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Borromean Rings and Embedding Obstructions

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Abstract—This survey describes recent examples of incompleteness of the Van Kampen and the deleted product obstructions beyond the metastable case. The construction is an interesting example of the interplay between algebraic and geometric topology and one of its origins is the Borromean rings example.

1. INTRODUCTION

A classical problem in topology is to find conditions under which a given polyhedron is embeddable into $\mathbb{R}^m$ for any given m (see [27]). By general position, every n-polyhedron is embeddable in $\mathbb{R}^m$ for every $m \geq 2n+1$. A nice characterization of graphs, embeddable in $\mathbb{R}^2$ (in terms of "prohibited" subgraphs) was obtained by Kuratowski [20] (see also [41, 22]).

However, this approach can only be fruitful in dimension 2 [30]. For the problem of the embeddability of an n-dimensional polyhedron K in $\mathbb{R}^{2n}$ Van Kampen introduced a cohomological obstruction [17]. He also initiated a proof of its sufficiency for $n \geq 3$, which was then completed by Shapiro and Wu, using the Whitney trick [34, 44].

Subsequently, their results were generalized to the *metastable* case $m \geq \frac{3(n+1)}{2}$ by Haefliger and Weber, to smooth and PL embeddings of smooth n-manifolds and n-polyhedra, respectively, in $\mathbb{R}^m$ (in terms of the deleted product obstruction) [12, 42] (see also [13, 35–37]).

The metastable dimension restriction is necessary in the Haefliger theorem [25, 14; 26, §2], and it is sharp in the isotopy analog of the Haefliger–Weber theorem [10, 11; 23, Proposition 8.3].

Recently, Freedman, Krushkal, and Teichner have shown that the Van Kampen–Shapiro–Wu theorem fails for $n = 2$ [6]. Furthermore, Segal, Skopenkov, and Spiez showed that the metastable dimension restriction in the Weber theorem [33, 40] is sharp (see also [24, 16]), even if the deleted product obstruction is replaced by the p-fold deleted product obstruction.

The construction of their examples is an interesting interplay between algebraic and geometric topology. The purpose of this survey is to show how these examples were conceived. One of their origins was the Borromean rings example, which was also the origin of the examples in [10, 11; 23, Proposition 8.3].

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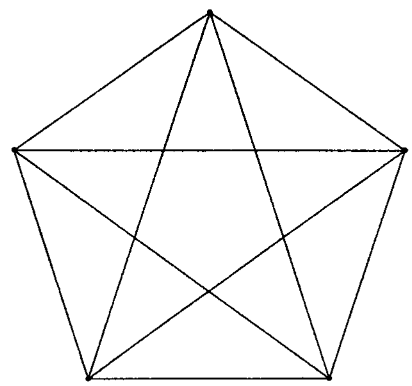

Fig. 1

2. THE VAN KAMPEN OBSTRUCTION

To explain the idea of the Van Kampen obstruction, we would like to sketch the proof of non-planarity of K_5 (the complete graph with 5 vertices). Take any general position map $f\colon K_5 \to \mathbb{R}^2$. Let v_f be the sum mod 2 of the numbers $|f\sigma \cap f\tau|$ of the intersection points of the f-images $f\sigma$ and $f\tau$, over all non-ordered pairs $\{\sigma, \tau\}$ of disjoint edges of K_5. For the map f shown in Fig. 1, $v(f) = 1$. Every general position map $f\colon K_5 \to \mathbb{R}^2$ can be transformed to any other such map through isotopies of $\mathbb{R}^2$ and "Reidemeister moves" for graphs in the plane from Fig. 2. For each edge of K_5 with vertices a and b, the graph $K_5 - \{a, b\}$, obtained by deleting from K_5 vertices a and b and interiors of the edges adjacent to a and b, is a circle. This is the very property of K_5 which is necessary for this proof. Therefore v_f is invariant under the "Reidemeister moves." Hence $v_f = 1$ for *each* general position map $f\colon K_5 \to \mathbb{R}^2$, and so K_5 is nonplanar.

Now let us discuss some generalizations of the above proof which will be used in the sequel. This proof actually implies a stronger assertion. Let e be an edge of K_5 and Σ the cycle in K_5 formed by the edges of K_5 disjoint with e. Then $K_5 - \mathring{e}$ is embeddable into $\mathbb{R}^2$ and, for each embedding $g\colon K_5 - \mathring{e} \to \mathbb{R}^2$, the g-images of the ends of e (the 0-sphere) lie on different sides from $g\Sigma$. Similarly, one can prove that the graph K_{33} (three houses and three wells) is not embeddable into $\mathbb{R}^2$ and that the 2-skeleton K of the 6-simplex is not embeddable into $\mathbb{R}^4$. Moreover, let e be a 2-simplex of K and $P = K - \mathring{e}$. Then P embeds in $\mathbb{R}^4$ and P contains two disjoint spheres Σ^2 and $\Sigma^1 = \partial e$ such that, for each embedding $P \to \mathbb{R}^4$, the images of these spheres link with a nonzero (actually, 1) linking number [7].

Now we are in a position to define the Van Kampen obstruction $v(K)$. Throughout this chapter we shall omit $\mathbb{Z}_2$-coefficients from the notation of the (co)chain and (co)homology groups. Fix a triangulation T of K. For any general position PL map $f\colon K \to \mathbb{R}^2$ and disjoint edges σ and τ of T, let $v_f(\sigma, \tau) = |f(\sigma) \cap f(\tau)| \mod 2$.

Let $\widetilde{K} = \bigcup\{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \varnothing\}$ be the *simplicial deleted product* of K. We denote it by $\widetilde{K}$, not by $\widetilde{T}$, because its equivariant homotopy type depends only on K, not on T [15]. The group $\mathbb{Z}_2$ acts on $\widetilde{K}$ by exchanging factors. Let $K^* = \widetilde{K}/\mathbb{Z}_2$. Then $v_f \in C^2(K^*)$. This v_f is invariant under isotopy of $\mathbb{R}^2$ and "Reidemeister moves" from Fig. 2a. The "Reidemeister move" from Fig. 2b adds to v_f the cochain, which is 1 on the class of $\alpha \times \beta$ for $v \in \alpha$, and 0 elsewhere. This cochain is a coboundary of the elementary cochain from $B^2(K^*)$ that assume value 1 on the class of $v \times \beta$, and 0 elsewhere. Then the equivalence class $v_2(K) \in H^2(K^*) = C^2(K^*)/B^2(K^*)$

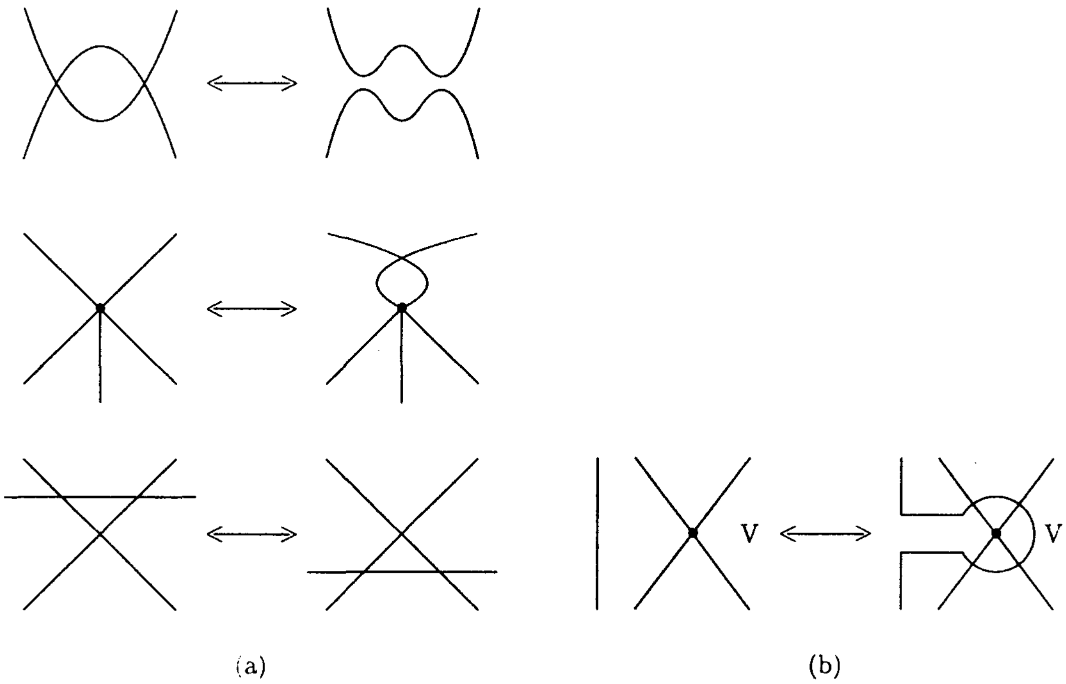

Fig. 2

of v_f does not depend on f. For a proof of this fact without the use of the "Reidemeister moves" see [6].

This $v_2(K)$ is the mod 2 Van Kampen obstruction for embeddability of K in $\mathbb{R}^2$. It is clear that, for all planar graphs K, one has $v_2(K) = 0$. Analogously one defines the mod 2 Van Kampen obstruction $v_2(K) \in H^{2n}(K^*)$ to embeddability of an n-polyhedron K in $\mathbb{R}^{2n}$.

The genuine Van Kampen obstruction $v(K)$ (with integer coefficients) is constructed as follows. Fix a triangulation of K and define $\widetilde{K}$ and K^* as above. Choose an orientation of $\mathbb{R}^{2n}$ and on n-simplexes of K. In fact, $v(K)$ depends on this choice, but only up to an automorphism of the group, in which $v(K)$ is defined. For any general position map $f\colon K \to \mathbb{R}^{2n}$ and any two disjoint oriented n-simplexes σ and τ of K, count an intersection as $+1$, when the orientation of $f\sigma$ followed by that of $f\tau$ agrees with that of $\mathbb{R}^{2n}$, and as -1 otherwise. Let $v_f \in C^{2n}(\widetilde{K}, \mathbb{Z})$ be the cocycle which counts the intersection of $f\sigma$ and $f\tau$ algebraically in this fashion. Clearly, $v_f(\sigma \times \tau) = (-1)^n v_f(\tau \times \sigma)$. So v_f is in the subgroup $C^{2n}_s(\widetilde{K}, \mathbb{Z})$ of $C^{2n}(\widetilde{K}, \mathbb{Z})$ formed by the cochains whose components corresponding to *symmetric* 2-cells $\sigma \times \tau$ and $\tau \times \sigma$ are equal (for odd n).or opposite (for even n). Then $v(K) \in H^{2n}_s(\widetilde{K}, \mathbb{Z}) = C^{2n}_s(\widetilde{K}, \mathbb{Z})/B^{2n}_s(\widetilde{K}, \mathbb{Z})$ is the class of v_f.

The above constructions can be generalized in several ways. Given an embedding $A \subset \partial B^m$ of a subpolyhedron A of K, we can define analogously the obstruction to extending the embedding of A to an embedding $K \to B^m$. Analogously, one can construct a difference element $u(f) \in H^{2n}_s(\widetilde{K}, \mathbb{Z})$ of an embedding $f\colon K \to \mathbb{R}^{2n+1}$. As it was pointed out by Shapiro, when $v(K) = 0$ (and hence K is embeddable in $\mathbb{R}^{2n}$ for $n \geq 3$), one can construct the 'second obstruction' to embeddability of K in $\mathbb{R}^{2n-1}$, etc. For a controlled analog of the Van Kampen obstruction see [3, §4; 28; 2, §4].

From the definition we can see that v_f "measures" the deviation of f from an embedding. So it is natural to conjecture that the vanishing of $v(K)$ implies the possibility of removing the singularities, and that the condition $v(K) = 0$ is not only necessary, but also sufficient for K to be embeddable in $\mathbb{R}^{2n}$.

Theorem 2.1 [17, 34, 44, 29, 6]. *For a finite n-polyhedron K to be embeddable in $\mathbb{R}^{2n}$ it is necessary that $v(K) = 0$. For $n \neq 2$ it is also sufficient, whereas for $n = 2$ it is not.*

The relative case of Theorem 2.1 is true for $n \geq 3$ and is false for $n = 2$. It would be interesting to know if it is true for $n = 1$.

Using the idea of obstruction, in [17, 43] it was proved that any PL or smooth n-manifold is PL or smoothly embeddable in $\mathbb{R}^{2n}$. For $n \geq 3$, an interesting corollary of Theorem 2.1 (and for $n = 2$ a separate result) is that every acyclic n-polyhedron is PL (if $n = 2$, Top) embeddable in $\mathbb{R}^{2n}$ [44, 19].

The Whitney trick, on which the proof of sufficiency in Theorem 2.1 for $n \geq 2$ is based, cannot be performed for $n = 2$ [18, 21]. The sufficiency in Theorem 2.1 for the case $n = 1$ is a corollary of the Kuratowski description of planar graphs.

However, Sarkaria has found a proof of this case based on the 1-dimensional Whitney trick [29]. He also asked if the sufficiency in Theorem 2.1 for the case $n = 2$ holds. Freedman, Krushkal and Teichner constructed an example showing that it does not.

3. CONSTRUCTION OF THE FREEDMAN–KRUSHKAL–TEICHNER EXAMPLE

To illustrate one of the main ideas let us first construct Borromean rings (i.e., three circles embedded into $\mathbb{R}^3$ such that every pair of them is unlinked but the three of them together are linked) using the non-commutativity of the fundamental group. In this section tilde does not denote the deleted product.

Take two circles Σ and $\tilde{\Sigma}$ in $\mathbb{R}^3$ far away from one another. Embed in $\mathbb{R}^3 - (\Sigma \sqcup \tilde{\Sigma})$ the Figure Eight (i.e., the wedge of two circles) C such that the inclusion $C \subset \mathbb{R}^3 - (\Sigma \sqcup \tilde{\Sigma})$ induces an isomorphism of fundamental groups. Take generators a and b of $\pi_1(C) = \pi_1(\mathbb{R}^3 - (\Sigma \sqcup \tilde{\Sigma}))$ represented by the two (arbitrarily oriented) circles of the Figure Eight.

Consider a map $S^1 \to C \subset \mathbb{R}^3$ representing the element $aba^{-1}b^{-1}$. By general position, there is an embedding $f\colon S^1 \to \mathbb{R}^3$ very close to this map. Then Σ, $\tilde{\Sigma}$, and $f(S^1)$ are Borromean rings. In fact, Σ and $\tilde{\Sigma}$ are unlinked by their definition. It is easy to take f so that Σ and $f(S^1)$, $\tilde{\Sigma}$ and $f(S^1)$ are unlinked (the reason for this is that f induce the zero homomorphism of the 1-dimensional homology groups). But f induces a nonzero homomorphism of the fundamental groups. Therefore Σ, $\tilde{\Sigma}$ and $f(S^1)$ are linked together.

From the existence of Borromean rings one can deduce the following folklore counterexample to the relative version of Theorem 2.1 for $n = 2$. Let $K = D^2 \sqcup D^2 \sqcup D^2$, $A = \partial D^2 \sqcup \partial D^2 \sqcup \partial D^2$, and $A \subset S^3 \cong \partial D^4$ be the Borromean rings. Since all the three Borromean rings are linked, it follows that the embedding $A \to \partial D^4$ cannot be extended to an embedding $K \to D^4$. But the unlinkedness of each pair of Borromean rings implies the vanishing of the relative Van Kampen obstruction to this extension. This is clear, since the Van Kampen obstruction counts double intersections but does not count triple intersections.

Now we are in a position to construct the Freedman–Krushkal–Teichner counterexample to Theorem 2.1 for $n = 2$ [6]. Let P be the 2-skeleton of the 6-simplex minus the interior of a 2-simplex from this 2-skeleton. Recall that P contains two disjoint spheres Σ^2 and Σ^1 such that for each embedding $P \to \mathbb{R}^4$ these spheres link with a nonzero (actually, 1) linking number. Let $\tilde{P}$ be a copy of P. For a subset $A \subset P$ we denote by $\tilde{A} \subset \tilde{P}$ its copy.

Embed P and $\tilde{P}$ in $\mathbb{R}^4$ standardly (i.e., so that both Σ^2 and $\tilde{\Sigma}^2$ are unknotted, Σ^2 and $\tilde{\Sigma}^2$ are unlinked, and Σ^2 and Σ^1, $\tilde{\Sigma}^2$ and $\tilde{\Sigma}^1$ are standardly linked spheres) and far away from one another. Take any point $x \in \Sigma^1$ and push a finger from x to $\tilde{x}$ to obtain an embedding $P \vee \tilde{P} \subset \mathbb{R}^4$.

Let $C = \Sigma^1 \vee \tilde{\Sigma}^1$ be a Figure Eight (with the base point $x = \tilde{x}$). Then the inclusion $C \subset \mathbb{R}^4 - (\Sigma^2 \sqcup \tilde{\Sigma}^2)$ induces an isomorphism of the fundamental groups. Take generators a and b of $\pi_1(C)$ represented by the two (arbitrarily oriented) circles of the Figure Eight. Take a map $h\colon S^1 \to C$ representing the element $aba^{-1}b^{-1}$. Let K be the mapping cone of the composition of h with the inclusion $C \subset P \vee \tilde{P}$ (i.e., $K = D^2 \bigcup_{h\colon \partial D^2 \to C}(P \vee \tilde{P})$).

Then K is nonembeddable into $\mathbb{R}^4$ although $v(K) = 0$. For a detailed proof see [6]. The reason for $v(K) = 0$ is that the Van Kampen obstruction preserves the homology property that $aba^{-1}b^{-1}$ is null-homologous and loses the homotopy property that $aba^{-1}b^{-1}$ is not null-homotopic.

Let us sketch the proof of nonembeddability of K into $\mathbb{R}^4$. Suppose to the contrary, that there exists an embedding $g\colon K \to \mathbb{R}^4$. If both $g\Sigma^2$ and $g\tilde{\Sigma}^2$ are unknotted in $\mathbb{R}^4$, then it follows from the property of P that the map $C \to gC \subset \mathbb{R}^4 - g(\Sigma^2 \sqcup \tilde{\Sigma}^2)$ induces a monomorphism of fundamental groups. In general case this is proved using the Stallings theorem on central series of groups. But the element $aba^{-1}b^{-1}$, which is nonzero in $\pi_1 C$, goes to a loop in $\mathbb{R}^4 - g(\Sigma^2 \sqcup \tilde{\Sigma}^2)$, which is extendible to gD^2 and hence null-homotopic. Contradiction.

The example of Borromean rings suggests that one can introduce an obstruction to (relative) embeddability, analogous to Van Kampen's, but deduced from triple (quadruple,...) intersections. And that this obstruction is sufficient to embeddability, even when the Van Kampen obstruction fails to be such. Although such obstructions can really be defined [23], they surprisingly give no more information on the embeddability of a polyhedron into $\mathbb{R}^m$ (cf. Section 4).

4. THE DELETED PRODUCT OBSTRUCTION

Let $\widetilde{K} = \{(x, y) \in K \times K \mid x \neq y\}$ be *the deleted product* of K. An embedding $f\colon K \to \mathbb{R}^m$ induces a map $\tilde{f}\colon \tilde{X} \to S^{m-1}$, defined by $\tilde{f}(x, y) = \frac{f(x)-f(y)}{|f(x)-f(y)|}$. This map is equivariant with respect to the involution $t(x, y) = (y, x)$ on $\widetilde{K}$ and the antipodal involution a on S^{m-1}. The nonexistence of an equivariant map $\widetilde{K} \to S^{m-1}$ is the *deleted product obstruction* to embeddability of K into $\mathbb{R}^m$. If K is a polyhedron with a triangulation T, then the simplicial deleted product of K (cf. Section 2) is an equivariant retract of $\widetilde{K}$, so we shall not distinguish between them.

The existence of an equivariant map $\widetilde{K} \to S^{m-1}$ is equivalent to the existence of a cross-section of the bundle $g\colon \widetilde{K} \times S^{m-1}/(t \times a) \xrightarrow{S^{m-1}} \widetilde{K}/t$. Here, the map g is defined by $g[(x, y), \alpha] = [(x, y)]$. So, if K is either a polyhedron or a smooth manifold, then obstruction theory can be applied. In particular, the Van Kampen obstruction is just the first obstruction to the existence of such a cross-section.

Theorem 4.1 [42, 40]. *If an n-polyhedron K is TOP embeddable in $\mathbb{R}^m$, then there exists an equivariant map $\Phi\colon \widetilde{K} \to S^{m-1}$. For $m \geq \frac{3(n+1)}{2}$ this condition is also sufficient for PL embeddability. For each pair (m, n) such that $4 \leq m \leq \frac{3n}{2} + 1$, it is not sufficient* (*even for TOP embeddability*).

Theorem 4.1 has many corollaries [42]. For other proofs, see [13, 36] and, for a smooth case, see [12].

The dimension restriction $m \geq \frac{3(n+1)}{2}$ for the sufficiency in Theorem 4.1 is due to the Freudenthal Suspension theorem, the Penrose–Whitehead–Zeeman–Irwin Embedding theorem, the Zeeman Unknotting theorem and the general position arguments.

Using the approach of [13, §3, Proof of Proposition 1], it can perhaps be shown that the restriction due to the Penroze–Whitehead–Zeeman–Irwin Embedding theorem is not essential. Torunczyk and Spiez showed that the same is true for the Zeeman Unknotting theorem [38, 39].

Using Whitehead's generalization (the so-called 'hard part') of the Freudenthal Suspension theorem and the Whitehead higher-dimensional finger moves [8, §10], they also showed that the restriction due to the Freudenthal Suspension theorem is not sharp [39], see also [4, 5, 32] (note that the application of the higher-dimensional finger moves in this situation was first suggested by Ščepin).

This was the reason why, in 1992, Dranishnikov and Schepin suggested to the second author to prove the sufficiency in Theorem 4.1 for $m = \frac{3n}{2} + 1$. However, Segal and Spiez constructed a counterexample, using the same higher-dimensional finger moves. They showed that for each pair (m, n) such that $4 \leq m \leq \frac{3n}{2} + 1$ (except for a finite number), the sufficiency is not true [33]. Their example used a homotopy corollary of the Adams theorem on vector fields, and their exceptions were caused by the Adams exceptions 1, 3, and 7.

Some of the exceptions were treated by the second author (cf. [31]) using finger moves, the idea of [6], and the results of [33]. Subsequently, this construction was generalized independently by Segal–Spiez and the second author to obtain a simplification of [33], which did not use the Adams theorem, and therefore had no exceptions [40]. This example shows that the restriction $m \geq \frac{3(n+1)}{2}$ is actually necessary, in general, for validity of the *second* part of the proof of sufficiency in Theorem 4.1 (cf. [36]).

The examples of [33, 40] have a stronger property, which implies that, for $4 \leq m \leq \frac{3n}{2} + 1$, even the deleted product cube (or the p-fold deleted product) obstruction is insufficient for embeddability of n-polyhedra in $\mathbb{R}^m$. This property is that K be *quasi-embeddable* in $\mathbb{R}^m$, i.e., for each $\epsilon > 0$ there should exist a map $f\colon K \to \mathbb{R}^m$ whose preimages are of a diameter of less than ϵ.

This is a topological property (i.e., it does not depend on the metric of K), and for polyhedra K it is equivalent to the following: For each triangulation of K there is a map $f : K \to \mathbb{R}^m$ such that $f\sigma \cap f\tau = \varnothing$ for each disjoint simplexes σ and τ of this triangulation.

Clearly, quasi-embeddability of a polyhedron K in $\mathbb{R}^m$ implies $v(K) = 0$ for $m = 2n$, and implies the existence of an equivariant map from the p-fold simplicial deleted product of K (which is an equivariant retract of the genuine p-fold deleted product) to that of $\mathbb{R}^m$. Thus the Van Kampen and the deleted product obstructions are actually obstructions to quasi-embeddability, not embeddability.

Problem 4.2. *Suppose that K is any n-polyhedron and there exists an equivariant map $\widetilde{K} \to S^{m-1}$, or else from the p-fold (simplicial) deleted product of K to that of $\mathbb{R}^m$. Is K then quasi-embeddable into $\mathbb{R}^m$ (at least for $m = \frac{3n}{2} + 1$)?*

It is interesting that, for *PL manifolds*, the dimensional restriction in Theorem 4.1 can be weakened:

Theorem 4.3 [35, 37]. *If N is any closed $(3n-2m+2)$-connected PL n-manifold, $m - n \geq 3$ and there exists an equivariant map $\widetilde{N} \to S^{m-1}$, then there is a PL embedding of N into $\mathbb{R}^m$.*

APPENDIX
BORROMEAN RINGS AND BOY IMMERSION

The following was proved in [1] (see also [2, 9]). Let $h\colon \mathbb{R}P^2 \to \mathbb{R}^3$ be a Boy immersion. Fix any orientation on S^2 and on the double point (immersed) circle Δ of h. Take a small ball

$D^3 \subset \mathbb{R}^3$ containing the triple point of immersion $h_1 = h \circ r$, where $r: S^2 \to \mathbb{R}P^2$ is the standard double covering. Let $\bar{h}: (S^2 - h_1^{-1}\mathring{D}^3) \to \mathbb{R}^3 \times \mathbb{R}$ be a generic smooth map such that $(\pi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3) \circ \bar{h} = h_1$ and, for each two points $x, y \in S^2 - h_1^{-1}D^3$, if $hx = hy$ and $\bar{h}x > \bar{h}y$, then the following three vectors form a *positive* basis of $\mathbb{R}^3$ at the point $hx = hy$: the orientation vector of Δ, the normal vector to a small sheet of $h_1(S^2)$ containing x, and the normal vector to a small sheet of $h_1(S^2)$ containing y. Then $\bar{h}|_{h_1^{-1}\partial D^3} \to \partial D^3 \times \mathbb{R}$ form Borromean rings (after the identification ∂D^3 [illegible] $\mathbb{R}^3$ [illegible]

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