i were constructed in [137]. Example 4.5.e$''$ was constructed in [136]; cf. [119], Example 1.5.

Examples 4.5.e and e$'$ show that the connectedness assumption in the closed case of Theorem 4.2.e cannot be significantly weakened. Indeed, using these examples and the tables in [148], we see that (EXI.e) is false for $N = S^q \sqcup S^q$, $q = 6, 14$, and $m = \frac{3q}{2} + 1$ (by Example 4.4.e) and for $N = S^1 \times S^q$ in the following cases. (The first seven cases are obtained from Example 4.5.e, and the last three, from conjectural Example 4.5.e$'$; see also [137].)

$q =$	6	14	12	13	26	29	28	13	11	12
$m =$	10	22	18	19	38	43	40	21	17	18
$m =$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$	$\frac{3n-3}{2}$	$\frac{3n-4}{2}$	$\frac{3n-5}{2}$	$\frac{3n-4}{2}$	$\frac{3n-7}{2}$	$\frac{3n}{2}$	$\frac{3n-2}{2}$	$\frac{3n-3}{2}$

Let us now state the classification versions of the above results.

Theorem 4.6. *If N is an n-polyhedron (respectively, a smooth n-manifold), then assertions CLA are true under the following conditions:*

e) $m \geqslant \frac{3n}{2} + 2$ (*respectively, either* $m \geqslant \frac{3n}{2} + 2$, *or* $(N, \partial N)$ *is* $(d-1)$-*connected,* $\pi_1 \partial N = 0$, *and* $m \geqslant 7$);

i) $m \geqslant \frac{3n}{2} + 2$ (*respectively, either* $m \geqslant \frac{3n}{2} + 1$, *or* $(N, \partial N)$ *is homologically* $(d-1)$-*connected*).

Theorem 4.7. *If N is an n-polyhedron, then the piecewise linear cases of assertions CLA are true under the following conditions:*

e) *N is a piecewise linear n-manifold, $m \geqslant n + 3$, and either N is closed and $(d+1)$-connected, or $(N, \partial N)$ is d-connected and $\pi_1 \partial N = 0$;*

i) *either $m \geqslant \frac{3(n+1)}{2}$, or N is a piecewise linear n-manifold, $(N, \partial N)$ is d-connected, $\pi_1 \partial N = 0$, and $n \geqslant 6$.*

For $d = 0$, it suffices to require *homological* 1-connectedness in Theorem 4.7 [137]. Theorem 4.6.e was proved in [46], Theorem 1$'$, and [156], Theorem 1$'$; see also [55], Corollary 1, and [135], § 3. Theorem 4.6.i was proved in [52] and [55], footnote on p. 3. Theorem 4.7 was proved in [135] and [137].

An interesting corollary of [103] was proved in [160] for graphs and in [136] for the general case: two embeddings $f, g \colon K \to \mathbb{R}^2$ of a Peano continuum K are isotopic if and only if $\widetilde{f} \simeq_{\mathrm{eq}} \widetilde{g}$.

The remarks on smoothing of the embedding in Theorem 4.2.e, the connectedness assumptions in Theorem 4.2.e, and the relative and approximation versions of Theorem 4.2 remain valid for Theorem 4.7.e, Theorem 4.7.e, and Theorem 4.7, respectively.

Corollary 4.8. 1) *The smooth case of assertion* (CLA.e) *is true for a homologically* 1-*connected closed smooth* $(8l-2)$-*manifold* N *and* $m = 12l - 2$.

2) *If* $p \leqslant q$ *and* $m \geqslant \frac{3q}{2} + p + 2$ $(m \geqslant \frac{3(q+p)}{2} + 2)$, *then the set of piecewise linear* (*smooth*) *embeddings* $S^p \times S^q \to \mathbb{R}^m$ *is in a one-to-one correspondence, up to piecewise linear* (*smooth*) *isotopy, with the set of elements of the group* $\pi_q(V_{m-q,p+1}) \oplus \pi_p(V_{m-p,q+1})$.

Corollary 4.8 was proved in [137]. (For $m = 2q + p + 1$, Corollary 4.8.2 was proved in [64].) For the computation of $\pi_q(V_{ab})$, see [79], [111], and [155].

Example 4.9. e) The piecewise linear case of assertion (CLA.e) is false for the disconnected manifold $N = S^n \sqcup S^n \sqcup S^n$ and $m = \frac{3(n+1)}{2}$.

eie) The smooth case of assertion (CLA.e) (and of (CLA.ie) in § 11) is not true for $m = \frac{3(n+1)}{2}$ and $N = S^n$ (which is, of course, $(d+1)$-connected).

e′) The piecewise linear case of assertion (CLA.e) is false for each pair (m, n) such that $n+3 \leqslant m \leqslant \frac{3(n+1)}{2}$ but $m-n \notin \{4, 8\}$ and the polyhedron $N = S^n \sqcup S^{2m-2n-3}$ (a disconnected piecewise linear manifold for $m = \frac{3(n+1)}{2}$).

i) The case of piecewise linear (smooth) manifolds in assertion (CLA.i) is false for $m = \frac{3n}{2} + 1$ $(m = \frac{3n+1}{2})$ and $N = S^n$.

Example 4.10. e) For each integer $l = 2$ or $l \geqslant 4$, if $\Sigma^3 : \pi_{4l+1}(S^{2l}) \to \pi^S_{2l+1}$ is epimorphic, then neither the piecewise linear nor the smooth case of assertion (CLA.e) is true for $N = S^1 \times S^{4l+1}$ and $m = 6l + 3 = \frac{3n}{2}$.

e′) The piecewise linear case of assertion (CLA.e) is false for any pair (m, n) such that $n + 2 \leqslant m \leqslant \frac{3(n+1)}{2}$ and the polyhedron $N = S^n \vee S^n \sqcup S^{2m-2n-3}$.

Example 4.9.e was constructed in [45] and [102], Proposition 8.3. Example 4.9.eie was constructed in [43] and [47], §8.14. Example 4.9.e′ actually follows from [45] and [47] (see the proof in [137]). Example 4.9.i actually follows from [49] and [50] (see the proof in [137]). Example 4.10 was constructed in [137].

Example 4.10.e shows that the connectedness assumption in Theorem 4.7.e cannot be significantly weakened: if $N = S^p \times S^q$, then assertion (CLA.e) is true for $m \geqslant \frac{3q}{2} + p + 2$ but can be false for $m = \frac{3q+3}{2}$. The hypothesis of Example 4.10.e holds for $l \leqslant 15$, and we conjecture that it holds for $l = 2$ or $l \geqslant 4$. We conjecture that the piecewise linear (smooth) case of assertion (EXI.i) is also false for piecewise linear (smooth) manifolds and $m = \frac{3n+1}{2}$ $\left(m = \frac{3n}{2}\right)$.

The example of Borromean rings in the beginning of § 6 (cf. [120]) suggests that one can introduce an obstruction to (relative) embeddability similar to the van Kampen and deleted product obstructions but obtained from triple (quadruple,...) intersections. Moreover, the vanishing of this obstruction will be sufficient for embeddability even if this is not the case for the van Kampen and deleted product obstructions. A possible candidate for a necessary condition for embeddability for the case in which the deleted product necessary condition fails to be sufficient is the deleted G-product condition. It can be stated as follows. Let G be a subgroup of the permutation group S_p, and let

$$\widetilde{N}_G = \{(x_1, \dots, x_p) \in N^p \mid x_i \neq x_{\sigma(i)} \text{ for each } \sigma \in G, i = 1, \dots, p\}.$$

The space $\widetilde{N}_G$ is called the *deleted G-product* of N. The group G obviously acts on $\widetilde{N}_G$. For an embedding $f\colon N \to \mathbb{R}^m$, the map $\widetilde{f}_G\colon \widetilde{N}_G \to \widetilde{\mathbb{R}^m}_G$ given by the formula $\widetilde{f}_G(x_1, \dots, x_p) = (fx_1, \dots, fx_p)$ is well defined. Clearly, $\widetilde{f}_G$ is G-equivariant. Thus, the existence of a G-equivariant map $\Phi\colon \widetilde{N}_G \to \widetilde{\mathbb{R}^m}_G$ is *the deleted G-product condition for embeddability of N in $\mathbb{R}^m$*. This approach works well in the theory of link maps [102]. In contrast, Examples 4.4 and 4.5.ε show that not only the deleted product condition, but also the deleted G-product condition (for any group G) is insufficient for embeddability.

The deleted G-product conditions for immersibility, isotopy, and regular homotopy can be defined in a similar way. Example 4.5.i apparently shows that the deleted G-product condition is not sufficient for immersibility. Examples 4.9.eie, e$'$ and 4.10.e$'$ show the incompleteness of the deleted G-product condition for isotopy (for any group G). We note that Example 4.9.e does not show the incompleteness. Example 4.9.i shows that the deleted G-product condition (for any group G) is not sufficient for regular homotopy.

§5. Appendix. Basic embeddings in the plane

In the papers dealing with the solution of Hilbert's 13th problem, A. N. Kolmogorov [87] and V. I. Arnold [6] proved that any continuous function of n variables defined on a compact subset of $\mathbb{R}^n$ can be represented as a superposition of continuous functions of one variable and addition (for a popular exposition, see [5]). Ostrand extended this theorem to arbitrary n-dimensional compacta [110]. In fact, it is in these papers that the notion of basic embedding, explicitly introduced in [144], appeared for the first time. An embedding $K \subset \mathbb{R}^m$ is said to be *basic* if for any continuous function $f\colon K \to \mathbb{R}$ there are continuous functions $g_1, \dots, g_m\colon \mathbb{R} \to \mathbb{R}$ such that $f(x_1, \dots, x_m) = g_1(x_1) + \dots + g_m(x_m)$ for any $(x_1, \dots, x_m) \in K$.

Theorem 5.1 ([87], [6], [110], [145], [94]). *Any n-dimensional compact space is basically embeddable in $\mathbb{R}^{2n+1}$ and is not basically embeddable in $\mathbb{R}^{2n}$ for $n > 1$.*

It is of interest to compare this theorem with the Nöbeling–Menger–Pontryagin theorem [76] on the embeddability of any n-dimensional compact space in $\mathbb{R}^{2n+1}$ and with the example of an n-dimensional polyhedron not embeddable in $\mathbb{R}^{2n}$ [28], [80]. Obviously, K is basically embeddable in $\mathbb{R}$ if and only if K is topologically embeddable in $\mathbb{R}$. It follows from Theorem 5.1 that a compact space K is basically embeddable in $\mathbb{R}^m$ for $m > 2$ if and only if $\dim K < m/2$. Thus, the only remaining case to be examined is $m = 2$. The problem of characterization of compact spaces basically *embedded* in the plane was stated as early as in [5] and was solved in [145]: a compact space K is basically embedded in the plane if and only if $E^n(K) = \varnothing$ for some n. Here

$$E(Z) = \left\{z \in Z :\ |Z \cap p_x^{-1} p_x z| \geqslant 2 \text{ and } |Z \cap p_y^{-1} p_y z| \geqslant 2\right\}$$

and p_x and p_y are the projections on the coordinate axes in the plane. It is convenient to state the characterization [134] of arcwise connected compact sets basically *embeddable* in the plane first for graphs and then for the general case. The conjecture about the embeddability of (not necessarily arcwise connected) continua in the plane can be found in [134].

Theorem 5.2 ([134]; cf. [89], [98], and [118], § 2). *A finite graph is basically embeddable in the plane if and only if either of the following two equivalent conditions holds*:

(a) *it does not contain subgraphs homeomorphic to* S, C_1, *or* C_2 (Fig. 5.1.a), *that is, a circle, a five-pointed star, or a cross with branched endpoints*;

(b) *it is contained in one of the graphs* V_n (Fig. 5.1.b).

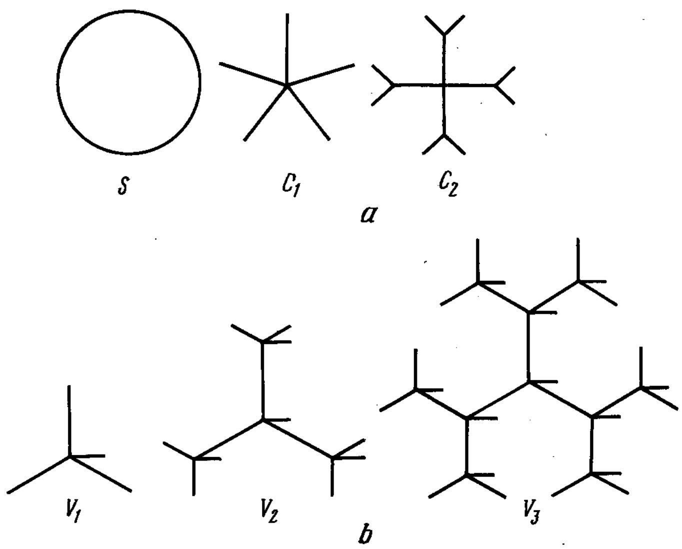

Figure 5.1

Theorem 5.3 ([90]; see also [15] and [91]). *A finite graph K admits a basic embedding $K \subset \mathbb{R} \times T_n$ (that is, an embedding such that for any continuous function $f\colon K \to \mathbb{R}$ there are continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon T_n \to \mathbb{R}$ such that $f(x,y) = g(x) + h(y)$ for any point $(x,y) \in K$) if and only if K is a tree and either $\delta(K) < n$, or $\delta(K) = n$ and K has a horrible vertex with a hanging edge.*

Here T_i is an i-od, that is, an i-pointed star; a vertex of K is said to be *horrible* if its degree is greater than 4 and *awful* if its degree is equal to 4 and it is not an endpoint of a hanging edge. The *defect* of a graph K is defined as the sum $\delta(K) = (\deg A_1 - 2) + \dots + (\deg A_k - 2)$, where $A_1, \dots, A_k$ are all horrible and awful vertices of K.

Let us introduce the notation and definitions needed in the characterization of arcwise connected continua basically embeddable in the plane. Let I be the segment $[0,1]$. By $\operatorname{Int} J$ we denote an arc J with endpoints deleted. We say that an arc s

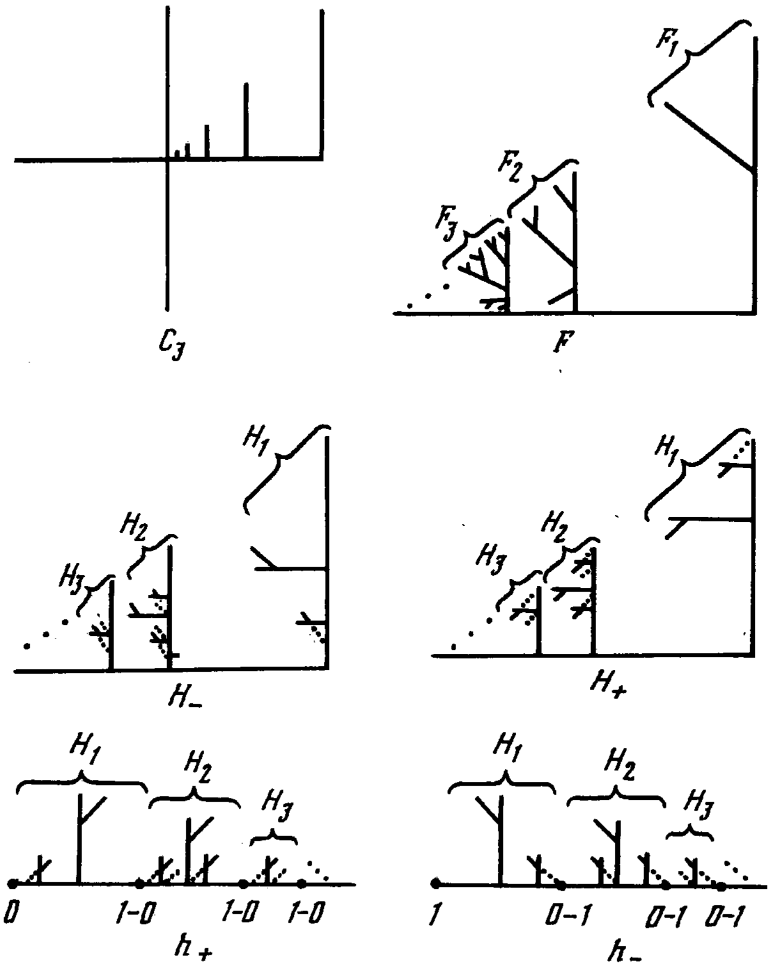

Figure 5.2

connects points A and B if $s \cap A$ and $s \cap B$ are distinct endpoints of s. A sequence of sets is called a *null sequence* if their diameters tend to zero. Let C_3 be a cross with a null sequence of arcs attached to one of its branches and converging to its centre (Fig. 5.2), let C_4 be a cross with a sequence of points converging to its centre (Fig. 5.3), and let B be the union of I with a null sequence of arcs each having an endpoint attached to $\mathring{I}$ (Fig. 5.3). Obviously, the topological type of B is independent of possible variations in the construction. Further, suppose that F_1 is a triod and F_{n+1} is obtained from F_n by branching each endpoint of F_n (Fig. 5.3); H_n is the union of I with a null sequence of triods each having an endpoint attached to I at a point of the set $D_n = \{3^{-l_1} + \cdots + 3^{-l_s} \mid s \leqslant n,\ 0 < l_1 < \cdots < l_s$, the l_i are integers$\}$ (Fig. 5.3); F is the union of I with the null sequence of sets F_n having an endpoint attached to the point $1/n \in I$ (Fig. 5.2); H_+ and H_- are the unions of I with the null sequence of continua H_n connected to the points

$1/n \in I$ by arcs that intersect H_n at the points $1 \in I \subset H_n$ and $0 \in I \subset H_{n-1}$, respectively (Fig. 5.2); h_+ and h_- are obtained from the null sequence of continua H_n by pasting together the points $1 \in I \subset H_n$ and $0 \in I \subset H_{n-1}$ ($0 \in I \subset H_n$ and $1 \in I \subset H_{n-1}$, respectively) (Fig. 5.2). A sequence $I_1 \subset I_2 \subset \cdots \subset I_n = K$ such that I_{l+1} is obtained by attaching a null sequence of arcs, each at one endpoint, to I_l is called a *filtration*. These arcs will be called *arcs of order* l.

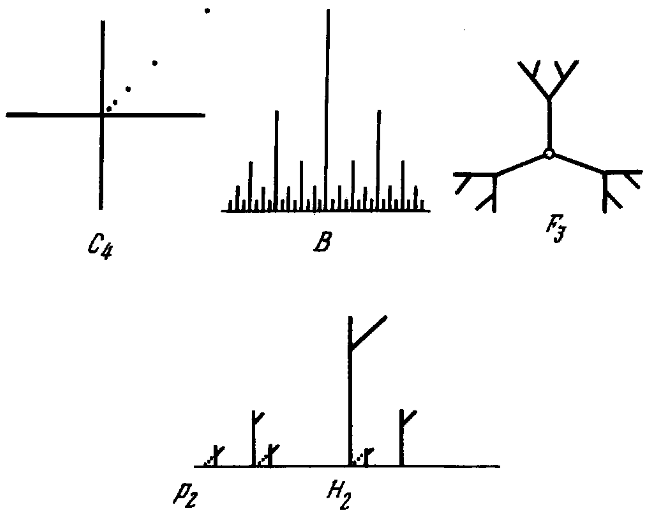

Figure 5.3

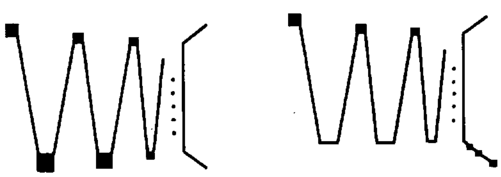

Figure 5.4

Theorem 5.4 [134]. *An arcwise connected compact space K is basically embeddable in the plane if and only if it is locally connected* (*that is*, *is a Peano space*) *and any of the following three equivalent conditions holds.*

(1) *K does not contain S, C_2, C_4, and B as subcompacta and contains only finitely many subcontinua F_n and H_n* (see Figs. 5.1 and 5.3).

(2) (Cf. [18], [19].) *K does not contain any of the continua S, C_1, C_2, C_3, B, F, H_+, H_-, h_+, and h_-* (Figs. 5.2 and 5.3) *as a subcontinuum.*

(3) *There is a filtration $I_1 \subset \cdots \subset I_n = K$ and a positive integer M such that the following properties hold for any arc s of order $l-1$ ($s = I_1$ if $l = 1$), where $\{s_m\}$ is the set of all arcs of order $\geq l$ having non-empty intersection with s and $R = s \cap (\bigcup_m s_m)$:*

(I) $R \subset \operatorname{Int} s$;

(C) *at most two of the arcs* s_m *are attached to each point of* s, *and if there are exactly two*, *then one of them is hanging* (*that is*, *none of the arcs of order* $l+1$ *or greater has non-empty intersection with it*), *and the point to which these two arcs are attached has a neighbourhood in* K *homeomorphic to the cross*;

(B) $\operatorname{Cl} R$ *is nowhere dense in* S;

(H) $h^M Q = \varnothing$, *where* $Q = \{x \in R \mid \text{if } x \in s_m, \text{then } s_m \text{is not hanging}\}$ *and* hZ *is the set obtained from* Z *by removing points isolated in* Z.

II. Completeness of the deleted product obstruction

Notation

We use the notation of [125]. A superscript on a polyhedron indicates its dimension. The simplices of any triangulation T are assumed to be linearly ordered in ascending order of dimension. The lexicographic ordering is considered on $T \times T$. For a map $f\colon N \to \mathbb{R}^m$,

$$\widetilde{\Delta}(f) = \{(x,y) \in N \times N \mid x \neq y,\ fx = fy\} \quad \text{and} \quad \Sigma(f) = \{x \in N : |f^{-1}fx| \geqslant 2\}$$

denote the *singular sets* of f. The precise definition of a *regular neighbourhood* is given in § 8 (after the proof of the piecewise linear case of Theorem 3.1), but to understand the ideas of proofs presented, in particular, before § 8, it suffices to treat a regular neighbourhood as a sufficiently small neighbourhood without 'unnecessary holes'. By $\operatorname{link}(\,\cdot\,,\,\cdot\,)$ we denote the linking coefficient.

§6. The construction of Example 4.5.ε

In this section, $\widetilde{P}$ stands for a copy of P rather than the deleted product of P. (The copy of a subset $A \subset P$ is denoted by $\widetilde{A} \subset \widetilde{P}$.)

Let us illustrate one of the main ideas used in the construction of the Freedman–Krushkal–Teichner example (Theorem 2.1.a for $n = 2$ and Example 4.5.ε for $m = 4$, $n = 2$) [29]. To this end, let us construct three circles embedded in $\mathbb{R}^3$ so that any two of them are unlinked but all three are linked together. Such triples of circles in $\mathbb{R}^3$ will be called *generalized Borromean rings.* Our construction is based on the fact that a fundamental group need not be commutative. We take two unknotted circles Σ, $\widetilde{\Sigma}$ in $\mathbb{R}^3$ far away from each other. In $\mathbb{R}^3 - (\Sigma \sqcup \widetilde{\Sigma})$, we embed the figure eight, that is, the wedge of two circles C, so that the inclusion $C \subset \mathbb{R}^3 - (\Sigma \sqcup \widetilde{\Sigma})$ induces an isomorphism of the fundamental groups. We take the generators a and b of $\pi_1(C) = \pi_1(\mathbb{R}^3 - (\Sigma \sqcup \widetilde{\Sigma}))$ represented by the two (arbitrarily oriented) circles of the figure eight. We consider a map $S^1 \to C \subset \mathbb{R}^3$ representing the element $aba^{-1}b^{-1}$. By a general position argument, there is an embedding $f\colon S^1 \to \mathbb{R}^3$ very close to this map. Then Σ, $\widetilde{\Sigma}$, and $f(S^1)$ are generalized Borromean rings. Indeed, Σ and $\widetilde{\Sigma}$ are unlinked by definition. One can readily choose an f such that Σ and $f(S^1)$, as well as $\widetilde{\Sigma}$ and $f(S^1)$, are unlinked. (This follows from the fact that f induces the zero homomorphism of the one-dimensional homology groups.) But f induces a non-zero homomorphism of the fundamental groups. Therefore Σ, $\widetilde{\Sigma}$, and $f(S^1)$ are linked.

From the existence of generalized Borromean rings, one can derive the following counterexample to the relative version of Theorem 2.1.a for $n = 2$. Let $K = D^2 \sqcup D^2 \sqcup D^2$ and $A = \partial D^2 \sqcup \partial D^2 \sqcup \partial D^2$. Next, let $A \subset S^3 \cong \partial D^4$ be generalized Borromean rings. Since all three rings are linked, it follows that the embedding $A \to \partial D^4$ cannot be extended to an embedding $K \to D^4$. But since any two of the Borromean rings are unlinked, we see that the corresponding relative van Kampen obstruction to this extension vanishes. Indeed, the van Kampen obstruction takes account of double intersections but ignores triple intersections.

Now we are in a position to construct the Freedman–Krushkal–Teichner example. Let P be the 2-skeleton of the 6-simplex minus the interior of some 2-simplex of this 2-skeleton. We recall (see the third paragraph of § 2) that P contains two disjoint spheres Σ^2 and Σ^1 such that for each embedding $P \to \mathbb{R}^4$ these spheres are linked with a non-zero linking coefficient (actually equal to ± 1). We embed P and $\widetilde{P}$ in $\mathbb{R}^4$ in the standard way (thus, Σ^2 and $\widetilde{\Sigma}^2$ are unknotted, and Σ^2 and Σ^1, as well as $\widetilde{\Sigma}^2$ and $\widetilde{\Sigma}^1$, are standard linked spheres) far away from each other. Then Σ^2 and $\widetilde{\Sigma}^2$ are unlinked. We take an arbitrary point $x \in \Sigma^1$, join x to $\widetilde{x}$ by an arc, and pull these points to each other along this arc (Fig. 6.1). We obtain an embedding $P \vee \widetilde{P} \subset \mathbb{R}^4$. Let $C = \Sigma^1 \vee \widetilde{\Sigma}^1$ be a figure eight (with base point $x = \widetilde{x}$). Then the inclusion $C \subset \mathbb{R}^4 - (\Sigma^2 \sqcup \widetilde{\Sigma}^2)$ induces an isomorphism of the fundamental groups. We take the generators a and b of the group $\pi_1(C)$ represented by the two (arbitrarily oriented) circles of the figure eight and consider a map $h\colon S^1 \to C$ representing the element $aba^{-1}b^{-1}$. Let K be the mapping cone of the composition of h with the inclusion $C \subset P \vee \widetilde{P}$ (that is, $K = D^2 \cup_{h\colon \partial D^2 \to C} (P \vee \widetilde{P})$).

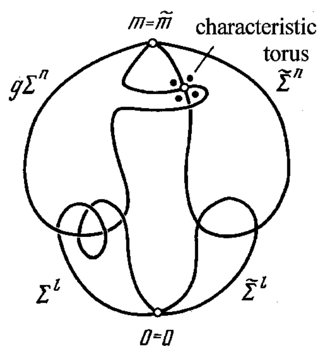

Figure 6.1

Then K is non-embeddable in $\mathbb{R}^4$, although $v(K) = 0$. For a detailed proof, see [29]. The reason for the equality $v(K) = 0$ is that the van Kampen obstruction takes account of the homology property (the cycle $aba^{-1}b^{-1}$ is null-homologous) but fails to recognize the finer *homotopy* property (the cycle $aba^{-1}b^{-1}$ is not null-homotopic). Let us sketch the proof of non-embeddability of K in $\mathbb{R}^4$. Suppose the contrary: there is an embedding $g : K \to \mathbb{R}^4$. If both $g\Sigma^2$ and $g\widetilde{\Sigma}^2$ are unknotted in $\mathbb{R}^4$, then it follows from the properties of P that the map $C \to gC \subset \mathbb{R}^4 - g(\Sigma^2 \sqcup \widetilde{\Sigma}^2)$ induces a monomorphism of the fundamental groups. In the general case, this can be proved on the basis of the Stallings theorem on lower central series of groups [143], [29]. But the element $aba^{-1}b^{-1}$, which is non-zero in $\pi_1 C$, is taken to a loop in $\mathbb{R}^4 - g(\Sigma^2 \sqcup \widetilde{\Sigma}^2)$ extendible to gD^2 and hence null-homotopic. This contradiction proves the non-embeddability of K in $\mathbb{R}^4$.

Now we present the construction of Example 4.2.ε, which is a higher-dimensional generalization of the Freedman–Krushkal–Teichner example. A polyhedron K is said to be *quasi-embeddable* in $\mathbb{R}^m$ if for each triangulation T of K there is a map $f\colon K \to \mathbb{R}^m$ (which is called an almost embedding) such that $f\sigma \cap f\tau = \varnothing$ for any $\sigma \times \tau \in \widetilde{T}$. This definition is non-standard but is equivalent to the standard definition (§ 1). It suffices to construct an n-polyhedron R quasi-embeddable but not embeddable in $\mathbb{R}^m$. Let $l = m - n - 1$. It suffices to construct Example 4.2.ε for $m = n + 2 \geq 4$ and for $n + 3 \leq m \leq \frac{3n}{2} + 1$.

Lemma 6.1 (cf. [29], Lemma 6 and [131], Lemma 1.4)**.** *For each $l \geqslant \frac{n}{2}$, there is an n-polyhedron K containing two disjoint wedges $\Sigma^n \vee \widetilde{\Sigma}^n$ and $\Sigma^l \vee \widetilde{\Sigma}^l$ of spheres such that*:

a) *for any piecewise linear embedding $K \to \mathbb{R}^m$, the pairs $\Sigma^n, \widetilde{\Sigma}^l$ and $\widetilde{\Sigma}^n, \Sigma^l$ are not linked and the homological linking coefficients of the pairs Σ^n, Σ^l and $\widetilde{\Sigma}^n, \widetilde{\Sigma}^l$ are non-zero* (*for $m = n + 2$, these linking coefficients are odd*);

b) *there is a piecewise linear embedding $K \to \mathbb{R}^m$ for which the wedge $\Sigma^n \vee \widetilde{\Sigma}^n$ is unknotted in $\mathbb{R}^m$.*

Proof. Let $\Delta^k_{a_0\dots a_s}$ be the k-skeleton of the s-simplex with vertices $a_0 \dots a_s$. We set

$$P = \Delta^n_{12\dots m+2} \cup \mathrm{Con}(\Delta^l_{12\dots m+2}, 0) \text{ and } Q = \Delta^n_{12\dots m+2} \cup \mathrm{Con}(\Delta^l_{12\dots m+2} \setminus \mathring{\Delta}^l_{12\dots l+1}, 0).$$

Let $\Sigma^l = \partial\Delta^{l+1}_{01\dots l+1}$ and $\Sigma^n = \partial\Delta^{n+1}_{l+2\dots m+2}$ be spheres in Q. We set

$$K = Q \bigcup_{0=\widetilde{0}, m=\widetilde{m}} \widetilde{Q}.$$

The unlinkedness in 6.1.a follows from the fact that Σ^l (respectively, $\widetilde{\Sigma}^l$) bounds the disc $\Delta^{l+1}_{01\dots l+1}$ (respectively, $\widetilde{\Delta}^{l+1}_{01\dots l+1}$) in $K - \widetilde{\Sigma}^n$ (respectively, in $K - \Sigma^n$).

To prove the second part of 6.1.a for $m > n + 2$, it suffices to show that P is not embeddable in $\mathbb{R}^m$ (cf. [131], proof of Lemma 1.4). Hence it suffices to prove that there are no equivariant maps $\widetilde{P} \to S^{m-1}$. This follows from [131], the construction before Lemma 1.2 and Lemma 1.2 itself and from [161]. For $m = n + 2$, the second part of 6.1.a follows from [131], Lemma 1.4.

In the first two paragraphs of the proof of Lemma 1.1 in [131], it was actually proved that Q is embeddable in $\mathbb{R}^m$. Since $m \geqslant n + 2$, it follows that there is an embedding $K \subset \mathbb{R}^m$. If $m > n + 2$, then 6.1.b holds for *any* embedding $K \subset \mathbb{R}^m$ ([96], Theorem 8). If $m = n + 2$, then for our embedding $Q \subset \mathbb{R}^m$ the sphere Σ^n is unknotted in $\mathbb{R}^m$. We can embed two copies of Q in $\mathbb{R}^m$ far from each other. Let us join two points of Σ^n and $\widetilde{\Sigma}^n$ by an arc and pull the points of the spheres together along this arc. By performing the same construction for Σ^l and $\widetilde{\Sigma}^l$, we obtain the desired embedding.

Lemma 6.2 (cf. [131], § 2). *Let K be the polyhedron and $K \to \mathbb{R}^m$ the embedding described in Lemma* 6.1. *Let $D^n \subset \Sigma^n$ and $\widetilde{D}^n \subset \widetilde{\Sigma}^n$ be piecewise linear discs in the interiors of some n-simplices of some triangulation of K such that these simplices contain the unique common point of Σ^n and $\widetilde{\Sigma}^n$. Then there is a piecewise linear map $g : K \to \mathbb{R}^m$ such that*:

a) $g\big|_{K-\mathring{D}^n}$ *is an inclusion and* $g\big|_{K-\mathring{\widetilde{D}}^n}$ *is an embedding, but* $g(D^n) \cap g(\widetilde{D}^n) \neq \varnothing$;

b) *the Whitehead product of the* (*arbitrarily oriented*) *inclusions of* Σ^l *and* $\widetilde{\Sigma}^l$ *in* $\Sigma^l \vee \widetilde{\Sigma}^l \to \mathbb{R}^m - g(\Sigma^n \vee \widetilde{\Sigma}^n)$ *is homotopic to zero.*

Proof. We take points $a \in \mathring{D}^n$ and $\widetilde{a} \in \mathring{\widetilde{D}}^n$ and a small arc $s \subset \mathbb{R}^m$ joining a to $\widetilde{a}$. By a general position argument, $s \cap K = \{a, \widetilde{a}\}$. We make a finger move of D^n along s (that is, we construct a new embedding $D^n \to \mathbb{R}^m$ obtained from the old one by pushing an n-dimensional finger from D^n along the arc s; see Fig. 6.1). We obtain a new piecewise linear map $g : K \to \mathbb{R}^m$ such that the property in Lemma 6.2.a holds. By a general position argument, $\dim(g(D^n) \cap \widetilde{D}^n) \leqslant 2n - m$ and $g(D^n)$ intersects $\widetilde{D}^n$ transversally. We can represent a regular neighbourhood B^m of an arbitrary point c of this intersection as the product $B^{2n-m} \times B^{l+1} \times B^{l+1}$ of balls, where $B^{2n-m} \times 0 \times 0$ corresponds to the intersection and $B^{2n-m} \times B^{l+1} \times 0$ and $B^{2n-m} \times 0 \times B^{l+1}$ correspond to $g(D^n)$ and $\widetilde{D}^n$, respectively. (We denote the centre of B^k by 0.) In a neighbourhood of the point c, we have the torus $0 \times \partial B^{l+1} \times \partial B^{l+1}$ (which is called *distinguished* or *characteristic*). By Lemma 6.1.b, $S^m - \Sigma^n \vee \widetilde{\Sigma}^n \simeq S^l \vee S^l$. Let α and $\widetilde{\alpha}$ be the elements of the group $\pi_l(\mathbb{R}^m - \Sigma^n \vee \widetilde{\Sigma}^n)$ represented by the homeomorphisms $S^l \to y \vee S^l$ and $S^l \to S^l \vee y$ ($y \in S^l$), respectively (with some orientations). With appropriate orientations, the inclusions of $0 \times \partial B^{l+1} \times y$ and $0 \times y \times \partial B^{l+1}$ in $\mathbb{R}^m - \Sigma^n \vee \widetilde{\Sigma}^n$ are homotopic to the spheroids α and $\widetilde{\alpha}$, respectively. Since the map

$$[\alpha, \widetilde{\alpha}] \colon S^{2l-1} \to S^l \vee S^l \cong (0 \times y \times \partial B^{l+1}) \vee (0 \times \partial B^{l+1} \times y)$$

extends to a map $B^{2l} \to 0 \times \partial B^{l+1} \times \partial B^{l+1}$ ([14], [86], [30]), it follows that $[\alpha, \widetilde{\alpha}]$ is null-homotopic in $\mathbb{R}^m - g(\Sigma^n \vee \widetilde{\Sigma}^n)$. Let $p = \operatorname{link}(\Sigma^l, \Sigma^n)$ and $\widetilde{p} = \operatorname{link}(\widetilde{\Sigma}^l, \widetilde{\Sigma}^n)$. By Lemma 6.1.a, both p and $\widetilde{p}$ are non-zero. The inclusions of Σ^l and $\widetilde{\Sigma}^l$ in $\mathbb{R}^m - \Sigma^n \vee \widetilde{\Sigma}^n$ represent the elements $p\alpha$ and $\widetilde{p}\widetilde{\alpha}$ of $\pi_l(\mathbb{R}^m - \Sigma^n \vee \widetilde{\Sigma}^n)$, respectively. It follows that the assertion of Lemma 6.2.b holds, since $[p\alpha, \widetilde{p}\widetilde{\alpha}] = p\widetilde{p}[\alpha, \widetilde{\alpha}] = 0$ in $\mathbb{R}^m - g(\Sigma^n \vee \widetilde{\Sigma}^n)$.

Lemma 6.3 (cf. [131], Lemmas 2.1 and 2.2 and [29], §§ 3.2 and 4). *Let $g \colon K \to \mathbb{R}^m$ be the map described in* Lemma 6.2 *and $r \colon B^{2l} \to \mathbb{R}^m - g(\Sigma^n \vee \widetilde{\Sigma}^n)$ a piecewise linear map such that $r|_{\partial B^{2l}} \colon \partial B^{2l} \to \Sigma^l \vee \widetilde{\Sigma}^l$ represents the Whitehead product of the inclusions $\Sigma^l \subset \Sigma^l \vee \widetilde{\Sigma}^l$ and $\widetilde{\Sigma}^l \subset \Sigma^l \vee \widetilde{\Sigma}^l$. We set*

$$Y = (K - \mathring{D}^n) \cup r(B^{2l}) \cup g(D^n) \subset \mathbb{R}^m \text{ and } R = (K - \mathring{D}^n) \cup r(B^{2l}) \bigcup_{\partial B^n = \partial D^n} B^n.$$

Then $\dim R = n$ *and R is quasi-homeomorphic to Y* (*hence, R is quasi-embeddable in $\mathbb{R}^m$*) *but is not topologically embeddable in $\mathbb{R}^m$.*

Proof. Since $m \leqslant \frac{3n}{2} + 1$, it follows that $2l \leqslant n$ and hence $\dim Y = \dim R = n$. We have $R \supset (K - \overset{\circ}{D}{}^n) \cup B^n \cong K$. Therefore, by the construction of the balls D^n and $\widetilde{D}^n$, it follows that the balls B^n and $\widetilde{D}^n$ are contained in the interiors of some adjacent n-simplices of some triangulation of R. Consequently, there is an obvious map $R \to Y$ whose singular set is contained in $B^n \cup \widetilde{D}^n$ and hence in the interiors of two adjacent simplices of some triangulation of R. Therefore, just as in [131], Lemma 2.1, R is quasi-homeomorphic to Y.

Suppose that there is an embedding $h : R \to S^m$. For $m > n + 2$ by [12], and for $m = n + 2$ by the remark below, we can assume that h is a piecewise linear embedding. Let $\Sigma_1^n = (\Sigma^n - \overset{\circ}{D}{}^n) \bigcup_{\partial B^n = \partial D^n} B^n \subset R$. The map $h \circ r|_{\partial B^{2l}}$ can be extended to the map $h \circ r \colon B^{2l} \to S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n)$. Hence $h \circ r|_{\partial B^{2l}}$ is homotopically trivial in $S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n)$. Now we shall show the contrary, thus arriving at a contradiction.

For the case $m > n + 2$, just as in the proof of Lemma 6.1.b, we have $S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n) \simeq S^l \vee S^l$. Let $q = \operatorname{link}(h\Sigma^l, h\Sigma_1^n) \neq 0$ and $\widetilde{q} = \operatorname{link}(h\widetilde{\Sigma}^l, h\widetilde{\Sigma}^n) \neq 0$. By β and $\widetilde{\beta}$ we denote the elements of $\pi_l(S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n))$ represented by the homeomorphisms $S^l \to y \vee S^l$ and $S^l \to S^l \vee y$ $(y \in S^l)$, respectively (with the chosen orientations). Hence the homotopy class of the map $h \circ r|_{\partial B^{2l}} \colon \partial B^{2l} \to S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n)$ can be treated as the element $q\widetilde{q}[\beta, \widetilde{\beta}]$ of $\pi_{2l-1}(S^m - h(\Sigma_1^n \vee \widetilde{\Sigma}^n))$. By the Hilton theorem ([115], supplement to Lectures 5 and 6, pp. 231 and 257), the map $\varphi \colon \pi_{2l-1}(S^{2l-1}) \to \pi_{2l-1}(S^l \vee S^l)$ defined by the formula $\varphi(\gamma) = [\beta, \widetilde{\beta}] \circ \gamma$ is an injection. This can also be proved by using the exact homotopy sequence ([62], §5.3). Hence $[\beta, \widetilde{\beta}]$ has infinite order. It follows that the element $q\widetilde{q}[\beta, \widetilde{\beta}]$ is non-trivial.

In the case $m = n + 2$, just as in [29], Lemma 7, it follows from the property of Lemma 6.1.a by Stallings' theorem [143] that the commutator of the maps $\Sigma^l \subset \Sigma^l \vee \widetilde{\Sigma}^l \subset \mathbb{R}^m - h(\Sigma^n \vee \widetilde{\Sigma}^n)$ and $\widetilde{\Sigma}^l \subset \Sigma^l \vee \widetilde{\Sigma}^l \subset \mathbb{R}^m - h(\Sigma^n \vee \widetilde{\Sigma}^n)$ is non-zero.

Remark for $n = 2$ and topological embeddings (cf. [131], proof of Lemma 2.2). There are arbitrarily close piecewise linear approximations $h' \colon R \to \mathbb{R}^m$ to the embedding h such that $h'\sigma \cap h'\tau = \varnothing$ for any two disjoint simplexes σ and τ of some triangulation of R. By a general position argument, we can assume for $m = n + 2$ that $h'|_{\Sigma^1}$ and $h'|_{\widetilde{\Sigma}^l}$ are piecewise linear embeddings. Hence, by [131], Lemma 1.4,

$$\operatorname*{link}_{\bmod 2}(h'\Sigma_1^n, h'\Sigma^1) = \operatorname*{link}_{\bmod 2}(h'\widetilde{\Sigma}^n, h'\widetilde{\Sigma}^1) = 1.$$

The unlinkedness of the pairs $\Sigma^n, \widetilde{\Sigma}^l$ and $\widetilde{\Sigma}^n, \Sigma^l$ can be proved by analogy with the piecewise linear case. The rest of the proof is similar to the piecewise linear case $(h \to h')$, since we have used only the property of Lemma 6.1.a but not the fact that h is an embedding.

§7. The Whitney trick

In this section, we prove smooth embeddability of smooth n-manifolds in $\mathbb{R}^{2n}$ (Theorem 3.1) and the criterion for embeddability of n-polyhedra into $\mathbb{R}^{2n}$ (Theorem 2.1.a) for $n \geqslant 3$. The proofs of these theorems are based on constructions that generalize Reidemeister moves (Fig. 2.2). The most important of these constructions corresponds to the second Reidemeister move (Fig. 2.2.b) and is called the *Whitney trick*. This construction is used not only in the theory of embeddings, but also in other branches of geometric topology. Let us illustrate the idea of the Whitney trick by the following example.

Sketch of proof of Theorem 3.1 *in the smooth category.* Using a higher-dimensional analogue of the first Reidemeister move (Fig. 2.2.a), we can modify an arbitrary smooth generic map $f\colon N \to \mathbb{R}^{2n}$ so that a single self-intersection point with a prescribed sign will be added. Hence there is a generic map $f\colon N \to \mathbb{R}^{2n}$ such that its singularities consist of an even number of isolated double points with zero algebraic sum.

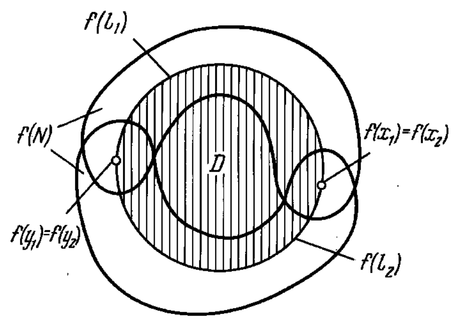

Figure 7.1

To complete the proof of Theorem 3.1, we 'kill' these double points pair by pair. This procedure is called the Whitney trick. We take double points $x_1, y_1, x_2, y_2 \in N$ such that $f(x_1) = f(x_2)$ and $f(y_1) = f(y_2)$. Next, we join x_1 to y_1 and x_2 to y_2 by arcs l_1 and l_2 so that these double points will have 'opposite signs' (Fig. 7.1). By a general position argument ($n \geqslant 2$), the restrictions $f|_{l_1}$ and $f|_{l_2}$ are embeddings, and l_1 and l_2 do not contain other double points of f. Since $2n \geqslant 4$, we can embed a 2-disc D in $\mathbb{R}^{2n}$ so that $\partial D = f(l_1) \cup f(l_2)$. By a general position argument ($n \geqslant 3$), we have $D \cap f(N) = \partial D$. Now we can move the image under f of a regular neighbourhood of l_1 in N along D so that the double points $f(x_1) = f(x_2)$ and $f(y_1) = f(y_2)$ cancel out (see the details in [84], [92]).

Let us introduce some notation needed below. Let K be an n-polyhedron with a triangulation T. A map $f\colon K \to \mathbb{R}^m$ is an embedding if and only if the following conditions hold:

$f|_\alpha$ is an embedding for each $\alpha \in T$; (7.1.a)

$f\alpha \cap f\beta = \varnothing$ for each $\alpha \times \beta \subset \widetilde{T}$; (7.1.b)

$f\alpha \cap f\beta = f(\alpha \cap \beta)$ for any $\alpha, \beta \in T$ such that $\alpha \cap \beta \neq \varnothing$. (7.1.c)

Proof of Theorem 2.1.a *for* $n \geqslant 3$. We take a generic map $\varphi\colon K \to \mathbb{R}^m$ that is linear on the simplices of some triangulation T of K. Then (7.1.a) holds with f replaced by φ. By a general position argument, (7.1.b) and (7.1.c) hold with f replaced by φ unless $\dim\alpha = \dim\beta = n$. The proof of Theorem 2.1.a consists of three parts, which are described in Lemmas 7.1, 7.2, and 7.3. The proof of Lemma 7.1 is based on the *van Kampen finger moves*, which generalize the fifth Reidemeister move (Fig. 2.2.e). Lemma 7.2 is proved by induction over pairs of n-simplices of $\widetilde{T}$ with an application of the Whitney trick. (We omit the details, which can be found in [29].) The proof of Lemma 7.3 is based on a higher-dimensional generalization of the fourth Reidemeister move (Fig. 2.2.d).

Lemma 7.1 (cf. [29], Lemma 2)**.** *Let K be an n-polyhedron with a triangulation T such that $v(K) = 0$. Let $\varphi\colon K \to \mathbb{R}^{2n}$ be a map such that* (7.1.a) *holds with f replaced by φ. Then there is a generic map $f\colon K \to \mathbb{R}^{2n}$ such that* (7.1.a) *holds and*

$$f\alpha \cdot f\beta = 0 \text{ for each } \alpha \times \beta \subset \widetilde{T}. \tag{7.2}$$

Proof. The condition $v(K) = 0$ implies that v_φ is an equivariant coboundary. Hence v_φ is equal to the sum of some 'elementary' equivariant coboundaries $\delta(\sigma^n \times \nu^{n-1})$ $(\sigma^n \times \nu^{n-1} \in \widetilde{T})$. By applying van Kampen finger moves, that is, higher-dimensional generalizations of the fifth Reidemeister move (Fig. 2.2.e), we obtain the desired map f.

Lemma 7.2 ([29], Lemma 4)**.** *Let $n \geqslant 3$, and let K be an n-polyhedron with a triangulation T. Next, let $\varphi\colon K \to \mathbb{R}^{2n}$ be a map such that* (7.1.a) *and* (7.2) *hold with f replaced by φ. Then there is a generic map $f\colon K \to \mathbb{R}^{2n}$ such that* (7.1.a) *and* (7.1.b) *hold.*

Lemma 7.3 (cf. [29], Lemma 5)**.** *Suppose that $n \geqslant 3$, K is an n-polyhedron with a triangulation T, and $\varphi\colon K \to \mathbb{R}^{2n}$ is a map such that* (7.1.a) *and* (7.1.b) *hold with f replaced by φ. Then there is an embedding $f\colon K \to \mathbb{R}^{2n}$.*

Proof. We can assume by induction that (7.1.c) holds with f replaced by φ for each pair (α, β) except for (σ^n, τ^n). Furthermore, we can assume that $\varphi\mathring{\sigma}^n \cap \varphi\mathring{\tau}^n$ is a point (say, p). Let v be a point of $\sigma^n \cap \tau^n$. We take piecewise linear arcs $l_1 \subset v \cup \mathring{\sigma}^p$ and $l_2 \subset \mathring{\tau}^q$ joining v to the inverse images of p and containing no singular points of φ in their interiors (Fig. 7.2). Then $\varphi(l_1 \cup l_2)$ is a circle. Since $n \geqslant 3$, it follows that this circle bounds a piecewise linear embedded 2-disc $D \subset \mathbb{R}^{2n}$. By a general position argument $(n + 2 < 2n)$, $\mathring{D} \cap \varphi K = \varnothing$. A regular neighbourhood of D in $\mathbb{R}^m$ $\operatorname{rel} \varphi v$ is a piecewise linear $2n$-ball D^{2n}. The inverse image $\varphi^{-1}D^{2n}$ is a regular neighbourhood of $l_1 \cup l_2$ in K $\operatorname{rel} v$ and is homeomorphic to the wedge $D^n \vee D^n$ of two piecewise linear n-balls with a common point on their boundaries. By [96], the restriction $\varphi : \partial D^n \vee \partial D^n \to \partial D^{2n}$ is unknotted. Hence it can be extended to an embedding $h : D^n \vee D^n \to D^{2n}$. To complete the proof, we set f equal to φ on $K - (D^n \vee D^n)$ and to h on $D^n \vee D^n$.

Van Kampen invented finger moves for the proof of Lemma 7.1. We need a modification of his construction, which we demonstrate by proving the case $m = 2n+1$ of Theorem 4.1.e. Let us introduce some notation. For maps $E, G\colon D^p \times D^q \to S^{m-1}$ and a homotopy $h\colon \partial(D^p \times D^q) \times I \to S^{m-1}$ such that $h_0 = E$ and $h_1 = G$ on $\partial(D^p \times D^q)$, we define a map $H_{EhG}\colon \partial(D^p \times D^q \times I) \to S^{m-1}$ by setting

$$H_{EhG}\big|_{D^p \times D^q \times 0} = E, \quad H_{EhG}\big|_{D^p \times D^q \times 1} = G, \quad H_{EhG}\big|_{\partial(D^p \times D^q) \times I} = h.$$

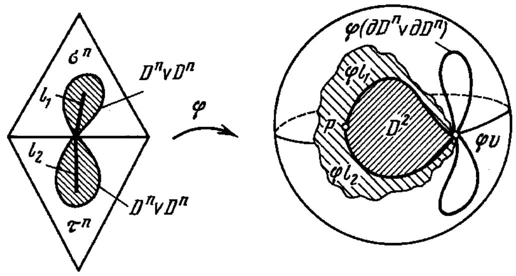

Figure 7.2

If $E = G$ on $\partial(D^p \times D^q)$, then $H_{EG} = H_{EiG}$, where i stands for the constant homotopy. For any maps $f, f' : D^p \sqcup D^q \to D^m$ such that $f'D^p \cap f'D^q = \varnothing$ and $f' = f$ on $D^p \sqcup \partial D^q$, we define a map

$$h_{ff'}\colon S^q \cong D^q \bigcup_{\partial D^q = \partial D^q_+} D^q_+ \to D^m - fD^p$$

by setting

$$h_{ff'}(x) = \begin{cases} f(x), & x \in D^q, \\ f'(x), & x \in D^q_+. \end{cases}$$

Here D^q_+ is a copy of D^q. If $f|_{D^p} = f'|_{D^p}$ is an embedding and fD^p is unknotted in D^m (say, for $m-p \geqslant 3$), then $D^m - fD^p \simeq S^{m-p-1}$. Indeed, let us take an embedding $g\colon D^{m-p} \to D^m$ such that gD^{m-p} intersects fD^p transversally at exactly one point. Then $g|_{S^{m-p-1}} : S^{m-p-1} \to D^m - fD^p$ is a homotopy equivalence, and the induced isomorphism of homotopy groups is independent of g (up to a factor of ± 1). We have $[h_{ff'}] \in \pi_q S^{m-p-1}$. Since $f' = f$ on $D^p \sqcup \partial D^q$, it follows that there is a homotopy $f_t\colon D^p \sqcup D^q \to D^m \operatorname{rel} D^p \sqcup \partial D^q$ from f to f'. Let $\widetilde{f}$, $\widetilde{f}_t$, and $\widetilde{f}'$ be the restrictions of these maps to $D^p \times D^q$, $\partial(D^p \times D^q)$, and $D^p \times D^q$, respectively. By [153], Lemma 1,

$$[H_{\widetilde{f}\widetilde{f}_t\widetilde{f}'}] = (-1)^{m-p}\Sigma^p[h_{ff'}] \in \pi_{p+q}S^{m-1}. \tag{7.3}$$

Proof of the case $m = 2n+1$ *of Theorem* 4.1.e. We present the proof for $n = 1$. (The general case can be proved in a similar way.) By a general position argument, every graph K can be embedded in $\mathbb{R}^3$, and we only need to prove that any map $\Phi\colon \widetilde{K} \to S^{m-1}$ can be realized by an embedding, that is, that there is an embedding $f\colon K \to \mathbb{R}^3$ such that $\widetilde{f} \simeq_{\mathrm{eq}} \Phi$. This assertion does not follow by a general position argument. Let T be a triangulation of K. Let us prove that for each pair (σ, τ) of edges of T such that $\sigma \leqslant \tau$ there is an embedding $f\colon K \to \mathbb{R}^3$ such that $\widetilde{f} \simeq_{\mathrm{eq}} F$ on

$$J = \bigcup \{\alpha \times \beta \cup \beta \times \alpha \subset \widetilde{T} \mid (\alpha, \beta) < (\sigma, \tau)\}.$$

Theorem 4.1.e will then follow by taking $\sigma = \tau$ to be the last simplex of T.

We prove the desired assertion by induction on (σ, τ). If σ and τ are the first edges of T, then $\dim J = 1$, and hence the assertion holds by a general position argument. Now suppose by induction that $\widetilde{f}|_J \simeq_{\mathrm{eq}} F|_J$ for an embedding $f\colon K \to \mathbb{R}^3$. If $\sigma \cap \tau \neq \varnothing$, then there is nothing to prove, so we suppose that $\sigma \cap \tau = \varnothing$. Let us take points $a \in \mathring{\sigma}$ and $b \in \mathring{\tau}$ and join their images fa and fb by an arc $l \subset \mathbb{R}^3$ such that $l \cap fK = \{fa, fb\}$. Let D^3 be a small regular neighbourhood of l in $\mathbb{R}^3$. Then $f^{-1}D^3$ is the disjoint union of some arcs $u \subset \mathring{\sigma}$ and $v \subset \mathring{\tau}$ that are regular neighbourhoods of a and b in K. We can assume that fu is unknotted in D^3. To carry out the inductive step, we shall wind the arc $f|_v$ several times around fu in D^3 (Fig. 7.3).

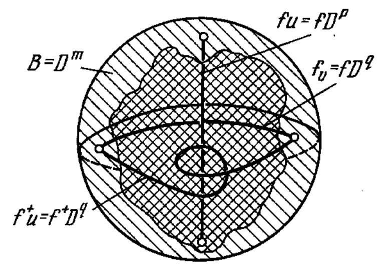

Figure 7.3

By the equivariant analogue of the Borsuk homotopy extension theorem, there is an equivariant extension $\Psi\colon \widetilde{K} \to S^2$ of $\widetilde{f}|_{J\cup(\sigma\times\tau-\mathring{u}\times\mathring{v})}$ such that $\Psi \simeq_{\mathrm{eq}} \Phi$. Hence $\Psi = \overline{f}$ on $\partial(u \times v)$.

We consider an embedding $f^+\colon u \sqcup v \to D^3$ such that $f^+ = f$ on $u \sqcup \partial v$ and an arbitrary homotopy $f_t \operatorname{rel} u \sqcup \partial v$ between f and f^+. By (7.3), it follows that

$$[H_{\Psi\widetilde{f_t}\widetilde{f^+}}] = [H_{\Psi\widetilde{f}}] + [H_{\widetilde{f}\widetilde{f_t}\widetilde{f^+}}] = [H_{\Psi\widetilde{f}}] + \Sigma[h_{ff^+}] \in \pi_2(S^2).$$

(All maps in the subscripts in this formula are restricted to $u \times v$ or $\partial(u \times v)$.) For *every* element $\beta \in \pi_1(S^1)$, there is an embedding $f^+\colon v \to D^3 - fu$ such that $[h_{ff^+}] = \beta$. Therefore, by the Freudenthal suspension theorem, there is a map $f^+\colon v \to D^3 - fu$ such that $[H_{\Psi\widetilde{f_t}\widetilde{f^+}}] = 0$. We extend this f^+ to the whole of K by f. We obtain a map $f^+\colon K \to \mathbb{R}^3$ for which $\widetilde{f^+} \simeq_{\mathrm{eq}} \Psi \simeq_{\mathrm{eq}} \Phi$ on the whole of $J \cup \sigma \times \tau$. This completes the inductive step and proves the desired assertion.

§8. Engulfing

The idea of engulfing was first introduced by Zeeman and Stallings to prove the Poincaré conjecture for $n > 4$ in the piecewise linear and topological categories [141], [164]. This idea has become one of the most important tools of geometric topology, particularly in the theory of embeddings and isotopy. Engulfing from a polyhedron C can be visualized as follows: tentacles protrude from C and engulf a polyhedron K; the dimension of a tentacle is only one greater than that of K. In the proof of Theorem 3.1, the role of the union of these tentacles will be played by the *trace* of K upon *collapse* to C. The dimension of new intersections of K with the tentacles is lower than the original dimension, and so the new intersections can be dealt with by induction (see the proof of Theorem 3.1 below). We shall illustrate the idea of engulfing by the proofs of Theorems 3.1 and 3.2.a in the piecewise linear case. Further discussion can be found in [59].

Proof of Theorem 3.1 *in the piecewise linear case.* Since each 2-manifold is a connected sum of tori and projective planes, it is embeddable in $\mathbb{R}^4$. Now we assume that $n \geqslant 3$. Without loss of generality, we can assume that N is connected. Consider a generic piecewise linear map $f\colon N \to \mathbb{R}^{2n}$. Then f has only finitely many *double points*, that is, points $x, y \in N$ such that $f(x) = f(y)$ but $x \neq y$. We denote these points by $x_1, y_1, \dots, x_p, y_p$. Then $f(x_i) = f(y_i)$ and f is an embedding outside $\{x_1, y_1, \dots, x_p, y_p\}$. To 'kill' x_1 and y_1, we take an arc $l \subset N$ joining x_1 to y_1 and containing no other points x_i, y_i. Thus $f(l)$ is a simple closed curve in $\mathbb{R}^{2n}$. Consider a 2-disc $D \subset \mathbb{R}^{2n}$ such that $\partial D = f(l)$ (Figs. 2.2.a and 8.1). Since $n + 2 < 2n$, we find by a general position argument that $D \cap f(N) = f(l)$. We take a regular neighbourhood B of D in R^{2n}. It is a $2n$-ball. If it is sufficiently small, then $B_0 = f^{-1}(B)$ is a regular neighbourhood of l in N. Hence B_0 is an n-ball. Since each ball is a cone over its boundary, it follows that the embedding $f\colon \partial B_0 \to \partial B$ can be conically extended to an embedding $F\colon B_0 \to B$ that agrees with f on ∂B_0. Using a similar argument for $i = 1, \dots, p$, we 'kill' all double points x_i, y_i and obtain an embedding of N in $\mathbb{R}^{2n}$.

To prove Theorem 3.2.a in the piecewise linear category, we generalize the above argument. To this end, let us introduce several important notions. We say that a polyhedron Y is obtained from a polyhedron K by an *elementary collapse* if $K = Y \cup B^n$ and $Y \cap B^n = B^{n-1}$, where B^{n-1} is a face of the ball B^n. This elementary collapse is said to be made *from* $\mathrm{Cl}(\partial B^n - B^{n-1})$ *along* B^n *to* B^{n-1}. A polyhedron K *collapses to* Y (in this case, we write $K \searrow Y$) if there is a sequence of elementary collapses $K = K_0 \searrow K_1 \searrow K_2 \searrow \cdots \searrow K_{n-1} \searrow K_n = Y$. A polyhedron K is *collapsible* if it collapses to a point. Clearly, the ball B^n is collapsible, since it collapses to the face B^{n-1} and so on by induction. Moreover, the cone cK over a compact polyhedron K is collapsible (to its vertex). Indeed, for each simplex $A \subset K$, the cone cA collapses from A to $c(\partial A)$, and hence $cK \searrow *$ by induction on the dimension of the simplices.

A collapse $K \searrow Y$ generates a deformation retraction $r\colon K \to Y$ given by the deformation of each ball B^n to the corresponding face B^{n-1}. There is a homotopy H_t between the identity map $K \to K$ and the deformation retraction $r\colon K \to Y$; it is induced by the collapse $K \searrow Y$. The *trace* of a subpolyhedron S of K under the collapse $K \searrow Y$ is the union of $H_t(S)$ over $t \in [0, 1]$.

Suppose that an embedding of a polyhedron K into a piecewise linear manifold M is given. A neighbourhood N of K in M is said to be *regular* if N is a compact bounded manifold and $N \searrow K$. A polyhedron can have several distinct regular neighbourhoods. But a regular neighbourhood is unique up to a homeomorphism [154] and even up to an isotopy equal to the identity on K [75]. The regular neighbourhood of a collapsible polyhedron is a ball [154]. The converse (if $K \subset B^n$ and $K \searrow *$, then $B \searrow K$) is true only in codimension $\geqslant 3$ [68]. We adopt the following convention. The first occurrence of the symbol $R_M(K)$ stands for 'a sufficiently small regular neighbourhood of K in M'. All subsequent occurrences of the same symbol refer to the same neighbourhood.

Proof of Theorem 3.2.a *in the piecewise linear category.* Let $k = 2n - m$ be the degree of connectedness of N and $f\colon N \to \mathbb{R}^m$ a generic map. We need to 'kill' the singular set $\Sigma(f)$. We have $\dim \Sigma(f) \le 2n - m = k$. For the most part, the proof deals with the construction of collapsible polyhedra $C \subset N$ and $D \subset \mathbb{R}^m$ such that $\Sigma(f) \subset C$ and $f^{-1}(D) = C$. The polyhedra C and D are analogues of the arc l and the disc D in the above argument. Once these polyhedra are constructed, the proof can be completed as follows. We choose regular neighbourhoods $B^n = R_N(C)$ and $B^m = R_{\mathbb{R}^m}(D)$ such that $f^{-1}B^m = B^n$. Since C and D are collapsible, B^n and B^m are balls. We define a new map $g\colon N \to \mathbb{R}^m$ as the cone over $f|_{\partial B^n}$ on B^n and by the formula $g = f$ outside $\mathring{B}^n$. Since f is an embedding outside B^n and $B^m \cap f(N) = f(B^n)$, it follows that g is an embedding.

Engulfing Lemma 8.1 ([141], [112])**.** *Suppose that N is a $(2k+2-n)$-connected closed n-manifold and $K \subset N$ is a k-polyhedron such that $n - k \geqslant 3$ and the inclusion $K \subset N$ is null-homotopic. Then K can be engulfed in N, that is, is contained in an n-ball $B \subset N$.*

Relative Engulfing Lemma 8.2. *Suppose that N is a k-connected closed piecewise linear n-manifold, $K \subset N$ is a k-polyhedron, and $C \subset N$ is a collapsible polyhedron. If $n - k \geqslant 3$, then K can be engulfed from C in N, that is, there is an $R_N(C) \supset K$.*

The construction of the polyhedra C and D. Since $\dim \Sigma(f) \leqslant k = 2n - m$, we have $n - \dim \Sigma(f) \geqslant n - k = m - n \geqslant 3$. We also note that any embedding of a k-polyhedron in a k-connected manifold is homotopic to a constant map. Hence, by the engulfing lemma, there is an n-ball $B \subset N$ containing $\Sigma(f)$. If $\dim \Sigma(f) < \frac{n}{2}$, then we take a generic cone C over $\Sigma(f)$ in B. If $\dim \Sigma(f)$ is arbitrary, then we define C to be the trace of $\Sigma(f)$ under a collapse of B to a point $x \notin \Sigma(f)$. In this case, C is called a *singular cone* over $\Sigma(f)$, and the point x is referred to as the *vertex* of C (Fig. 8.2). Likewise, we take a singular cone $D \subset \mathbb{R}^m$ over $f(C)$. Let $S' = f^{-1}(D) - C$.

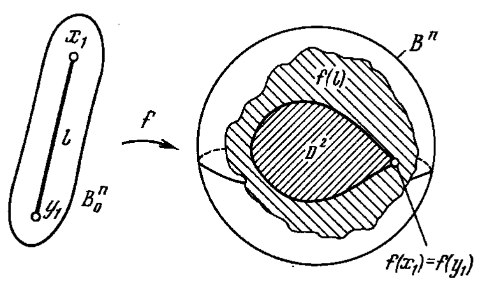

Figure 8.1

If $m \geqslant \frac{3(n+1)}{2}$, then we have $(k+2)+n < m$ and, by a general position argument, $D \cap f(N) = f(C)$, that is, $S' = \varnothing$. By definition, the singular cones C and D are collapsible.

If $m \geqslant \frac{3n}{2} + 1$, then S' need not be empty, but the dimension of S' does not exceed $(k+2) + n - m = 0$ by a general position argument. Thus, before 'killing'

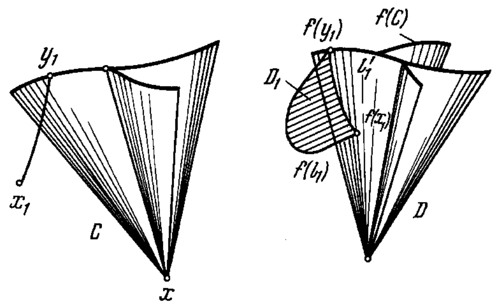

Figure 8.2

$\Sigma(f)$ in the same way as in the preceding, we must 'kill' the set S' of dimension 0. Let $S' = \{x_1, \dots, x_p\}$. We take an arc l_1 joining x_1 to an arbitrary point $y_1 \in C$ and an arc $l_1' \subset D$ joining $f(x_1)$ to $f(y_1)$. Then $f(l_1) \cup l_1'$ is a circle in $\mathbb{R}^m$. We choose a 2-disc D_1 bounded by this circle (Fig. 8.2). Since $2 + n < m$, we have $D_1 \cap f(N) = \partial(D_1)$ by a general position argument. By constructing such pairwise disjoint arcs $l_i \subset N$ and discs $D_i \subset \mathbb{R}^m$ for every x_i, we find that $f^{-1}(\widetilde{D}) = \widetilde{C}$, where $\widetilde{C} = C \cup l_1 \cup \dots \cup l_p$ and $\widetilde{D} = D \cup D_1 \cup \dots \cup D_p$. Let us show that $\widetilde{C}$ and $\widetilde{D}$ are collapsible. Indeed, a collapse of $\widetilde{C}$ to a point is obtained by first collapsing each l_i to y_i and then collapsing the singular cone C to its vertex. Likewise, $\widetilde{D} \searrow *$. We have constructed the desired collapsible polyhedra $\widetilde{C}$ and $\widetilde{D}$.

The above transition from $m \geqslant \frac{3n+3}{2}$ to $m \geqslant \frac{3n+2}{2}$ is the first step of an inductive procedure that allows us to construct the polyhedra C and D for $m \geqslant n+3$. At the jth inductive step, we have $m \geqslant \frac{3n+3-j}{2}$, and hence $\dim S' \leqslant j-1$. To 'kill' the polyhedron S', it suffices to find a 'membrane' of dimension $\leq j$ 'joining' S' to C. By the Relative Engulfing Lemma 8.2, there is a regular neighbourhood $B = R_N(C) \supset S'$. Let C' be the trace of S' under the collapse $B \searrow C$. Then C' collapses to $C' \cap C$. (Here C' plays a role similar to that of $l_1 \cup \dots \cup l_n$ in the first step.) We choose a similar membrane D' of dimension $\leq j+1$ joining $f(C')$ to D (Fig. 8.2). Then D' collapses to $D' \cap D$. However, the intersection of D' with $f(N)$ need not be contained in $f(C')$. Let $S'' = f^{-1}(D') - C'$. Then $\dim S'' \leqslant j+1+n-m \leqslant j-2$, since $m-n \geqslant 3$. We note that we have reduced the 'singular dimension': $\dim S'' \leqslant \dim S' - 1$. Hence, by analogy with the previous inductive step, we can 'kill' S'', that is, construct a polyhedron C'' containing S'' and collapsible to $C'' \cap (C' \cup C)$ and a polyhedron D'' collapsible to $D'' \cap (D' \cup D)$ such that $f^{-1}(D'') = C''$. Then the polyhedra $\widetilde{C} = C \cup C' \cup C''$ and $\widetilde{D} = D \cup D' \cup D''$ are collapsible, and $f^{-1}(\widetilde{D}) = \widetilde{C}$. This way we 'kill' S' and complete the jth inductive step. This argument remains valid for each $j \leqslant k = 2n-m$, since $\frac{3n+3-k}{2} = n+3$.

§9. The first part of the proof of Theorem 4.1.e

The proof of Theorem 4.1.e in the piecewise linear case consists of two steps, namely, Theorem 9.1 (an analogue of Lemmas 7.1 and 7.2) and Theorem 10.1 (an analogue of Lemma 7.3). Theorem 9.1 is proved by induction. It follows from Proposition 9.2 with $\sigma^p = \sigma^q =$ (the last simplex of T).

Theorem 9.1 [156]. *Suppose that K is an n-polyhedron with a triangulation T, $m \geq \frac{3(n+1)}{2}$, and $\Phi\colon \widetilde{K} \to S^{m-1}$ is an equivariant map. Then there is a piecewise linear map $f\colon K \to \mathbb{R}^m$ such that properties* (7.1.a) *and* (7.1.b) *hold and*

$$\widetilde{f}\big|_{\widetilde{T}} \simeq_{\mathrm{eq}} \Phi\big|_{\widetilde{T}}. \tag{9.1}$$

Proposition 9.2. *Under the hypotheses of* Theorem 9.1, *for each $\sigma^p \times \sigma^q \in \widetilde{T}$ such that $\sigma^p \leqslant \sigma^q$ there is a piecewise linear map $f : K \to \mathbb{R}^m$ such that* (7.1.a) *holds,* (7.1.b) *holds for $(\alpha,\beta) < (\sigma^p,\sigma^q)$, and*

$$\widetilde{f} \simeq_{\mathrm{eq}} \Phi \quad \text{on} \quad J_{\sigma^p\sigma^q} = \cup\{\alpha\times\beta \cup \beta\times\alpha \subset \widetilde{T} \mid (\alpha,\beta) < (\sigma^p,\sigma^q)\}. \tag{9.2}$$

First Ball Lemma 9.3. *If the conclusion of* Proposition 9.2 *holds, then there are piecewise linear balls $D^m \subset \mathbb{R}^m$, $D^p \subset \mathring{\sigma}^p$, and $D^q \subset \mathring{\sigma}^q$ such that:*

a) *$f|_{D^p}$ and $f|_{D^q}$ are proper embeddings in D^m;*
b) *$f\sigma^p \cap f\sigma^q \subset \mathring{D}^m$;*
c) *$D^p = \sigma^p \cap f^{-1}D^m$ and $D^q = \sigma^q \cap f^{-1}D^m$;*
d) *$D^m \cap fP = \varnothing$, where $P = K - \operatorname{st}\sigma - \operatorname{st}\tau$.*

Proof. We find from (7.1.b) that $f\sigma^p \cap f\partial\sigma^q = f\partial\sigma^p \cap f\sigma^q = \varnothing$ for $(\alpha,\beta) < (\sigma^p,\sigma^q)$. By a general position argument, $\dim(f\sigma^p \cap f\sigma^q) \leq p+q-m$. Let $C_1 \subset \sigma^p$ be the trace of the polyhedron $\sigma^p \cap f^{-1}\sigma^q$ under the collapse $\sigma^p \searrow$ (a point in $\mathring{\sigma}^p$). In a similar way, we define $C_2 \subset \mathring{\sigma}^q$. The polyhedra C_1 and C_2 are generalizations of the arcs l_1 and l_2 in the Whitney trick. They are collapsible, $f\sigma^p \cap f\sigma^q \subset fC_1 \cap fC_2$, and $\dim C_1, \dim C_2 \leq p+q-m+1$. By (7.1.b), for $(\alpha,\beta) < (\sigma^p,\sigma^q)$ we have $C_1 \cap P = \varnothing$. By a general position argument, $\dim(fP \cap f\sigma^q) \leqslant n+q-m$, and hence $\dim(fP \cap f\sigma^q) + \dim C_2 < q$ and $C_2 \cap P = \varnothing$.

We consider a sequence of collapses from some piecewise linear m-ball $J^m \subset \mathbb{R}^m$ containing $f(\sigma^p \cap \sigma^q)$ in its interior to a point in $\mathring{J}^m$. Let C be the trace of $C_1 \cup C_2$ under this collapse. The polyhedron C is a generalization of the disc D in the Whitney trick. It is collapsible, $C_1 \cup C_2 \subset C$, and $\dim C \leqslant p+q-m+2$. By a general position argument, $C \cap f\sigma^p = C_1$, $C \cap f\sigma^q = C_2$, and $C \cap fP = \varnothing$. Here we have used the inequality $m \geq \frac{3(n+1)}{2}$. Just as in the Whitney trick, we can now readily verify that the regular neighbourhoods of the polyhedra C, C_1, and C_2 in some sufficiently small (consistent) triangulations of $\mathbb{R}^m$, σ^p, and σ^q, respectively, are the desired balls.

Proof of Proposition 9.2. We take a generic map $f\colon K \to \mathbb{R}^m$ that is linear on the simplices of T. The map f already satisfies (7.1.a). By the inductive hypothesis (about (σ^p,σ^q)), we can assume that properties (7.1.a) and (9.2) hold and that property (7.1.b) holds for $(\alpha,\beta) < (\sigma^p,\sigma^q)$. We can also assume that f is in general position. Suppose that $p+q \geq m-1$. (Otherwise, the inductive step can be carried out by a general position argument.)

The first part of the proof (a generalization of the Whitney trick) is to obtain property (7.1.b) for $(\alpha,\beta) = (\sigma^p,\sigma^q)$. We take the piecewise linear balls D^m, D^p, and D^q from the First Ball Lemma. It follows from (7.1.b) and the property a) in Lemma 9.3 that $f\partial D^p \cap fD^q = f\partial D^q \cap fD^p = \varnothing$. Since $m-p \geqslant 3$, it follows that $D^m - fD^p \simeq S^{m-p-1}$ (see the remark before (7.3)). The homotopy class in $\pi_{q-1}S^{m-p-1}$ of the map $f|_{\partial D^q}\colon \partial D^q \to D^m - fD^p$ is called the *intersection number* of $f|_{\partial D^p}$ and $f|_{\partial D^q}$ and is denoted by $I(f|_{D^q}, f|_{D^q})$. We have

$$\Sigma^p I(f|_{D^p}, f|_{D^q}) = (-1)^{m-p}[\widetilde{f}|_{\partial(D^p\times D^q)}] = [\Phi|_{\partial(D^p\times D^q)}] = 0.$$

Here the first equality holds by [82] and [153], Proposition 1. The second equality holds since $\widetilde{f} \simeq \Phi$ on $\partial(D^p \times D^q)$ by the inductive hypothesis. The third equality holds since Φ is defined on $\widetilde{T} \supset D^p \times D^q$. By the Freudenthal suspension theorem, we find that the embedding $f|_{\partial D^q}$ extends to a *map* $f'\colon D^q \to D^m - fD^p$. Here we have used the inequality $2p+q \leqslant 2m-3$.

Since $2q-m+1 \leq m-p-2$, it follows from Theorem 3.2.a (actually, we need only the case $m \geqslant \frac{3(n+1)}{2}$ of this theorem) that f' is homotopic $\operatorname{rel}\partial D^q$ to an *embedding* $f^+\colon D^q \to D^m - fD^p$. Here we have once more used the inequality $p+2q \leq 2m-3$. Since $m-q \geqslant 3$, it follows that the relative version of Theorem 3.5 ([166], Corollary 1 to Theorem 9) that there is an ambient isotopy $h_t\colon D^m \to D^m \operatorname{rel}\partial D^m$ between $f|_{D^q}$ and f^+. We extend f^+ to K by the formula

$$f^+(x) = \begin{cases} h_1(f(x)) & \text{if } f(x) \in D^m \text{ and } x \in \gamma \text{ for some } \gamma \supset \sigma^q, \\ f(x) & \text{otherwise.} \end{cases}$$

One can readily check that f^+ satisfies (7.1.b) for $(\alpha,\beta) \leqslant (\sigma^p,\sigma^q)$, (9.2), and (7.1.a).

The second part of the proof (a generalization of van Kampen finger moves) is to obtain property (9.2) for $(\alpha,\beta) = (\sigma^p,\sigma^q)$ assuming properties (7.1.a) and (7.1.b) for $(\alpha,\beta) \leqslant (\sigma^p,\sigma^q)$ as well as property (9.2). We start from an analogue of the First Ball Lemma. By a general position argument, we can find points $a \in \mathring{\sigma}^p - \Sigma(f)$ and $b \in \mathring{\sigma}^q - \Sigma(f)$ such that the restrictions of f to some small neighbourhoods of a and b are embeddings. Since $x, y \leqslant m-2$, we can join the points fa and fb by an arc $l \subset \mathbb{R}^m$ such that $l \cap fK = \{fa, fb\}$. Let $D^m = R_{\mathbb{R}^m}(l)$. Then $f^{-1}D^m$ is the disjoint union of piecewise linear discs $D^p \subset \mathring{\sigma}^p$ and $D^q \subset \mathring{\sigma}^q$ that are regular neighbourhoods in K of a and b, respectively.

By the Borsuk homotopy extension theorem, there is an extension $\Psi\colon \widetilde{T} \to S^{m-1}$ of the map $\widetilde{f}|_{J\cup(\sigma^p\times\sigma^q - \mathring{D}^p\times\mathring{D}^q)}$ such that $\Psi \simeq \Phi$. It follows that $\Psi = \widetilde{f}$ on $\partial(D^p\times D^q)$. We can assume that $\Psi = \Phi$. By (7.3), for each map $f'\colon D^p \sqcup D^q \to D^m$ such that $f' = f$ on $D^p \sqcup \partial D^q$ and $f'D^p \cap f'D^q = \varnothing$ and for a homotopy f_t $\operatorname{rel} D^p \sqcup \partial D^q$ between f and f', we have

$$[H_{\Phi\widetilde{f}_t\widetilde{f}'}] = [H_{\Phi\widetilde{f}}] + [H_{\widetilde{f}\widetilde{f}_t\widetilde{f}'}] = [H_{\Phi\widetilde{f}}] + (-1)^{m-p}\Sigma^p[h_{ff'}] \in \pi_{p+q}S^{m-1}.$$

(Here Φ, $\widetilde{f}$, $\widetilde{f}^+$, and $\widetilde{f}_t$ stand for the restrictions of these maps to $D^p \times D^q$ and $\partial(D^p\times D^q)$, respectively.) Since $2p+q \leqslant 2m-3$, we have $q \leqslant 2(m-p-1)-1$, and Σ^p is an epimorphism by the Freudenthal suspension theorem. Since for *every* element $\beta \in \pi_q S^{m-p-1}$ there is a map (not necessarily an embedding) $f'\colon D^q \to D^m - fD^p$ such that $[h_{ff'}] = \beta$ and $f' = f$ on $D^p \sqcup \partial D^q$, it follows that we can take an f' such that $[H_{\Phi\widetilde{f}_t\widetilde{f}'}] = 0$. Here we have again used the inequality $p + 2q \leq 2m-3$.

The remaining part of the proof is the same as in the generalization of the Whitney trick.

§10. The second part of the proof of Theorem 4.1.e

Theorem 10.1 ([136]; cf. [153]). *Suppose that K is an n-polyhedron with a triangulation T, $m \geqslant \frac{3(n+1)}{2}$, and $\varphi\colon K \to \mathbb{R}^m$ is a piecewise linear map such that properties* (7.1.a) *and* (7.1.b) *hold and*

a) *for each α, there are homeomorphisms*

$$R_N(\alpha, \partial\alpha) \cong \operatorname{lk}\alpha * \alpha \quad \text{and} \quad R_{\mathbb{R}^m}(f\alpha, f\partial\alpha) \cong \operatorname{lk} f\alpha * f\alpha$$

(*for some small triangulation of $\mathbb{R}^m$ independent of α*) *such that $f \operatorname{lk}\alpha \subset \operatorname{lk} f\alpha$ and $f|_{R_N(\alpha,\partial\alpha)} = f|_{\operatorname{lk}\alpha} * f|_\alpha$.*

Then there is a piecewise linear embedding $f : K \to \mathbb{R}^m$ such that

$$\widetilde{f}|_{\widetilde{T}} \simeq_{\text{eq}} \varphi|_{\widetilde{T}}. \tag{10.1}$$

Here by lk we denote the *link* of a simplex in a triangulation [125]. The second (harder) part of the proof in [153] contains a mistake ([153], p. 24, lines 9 and 18), which is apparently technical and can be eliminated with the help of the same ideas. The proof in [136] also contains a mistake (property 10.1.a is not stated explicitly), which is rectified in the present paper. We note that property 10.1.a (or the property $f^{-1}R_{\mathbb{R}^m}(f\eta, f\partial\eta) \neq R_K(\eta, \partial\eta)$, which follows from 10.1.a) is false for an *arbitrary* generic piecewise linear map. But it holds for a map linear on simplices and is preserved in the course of the proof of Theorem 9.1. Therefore, Theorem 4.1.ε indeed follows from Theorem 9.1 (more precisely, from the version of Theorem 9.1 with property 10.1.a added to the assumptions and the conclusion) and Theorem 10.1. The proof of Theorem 10.1 presented here generalizes the proof of Lemma 7.3. Theorem 10.1 follows from Proposition 10.2 below with $\sigma^p = \sigma^q =$ (the last simplex of T).

Proposition 10.2. *Under the hypotheses of Theorem* 10.1, *for each $\sigma^p \times \sigma^q \in T \times T$ such that $\sigma^p \geqslant \sigma^q$ there is a piecewise linear map $f\colon K \to \mathbb{R}^m$ such that*

a) *$f\alpha \cap f\beta = \varnothing$ for each $\alpha \times \beta \in \widetilde{T}$,*

b) *$f|_\alpha$ is an embedding for each $\alpha \in T$,*

c) *$\widetilde{f}|_{\widetilde{T}}$ is equivariantly homotopic to $\widetilde{\varphi}|_{\widetilde{T}}$,*

d) *$f\alpha \cap f\beta = f(\alpha \cap \beta)$ for $(\alpha, \beta) < (\sigma^p, \sigma^q)$.*

Proof. The map φ already satisfies 10.2.a–10.2.c. We achieve 10.2.d by induction on (σ^p, σ^q). The base clause '$\sigma^p =$ (the first simplex of T)' follows by taking $f = \varphi$. Now we suppose that f satisfies 10.2.a–10.2.d. We can assume that f is in general position. Suppose that $p + q \geqslant m - 1$, $\sigma^q \not\subset \sigma^p$, and $\sigma^q \cup \sigma^p$ is not contained in the boundary of the same simplex of T. (Otherwise, the inductive step holds either by a general position argument or by the inductive hypothesis.) Let $D^r = f(\sigma^p \cap \sigma^q)$. By 10.2.b, D^r is a piecewise linear ball.

Second Ball Lemma 10.3. *There are piecewise linear-balls D^p, D^q, and $D^m \subset \mathbb{R}^m$ such that*

1) *$D^p \subset D^r \cup f\mathring{\sigma}^p$ and $D^q \subset D^r \cup f\mathring{\sigma}^q$;*

2) *$D^p = D^m \cap f\sigma^p$ and $D^q = D^m \cap f\sigma^q$ are properly embedded in D^m;*

3) $D^r = \partial D^p \cap \partial D^q$;
4) D^r *is unknotted in* ∂D^p *and* ∂D^q;
5) $\Sigma = \mathrm{Cl}((f\sigma^p \cap f\sigma^q) - D^r) \subset \mathring{D}^m \cup D^r$;
6) $D^m \cap X \subset D^r$, *where* $X = \bigcup f\{\alpha \in T | \alpha \cap \sigma^p = \varnothing$ *or* $\alpha < \sigma^q\}$.

Proof of Proposition 10.2 *on the basis of Second Ball Lemma* 10.3. We take the piecewise linear balls D^p, D^q, and D^m given by the Second Ball Lemma. We recall ([96], Theorem 9 and the preceding discussion) that if $m-3 \geqslant p$ and q, S^p and $S^q \subset S^m$, and $S^p \cap S^q = D^r$, where D^r is unknotted in S^p and S^q, then $S^p \cup S^q$ is unknotted in S^m. Hence we can assume that $\partial D^p \bigcup_{D^r} \partial D^q \subset \partial D^m$ is a standard embedding. By the relative version of Theorem 3.5 ([166], Corollary 1 to Theorem 9), we can assume that $(D^q, \partial D^q) \subset (D^m, \partial D^m)$ is a standard embedding. Hence, the embedding $\partial D^p \subset \partial D^m$ can be extended to a new embedding of D^p in $(\mathring{D}^m - D^q) \cup \partial D^p$. By the relative version of Theorem 3.5, this new embedding is ambient-isotopic to $D^p \subset D^m \operatorname{rel} \partial D^m$. Thus, there is an isotopy $h_t \colon D^m \to D^m \operatorname{rel} \partial D^m$ such that $D^q \cap h_1 D^p = D^r$. We define a map $f^+ \colon K \to \mathbb{R}^m$ by setting

$$f^+(x) = \begin{cases} h_1(f(x)) & \text{if } f(x) \in D^m \text{ and } x \in \alpha \text{ for some } \alpha \text{ containing } \sigma^p, \\ f(x) & \text{otherwise.} \end{cases}$$

Obviously, f^+ satisfies 10.2.a–10.2.c. Since $\sigma^q \cup \sigma^p$ is not contained in the boundary of the same simplex of T, $D^q \cap h_1 D^p = D^r$, and properties 10.3.5 and 10.3.6 hold, it follows that f^+ satisfies 10.2.d for $(\alpha, \beta) \leqslant (\sigma^p, \sigma^q)$. The inductive step is complete.

Collapsing Lemma 10.4. *If* A *and* F *are regular neighbourhoods of a polyhedron* Z *in a piecewise linear manifold* $M \operatorname{rel} Y$ *and* $A \subset F$, *then* $F \searrow A \operatorname{rel} Y$.

(This follows from [20], Theorem 3.1 and Addendum 3.4.)

Proof of the Second Ball Lemma. We carry out some preliminary constructions (cf. [153], § 6a). Let S be the link of some r-simplex $\subset \mathring{D}^r$ in some small triangulation of $\mathbb{R}^m$. Then S is a piecewise linear $(m-r-1)$-sphere and $R_{\mathbb{R}^m}(D^r, \partial D^r) \cong S * D^r$. By 10.2.b, $R_{\mathbb{R}^m}(D^r, \partial D^r) \cap f\alpha = R_{f\alpha}(D^r, \partial D^r)$ is taken by this homeomorphism to $(S \cap f\alpha) * D^r$ for each $\alpha \in T$. (For $\alpha \not\supset \sigma^p \cap \sigma^q$, each of these three sets is empty.) Furthermore, $S \cap f\alpha$ is a piecewise linear $(\dim \alpha - r - 1)$-ball for each $\alpha \in T$, $\alpha \supset \sigma^p \cap \sigma^q$.

We take distinct points $a \in (S \cap f\mathring{\sigma}^p) - X$ and $b \in (S \cap f\mathring{\sigma}^q) - X$. Since $m-r-1 \geqslant 2$ and $(n-r-1)+1 < m-r-1$, it follows by a general position argument that there is an arc $l \subset S$ joining a and b and such that $l \cap X = \varnothing$, $l \cap f\sigma^p = a$, and $l \cap f\sigma^q = b$. Let $\gamma = R_S(l) * D^r$. Then $\gamma \cap f\sigma^p$ and $\gamma \cap f\sigma^q$ are piecewise linear p- and q-balls, respectively (Fig. 10.1).

Let us construct D^p. By the inductive hypothesis, $f\sigma^p \cap f\partial\sigma^q = f\partial\sigma^p \cap f\sigma^q = D^r$. Hence $\Sigma \subset (f\mathring{\sigma}^p \cap f\mathring{\sigma}^q) \cup D^r$. Both $f\sigma^p$ and $(S * D^r) \cap f\sigma^p = (S \cap f\sigma^p) * D^r$ are regular neighbourhoods of $D^r \operatorname{rel} \partial D^r$ in $f\sigma^p$. By Collapsing Lemma 10.4, $f\sigma^p \searrow (S \cap f\sigma^p) * D^r \operatorname{rel} D^r$. Both $S \cap f\sigma^p$ and $R_{S \cap f\sigma^p}(a)$ are regular neighbourhoods of a in $S \cap f\sigma^p$. By Collapsing Lemma 10.4, $S \cap f\sigma^p \searrow R_{S \cap f\sigma^p}(a)$. Hence

$$(S \cap f\sigma^p) * D^r \searrow R_{S \cap f\sigma^p}(a) * D^r = \gamma \cap f\sigma^p \operatorname{rel} D^r.$$

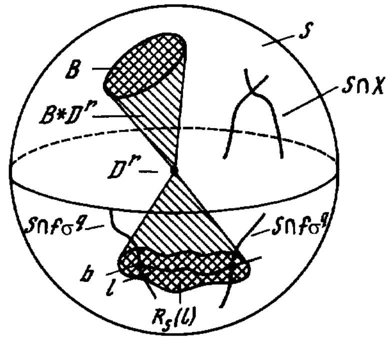

Figure 10.1

Let C_1 be the generic trace of Σ under the sequence of collapses (see Fig. 10.2)

$$f\sigma^p \searrow (S \cap f\sigma^p) * D^r \searrow \gamma \cap f\sigma^p \operatorname{rel} D^r.$$

Let $D^p = R_{f\sigma^p}((\gamma \cap f\sigma^p) \cup C_1, D^r)$. Then properties 10.3.1 and 10.3.4 hold for D^p, and moreover,

$$\begin{array}{llll} 1) & C_1 \subset f\sigma^p; & 4) & C_1 \cap X = \varnothing; \\ 2) & \Sigma \subset (\beta \cap f\sigma^p) \cup C_1; & 5) & D^p \cap X \subset D^r; \\ 3) & D^p \text{ is a piecewise linear } p\text{-ball}; & 6) & C_1 \cap f\sigma^q = \Sigma. \end{array} \tag{10.2}$$

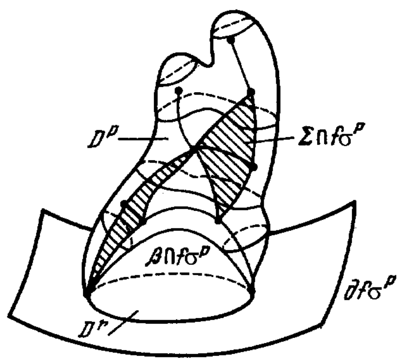

Figure 10.2

Indeed, properties (10.2.1) and (10.2.2) are obvious. Since $\Sigma \subset D^r \cup f\mathring{\sigma}^p$, it follows that $C_1 \subset D^r \cup f\mathring{\sigma}^p$, and hence 10.3.1 holds. Since $f\sigma^p$ is a piecewise linear manifold and $f\sigma^p \searrow (\gamma \cap f\sigma^p) \cup C_1 \operatorname{rel} D^r$, we see that $f\sigma^p$ is a regular neighbourhood of $(\gamma \cap f\sigma^p) \cup C_1$ in $f\sigma^p \operatorname{rel} D^r$ ([20], Theorem 9.1). Then, by Theorem 3.1 in [20], there is an isotopy $G_t\colon f\sigma^p \to f\sigma^p \operatorname{rel}(\gamma \cap f\sigma^p) \cup C_1$ between $G_0 = \mathrm{id}$ and a homeomorphism G_1 of $f\sigma^p$ onto $D^p \operatorname{rel}(\gamma \cap f\sigma^p) \cup C_1$. This implies (10.2.3). Moreover, $G_1|_{\partial f\sigma^p}$ is a homeomorphism of $\partial f\sigma^p$ onto $\partial D^p \operatorname{rel} D^r$.

Since D^r is unknotted in $\partial f\sigma^p$, we see that 10.3.4 is true for D^p. By a general position argument, $\dim\Sigma \leqslant 2n-m$. Thus $\dim C_1 \leqslant 2n-m+1$. By a general position argument and since $n+(2n-m)<m$, we have $\Sigma\cap X=\varnothing$. Again a general position argument and the inequality $n+(2n-m+1)<m$ imply (10.2.4). Since $l\cap X=\varnothing$, it follows that $\gamma\cap f\sigma^p\cap X=D^r$. This, together with (10.2.4), implies (10.2.5). By the definition of relative collapse, $C_1\cap D^r=\Sigma\cap D^r$. Because of this and by a general position argument ($n+(2n-m+2)<m$), we have (10.2.6).

Likewise, we can construct polyhedra C_2 and D^q such that properties 10.3.1, 10.3.4, and (10.2.1)–(10.2.6) hold with $C_1\to C_2$ and $p\to q$.

Let us construct D^m. We take a piecewise linear $(m-r-1)$-ball $B\subset S-(l\cup f\sigma^p\cup f\sigma^q)$. Then $S-\mathring{B}$ is a piecewise linear $(m-r-1)$-ball and $\sigma^m=(\mathbb{R}^m\cup\infty)-\operatorname{Int}(B*D^r)$ is a piecewise linear m-ball. By (10.2.1) and 10.1.a, $C_1\cap(S*D^r)\subset(S\cap f\sigma^p)*D^r$. Then $C_1\cap\operatorname{Int}(B*D^r)=\varnothing$, and hence $C_1\subset\mathring{\sigma}^m\cup D^r$. Likewise, $C_2\subset\mathring{\sigma}^m\cup D^r$. By analogy with the construction of D^p and D^q, let C be the generic trace of $C_1\cap C_2$ under the sequence of collapses

$$\sigma^m\searrow\sigma^m\cap(S*D^r)=(S-\mathring{B})*D^r\searrow R_S(l)*D^r=\gamma\operatorname{rel}D^r.$$

By analogy with (10.2.1)–(10.2.3), we can prove that $C\subset\sigma^m\cup D^r$, $C_1\cup C_2\subset\gamma\cup C$, and $D^m=R_{\sigma^m}(\gamma\cup C,D^r)$ is a piecewise linear m-ball. By analogy with (10.2.4), one can prove that $C\cap X=\varnothing$ by using (10.2.4) and the inequality $n+(2n-m+2)<m$. Then 10.3.6 is proved in the same way as (10.2.5). Property (10.2.6), in conjunction with a general position argument, implies

$$C\cap f\sigma^q=(C_1\cup C_2)\cap f\sigma^q=C_2\cup(C_1\cap f\sigma^q)=C_2\cup\Sigma=C_2.$$

Likewise, $C\cap f\sigma^p=C_1$. Therefore, $(\gamma\cup C)\cap f\sigma^p=(\gamma\cap f\sigma^p)\cup C_1$ and $(\gamma\cup C)\cap f\sigma^q=(\gamma\cap f\sigma^q)\cup C_2$. Because of this and since D^p, D^q, and D^m are regular neighbourhoods rel D^r of $(\gamma\cap f\sigma^p)\cup C_1$, $(\gamma\cap f\sigma^q)\cup C_2$, and $\gamma\cup C$ in the restrictions of the same triangulation of $\mathbb{R}^m$ to $f\sigma^p$, $f\sigma^q$, and σ^m, respectively, 10.3.2 follows. By (10.2.2) and the definitions of D^p, D^q, and Σ,

$$(\partial D^p-D^r)\cap(\partial D^q-D^r)\subset(f\mathring{\sigma}^p-\Sigma)\cap(f\mathring{\sigma}^q-\Sigma)=\varnothing.$$

Hence 10.3.3 is true. By (10.2.1), we have $\Sigma\subset(\gamma\cap f\sigma^p)\cup C_1\subset\gamma\cup C\subset\mathring{D}^m\cup D^r$, and so 10.3.5 is true.

§11. The idea of the proof of Theorem 4.2.e

The non-closed case of Theorem 4.2.e follows from Theorems 4.2.i and 11.1.a. The proof of the closed case of Theorem 4.2.e consists of three parts: the construction of an immersion $h\colon N-\mathring{B}^n\to\mathbb{R}^m$ (Theorem 4.2.i), the construction of a quasi-embedding $g\colon N\to\mathbb{R}^m$ (here we prove the weaker Theorem 11.1.b; see the complete proof in [137]), and the construction of an embedding $f\colon N\to\mathbb{R}^m$ (Lemma 11.2). When applying Lemma 11.2 to the proof of the property $f\simeq_{\mathrm{eq}}\Phi$ in Theorem 4.2.e, we need to take a triangulation T of N such that B is contained in a single simplex of T. Then $\widetilde{f} \simeq_{\text{eq}} \widetilde{g} \simeq_{\text{eq}} \Phi$ on $\widetilde{T}$, and $\widetilde{N}$ is equivariantly retracted onto $\widetilde{T}$ [63]). The non-closed case of Theorem 4.7.e follows from Theorem 4.7.i and an analogue of Theorem 11.1.b. The closed case of Theorem 4.7.e can be proved in the same way as Theorem 4.2.e, on the basis of the 'Concordance Implies Isotopy' theorem, and we do not present the proof here (see the complete proof in [137]).

Theorem 11.1. *Let N be an n-polyhedron (a smooth n-manifold).*

a) ([46], Theorems 2′a and 6.4, [153], Theorem 8, and [55], Corollary 5.(ii).) *Suppose that either $m \geqslant \frac{3(n+1)}{2}$ or the following properties hold: N is a piecewise linear (smooth) n-manifold, $(N, \partial N)$ is $[\frac{d}{3}]$-connected, $\pi_1 \partial N = 0$, and $m \geqslant 6$. If $h : N \to \mathbb{R}^m$ is a piecewise linear (smooth) immersion and $\Phi \colon \widetilde{N} \to S^{m-1}$ is an equivariant map such that $\widetilde{h} \simeq_{\text{eq}} \Phi$ on SN, then h is piecewise-linearly (smoothly) regularly homotopic to a piecewise linear (smooth) embedding $f \colon N \to \mathbb{R}^m$ such that $\widetilde{f} \simeq_{\text{eq}} \Phi$.*

b) *Suppose that $m \geqslant n+3$ and N is a closed d-connected piecewise linear manifold. If $h \colon N \to \mathbb{R}^m$ is a piecewise linear immersion and $\Phi \colon \widetilde{N} \to S^{m-1}$ is an equivariant map such that $\widetilde{h} \simeq_{\text{eq}} \Phi$ on SN, then h is piecewise-linearly regularly homotopic to a piecewise linear map $f \colon N \to \mathbb{R}^m$ such that $\Sigma(g)$ is contained in some piecewise linear n-ball $B \subset N$ and $\widetilde{f} \simeq_{\text{eq}} \Phi$ on $\widetilde{N} - \widetilde{B}$.*

The smooth case of Theorem 4.1.e was derived from that of Theorems 4.1.i and 11.1.a [46]. The piecewise linear case of Theorem 4.1.e was proved in this way in [137]. (Originally, it was proved in a similar but different way [153], [136].) For $A \subset N$, we write $A^* = A \times N \cup N \times A$. The following result was essentially proved in [58]. We present the proof for completeness.

Lemma 11.2 (cf. [135], Theorem 2.1.2)**.** *Suppose that N is a closed homologically d-connected piecewise linear n-manifold, $m - n \geqslant 3$, and $g \colon N \to \mathbb{R}^m$ is a map such that $\Sigma(g)$ is contained in some piecewise linear n-ball $B \subset N$. Then there is an embedding $f \colon N \to \mathbb{R}^m$ such that $f = g$ on $N - \mathring{B}$.*

The idea of a possible proof of Theorems 4.2.i and 4.7.i for piecewise linear manifolds. We follow the idea of [52]. The *equivariant Stiefel manifold* V_{mn}^{eq} is the space of maps $S^{n-1} \to S^{m-1}$ equivariant with respect to the antipodal involutions. The *piecewise linear Stiefel manifold* V_{mn}^{PL} is the space of equivariant piecewise linear embeddings $S^{n-1} \to S^{m-1}$. Suppose that we have proved that V_{mn}^{PL} is a retract of the space of all piecewise linear embeddings $S^{n-1} \to S^{m-1}$.

The set of equivariant maps $\Phi \colon SN \to S^{m-1}$ up to equivariant homotopy is in a one-to-one correspondence (up to fibrewise homotopy) with the set of sections of the V_{mn}^{eq}-bundle over N associated with TN. By [54], piecewise linear immersions $N \to \mathbb{R}^m$ are in a one-to-one correspondence (up to piecewise linear regular homotopy) with sections of the V_{mn}^{PL}-subbundle of the above bundle. The obstructions to a deformation of a section of the V_{mn}^{eq}-bundle to a section of the V_{mn}^{PL}-subbundle lie in

$$H^i(N, \pi_i(V_{mn}^{\text{eq}}, V_{mn}^{PL})_T) \cong H_{n-i}(N, \partial N, \pi_i(V_{mn}^{\text{eq}}, V_{mn}^{PL})).$$

(The coefficients are twisted in accordance with the orientable double cover of N; the case $\partial N = \varnothing$ is possible.) By the universal coefficients formula, it suffices to prove that $\pi_i(V_{mn}^{\text{eq}}, V_{mn}^{PL}) = 0$ for $0 \leqslant i \leqslant 2(m-n)-2$. From the exact sequence of the triple $V_{mn}^{\text{eq}} \supset V_{mn}^{PL} \supset V_{mn}$, one can see that it suffices to prove that $\pi_i(V_{mn}^{\text{eq}}, V_{mn}) = 0$ for $0 \leqslant i \leqslant 2(m-n)-2$ and $\pi_i(V_{mn}^{PL}, V_{mn}) = 0$ for $0 \leqslant i \leqslant 2(m-n)-3$. The former follows by [52], (1.1) (see the complete proof with misprints corrected in [137]) and the exact homotopy sequence of the pair $V_{mn}^{\text{eq}} \supset V_{mn}$ (even for $0 \leqslant i \leqslant 2(m-n)-1$). By [50], §§8.15, 10.2, and 11.2, and [47], $\pi_i(V_{mn}^{PL}, V_{mn}) = C_{i-1}^{m-n} = 0$ for $i \leqslant 2(m-n)-3$, $i \leqslant n$ (even if $m-n=2$).

Let us prove the latter fact without using the unpublished results of [50]. By [47], §4.6, and [106], Theorem 5.1, there are homomorphisms

$$\pi_i(V_{mn}) \xrightarrow{\alpha_i} \pi_i(G_m, G_{m-n}) \xleftarrow{\beta_i} \pi_i(V_{mn}^{PL})$$

such that α_i is an isomorphism for $0 \leqslant i \leqslant 2(m-n)-4$ and an epimorphism for $i = 2(m-n)-3$, and β_i is an isomorphism for $0 \leqslant i \leqslant 2m-n-4$ and an epimorphism for $i = 2m-n-3$. One can readily verify that the inclusion homomorphism $\rho_i^{PL}\colon \pi_i(V_{mn}) \to \pi_i(V_{mn}^{PL})$ coincides with $\beta_i^{-1} \circ \alpha_i$, and hence ρ_i^{PL} is an isomorphism for $0 \leqslant i \leqslant 2(m-n)-4$ and an epimorphism for $i = 2(m-n)-3$. It remains to apply the exact homotopy sequence of the pair $V_{mn}^{\text{eq}} \supset V_{mn}^{PL}$.

The complete proof of Theorems 4.2.i, which is also more direct and shorter (in the sense that it does not use the results on piecewise linear Stiefel manifolds), is given in [137]. This proof is in a sense similar both to the proof of the smooth case of Theorem 4.1.i ([139], [56], [52], [113]) and to the proof of the piecewise linear case of Theorem 4.1.i [55].

Proof of Theorem 11.1.b. If a triangulation T of a polyhedron N is given, we write $TN = \bigcup_{\sigma\cap\tau\neq\varnothing} \sigma \times \tau$. We take a triangulation T of N such that h is non-degenerate (that is, the restriction of h to every simplex of T is an embedding), $\widetilde{\Delta}(h) \cap TN = \varnothing$, and $\widetilde{h} \simeq_{\text{eq}} \Phi$ on $TN \cap \widetilde{T}$. By applying the method of the first part of the proof of Theorem 4.1.e, we can construct a regular homotopy from h to a T-immersion $g : N \to \mathbb{R}^m$ such that $g\sigma \cap g\tau = \varnothing$ for any disjoint simplices $\sigma, \tau \in T$ such that $\dim\sigma \leqslant \dim\tau$ and $\dim\sigma + 2\dim\tau \leqslant 2m-3$. Let $K = T^{(2m-2n-3)}$. Since $n+(2m-2n-3) \leqslant 2m-n-3$, it follows that $\widetilde{\Delta}(g)\cap K^* = \varnothing$. Since g is a non-degenerate immersion, it follows that $\widetilde{\Delta}(g) \cap R(K^*) = \varnothing$ and $\widetilde{g} \simeq_{\text{eq}} \Phi$ on $R(K^*) \cap \widetilde{T}$. (If h is an immersion, then $\widetilde{\Delta}(h)$ is closed in $N \times N$ and, in general, $\operatorname{Cl}_{N\times N} \widetilde{\Delta}(f) \subset \widetilde{\Delta}(f) \cup \operatorname{diag} N$. If $E \supset TN$, then $R_{\widetilde{T}}(E) \cap \widetilde{N}$ can be equivariantly retracted to $E \cap \widetilde{T}$ for any subpolyhedron $E \subset_{\text{eq}} N \times N$.) We note that $R(K^*) \cap \widetilde{T} \supset (R(K))^* \cap \widetilde{T}$. Therefore, $\Sigma(g) \cap R(K) = \varnothing$. Clearly, $N - R(K)$ is a regular neighbourhood in N of the skeleton U dual to K. Since $\dim U = n-1-(2m-2n-3) = d$ and N is d-connected, it follows by Engulfing Lemma 8.1 that U is contained in some piecewise linear n-ball in N. Therefore, by the theorem stating the uniqueness of a regular neighbourhood, $N - R(K)$ is also contained in some (possibly different) piecewise linear n-ball $B \subset N$. We have $\Sigma(g) \subset N - R(K) \subset B$ and $\widetilde{g} \simeq_{\text{eq}} \Phi$ on $(R(K))^* \cap \widetilde{N} \supset \widetilde{N} - (N - R(K))^{\sim} \supset \widetilde{N} - \widetilde{B}$.

Proof of Lemma 11.2. Let $M = \mathbb{R}^m - \operatorname{Int} R(g(N - \mathring{B}), g\partial B)$. Since N is homologically d-connected, we have by Alexander duality that

$$H_i(M) \cong H^{m-1-i}(\mathbb{R}^m - M) \cong H^{m-1-i}(N - \mathring{B}) \cong H_{n-m+1+i}(N) = 0$$

for $i \leqslant 2n-m+1$. Since $m-n \geqslant 3$, M is simply connected. Therefore, by the Hurewicz isomorphism theorem, M is $(2n-m+1)$-connected. Hence, by Irwin's Embedding Theorem 3.3, the embedding $g\colon \partial B \to \partial M$ extends to an embedding $f\colon B \to M$. Extending f by g outside B, we complete the proof.

§12. The idea of the proofs of isotopy versions

The *isotopy* versions of embeddability theorems can be reduced to their *boundary* versions (not to *relative* versions, as stated in [153]) by using the 'Concordance Implies Isotopy' Theorem (§ 1). To illustrate the idea, we present the following proof.

Proof of the piecewise linear case of Theorem 3.6.1. We consider embeddings $f, g\colon N^n \to \mathbb{R}^m$ and take a generic homotopy $H\colon N \times I \to \mathbb{R}^m \times I$ between f and g. Using the proof of Theorem 3.2, we modify the map H to obtain an embedding. Since f and g are embeddings, it follows that H is an embedding in some neighbourhoods of the bases $N \times 0 \sqcup N \times 1$. Therefore, we can make perturbations of H needed in the proof of Theorem 3.2 in $\operatorname{Int}(N \times I)$ to obtain an embedding H' coinciding with H in $N \times \{0,1\}$. This embedding $H'\colon N \times I \to \mathbb{R}^m \times I$ is a *concordance* between $f = H'|_{N\times 0}$ and $g = H'|_{N\times 1}$. Hence f and g are isotopic by the 'Concordance Implies Isotopy' Theorem.

For a subpolyhedron $A \subset K$, we set $A^* = \widetilde{A} \cup [A \times (K-A)] \cup [(K-A) \times A]$.

Proof of the piecewise linear case of Theorem 4.6.e. We take a generic homotopy $H : N \times I \to \mathbb{R}^m \times I$ between f and g. Using the Cylinder Lemma 12.2 (see below), we obtain an equivariant map $\Sigma\Phi \circ p : \widetilde{N \times I} \to S^m$ such that $\Sigma\Phi \circ p \simeq_{\mathrm{eq}} \widetilde{H}$ on $(N \times \{0,1\})^*$. Using the Theorem 12.1 (which is the piecewise linear case of the boundary version of Theorem 4.1.e) to $K = N \times I$ and $A = N \times \{0,1\}$, we obtain a concordance between f and g. Hence f and g are isotopic by the 'Concordance Implies Isotopy' Theorem.

Theorem 12.1. *Suppose that K is an n-polyhedron, $m \geqslant \frac{3(n+1)}{2}$, A is a subpolyhedron of K, B^m is a piecewise linear m-ball, $g\colon K \to B^m$ is a piecewise linear map such that $g|_A$ is an embedding in ∂B^m, and $g(K-A) \subset \mathring{B}^m$. There is an embedding $f\colon K \to B^m$ such that $f|_A = g|_A$ and $f(K-A) \subset \mathring{B}^m$ if and only if the equivariant map $\widetilde{g}\colon A^* \to S^{m-1}$ extends homotopically to an equivariant map $\Phi\colon \widetilde{K} \to S^{m-1}$.*

The proof of Theorem 12.1 is similar to that of Theorem 4.1.e.

For a polyhedron N with a *given* triangulation T, we identify $\widetilde{T}$ and $\widetilde{N}$ in this section. This will not lead to a misunderstanding.

Cylinder Lemma 12.2 ([136], Lemma 6.1; cf. [153], Lemma 7.1 and [135], Lemma 3.3)**.** *If a polyhedron N has a given triangulation and $N \times I$ is cell-subdivided as a product, then*

$$\frac{\widetilde{N \times I}}{N \times N \times 0 \times 1,\ N \times N \times 1 \times 0} \cong_{\mathrm{eq}} \Sigma(\widetilde{N} \times I).$$

Let $p\colon \widetilde{N\times I} \to \Sigma(\widetilde N \times I)$ *be the factor-projection. Then*

$$p^{-1}\Sigma(\widetilde N \times \{0,1\}) = \widetilde{N\times I} \cap (N\times N\times \partial(I\times I)).$$

Furthermore, suppose that $H\colon N\times I \to \mathbb{R}^m \times I \subset \mathbb{R}^{m+1}$ *is a level-preserving piecewise linear map and* $\varphi\colon \widetilde N \times \operatorname{diag} I \to S^{m-1}$ *is an equivariant homotopy between* $\widetilde{H|_{N\times 0}}$ *and* $\widetilde{H|_{N\times 1}}$. *Then* $\widetilde H \simeq_{\mathrm{eq}} \Sigma\varphi\circ p$ *on* $(N\times\{0,1\})^*$.

Proof. We recall that $\widetilde{N\times I}$ is a *cellular* deleted product. Therefore, for every $(x,s,y,t) \in \widetilde{N\times I}$ we have either $x\neq y$ or $\{s,t\} = \{0,1\}$. We define a map $p\colon \widetilde{N\times I} \to \Sigma(\widetilde N\times I)$ by the formula (Fig. 12.1)

$$p(x,s,y,t) = \begin{cases} [(x,y,\frac{s+t}{2}), s-t] & \text{if } x\neq y,\\ [\widetilde N\times I, s-t] & \text{if } x = y \ \ \text{(and hence } |s-t| = 1). \end{cases}$$

One can readily see that p is well-defined, equivariant, and surjective, and its only non-trivial inverse images are those of the vertices of the suspension, namely, $N\times N\times 0\times 1$ and $N\times N\times 1\times 0$. The assertion about $p^{-1}\Sigma$ can be verified easily.

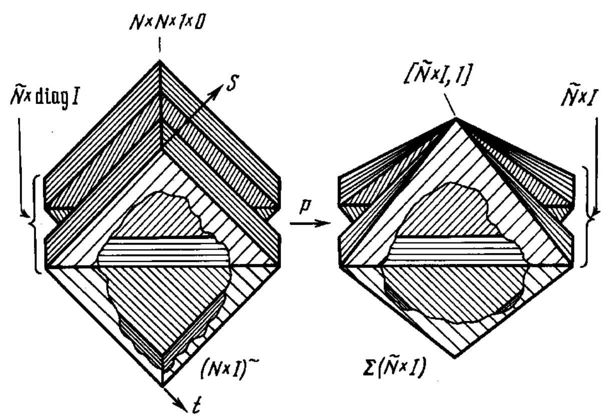

Figure 12.1

To prove the 'furthermore' part, we observe that $\Sigma\varphi\circ p(x,t,y,t) = \varphi(x,y,t)$. Hence $\Sigma\varphi\circ p = \widetilde H$ on $(N\times\{0,1\})^* \cap (\widetilde N\times \operatorname{diag} I)$. For $(x,s,y,t)\in (N\times\{0,1\})^*$ and $s<t$ $(s>t)$, both points $\Sigma\varphi\circ p(x,s,y,t)$ and $\widetilde H(x,s,y,t)$ are in the northern (southern) open hemisphere and hence are not antipodal. Therefore, $\Psi \simeq_{\mathrm{eq}} \widetilde H$ on $(N\times\{0,1\})^*$.

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