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ON HOMOGENEOUS COMPACTA IN EUCLIDEAN SPACE AND THE CLASSICAL HILBERT-SMITH CONJECTURE

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We outline the geometric approach to the Hilbert-Smith Conjecture which asserts that among locally compact groups only Lie groups can act effectively on manifolds. In particular, we discuss recent results of J. Malešič, E. V. Ščepin and the authors.

1. Introduction

The classical Hilbert 5^{th} problem — to show that every locally Euclidean group is a Lie group [18] — was solved in 1952 independently by Gleason [15] and Montgomery and Zippin [26]. A generalized Hilbert's 5^{th} problem, called the *Hilbert-Smith Conjecture*, asserts that among locally compact groups only Lie groups can act effectively on manifolds. It is known to be equivalent to a special case when the acting group G is the group of p-adic integers [6,27,34,36]. In 1946 Bochner and Montgomery proved this conjecture for groups G acting on a manifold M by diffeomorphisms [5]. The general case was attacked by methods of cohomological dimension theory, by constructions of wild Cantor sets in $\mathbb{R}^n$ with strong homogeneity properties and by investigating actions of p-adic integers on arbitrary compacta. If the group A_p acts effectively on an n-manifold M, then $\dim_{\mathbb{Z}_p} M/A_p = n+1$, $\dim_{\mathbb{Z}} M/A_p = n+2$ and $\dim_F M/A_p = n$ if F is a field and char $F \neq p$ [6,36] (for further discussion see [9]). Antoine's necklace [3] is an ambiently homogeneous (wild) Cantor set in $\mathbb{R}^3$. (For further examples in this direction see [7,32,33,35]). However, these examples cannot be extended to effective actions of A^p on $\mathbb{R}^n$. Dranišnikov constructed a free action of A_p on the universal Menger compactum μ^n [10]. See also [1,4,11].

2. Smooth Ambient Homogeneity

A simpler geometrical proof of the Hilbert-Smith conjecture for actions by diffeomorphisms was obtained in 1991 by E. V. Ščepin and the authors. Let us sketch the idea of this proof. Suppose that a locally compact group G acts effectively on an n-dimensional manifold M. Every orbit K of this action is *smoothly ambiently homogeneous*, i.e. for each x, $y \in K$ there exists a diffeomorphism $h : (M, K, x) \to (M, K, y)$. Let us prove that it follows from this property that K is a submanifold of M (then $G \cong K$ is a Lie group). It suffices to consider the case $M = \mathbb{R}^n$. If $\dim K = n$, the proof is obvious, so suppose that $\dim K < n$. Then by local compactness, some point of K is pinched by a round ball from the complement of K. By smooth ambient homogeneity, every point of K is pinched by a round cone from the complement of K. Using the Baire Category theorem, we conclude that the points from some open subset U of K are pinched by parallel isometric cones. Moreover, we may assume that $\operatorname{diam} U$ is less than the height of these cones, so every point of U is pinched by two cones, symmetric respective to this point.

If $\dim K = n-1$, then one can show that there is a Lipschitz chart $\mathbb{R}^{n-1} \to U \cap K$. This chart has a point of differentiability [13]. By ambient smooth homogeneity, K is thus a smooth $(n-1)$-submanifold of $\mathbb{R}^n$. Since the angle of the cone is arbitrarily close to π, we can avoid use of [13] by applying the Baire category theorem once again to get a differentiable chart $\mathbb{R}^{n-1} \to U \cap K$ at once [29]. If however, $\dim K < n-1$, we can pinch some point of $U \cap K$ by a round n-ball, not containing either of two cones, already pinching this point. Then we apply the induction (see [31]) to finish off the argument.

Our proof reveals a close relationship between the concept of homogeneity and the idea of taming subsets of manifolds, pinched by tangent balls. The latter problem was extensively investigated in the past, by various authors [8,14,16], [20]-[24].

3. Lipschitz Ambiently Homogeneous Fractals

The first idea to attack the Hilbert-Smith Conjecture for actions by Lipschitz homeomorphisms was to prove that a locally compact, Lipschitz ambient homogeneous subset K of $\mathbb{R}^n$ must be a submanifold of $\mathbb{R}^n$. Note that this is not true for arbitrary homeomorphisms by Antoine's example. In the Lipschitz case we can still pinch some point of K by a round n-ball from $\mathbb{R}^n \setminus K$. But then from Lipschitz ambient homogeneity it follows only that each point of K is pinched by a Lipschitz n-ball from $\mathbb{R}^n \setminus K$. This Lipschitz n-ball contains no round cones, pinching K from $\mathbb{R}^n \setminus K$. Using this geometric idea, Malešič [25] proved that the standard Cantor set in $\mathbb{R}^2$ is Lipschitz ambient homogeneous.

Thus the above conjecture on locally compact, Lipschitz ambient homogeneous subsets of $\mathbb{R}^n$ is false.

. Let us sketch his argument. Take a round disk A_ϕ, containing the standard Cantor set in $\mathbb{R}^2$. Take also round disks A_0 and A_1, containing two 'halves' $C \cap A_0$ and $C \cap A_1$ of C, so that the pairs $(A_0, C \cap A_1)$ and $(A_1, C \cap A_1)$ are isometric and both similar to (A_ϕ, C). Then there is an autodiffeomorphism h_ϕ of the plane, which is the identity outside $\operatorname{int} A_\phi$ and which exchanges $(A_0, C \cap A_0)$ and $(A_1, C \cap A_1)$ isometrically. The derivative of h_ϕ is continuous and therefore bounded, hence h_ϕ is Lipschitz. Similarly one can construct a Lipschitz homeomorphism h_0 (resp. h_1), which is the identity outside A_0 (resp. A_1) and which changes two quarters of C, lying in $C \cap A_0$ (resp. $C \cap A_1$) and so on. One can verify that compositions of the homeomorphisms (even infinite ones) are Lipschitz. Therefore, C is Lipschitz ambiently homogeneous.

Malešič also constructed Antoine's necklace, which is Lipschitz ambiently homogeneous in $\mathbb{R}^3$ [25]. For this purpose, all inscribed tori should be similar to the ambient one.

4. Actions by Lipschitz Homeomorphisms

Malešič [25] even constructed an action of the Cantor set $C = A_2$ on $\mathbb{R}^3$ whose restriction to $C \subset \mathbb{R}^3$ is just a multiplication. But this action is identical outside some ball. Is it possible to construct an effective action of C on $\mathbb{R}^3$? An intersection of self-similar objects like in Malešič's construction is called a fractal. This led Repovš and Ščepin [30] to apply the Hausdorff dimension to prove the Lipschitz case of the Hilbert-Smith conjecture.

We present an outline of their proof: Suppose that the group $G = A_p$ acts effectively on an n-manifold M and that ρ is a Riemann metric on M. Suppose also, that for every autohomeomorphism $g \in G$ of M, there exists $l_g > 1$ such that

$$\frac{1}{l_g} \leq \frac{\rho(g(x), g(y))}{\rho(x, y)} \leq l_g.$$

Then one can show that there is $L > 1$ such that $l_g \leq L$, for each $g \in G$. Using any probabilistic measure on G [17], define an equivariant metric ρ_G on M by

$$\rho_G(x,\ y) = \int_G \rho(\,g(x), g(y)\,)\, dg.$$

Let $p : M \to M/G$ be the projection. Define a metric on M/G by

$$\rho_G(\,p(x), p(y)\,) = \min_{g \in G}\{\ \rho_G(\,x, g(y)\,)\ \}.$$

We get an obvious contradiction:

$$n = \dim M \underset{(1)}{=} \dim_\rho M \underset{(2)}{=} \dim_{\rho G} M \geq$$
$$\underset{(3)}{\geq} \dim_{\rho G}(M/G) \underset{(4)}{\geq} \dim(M/G) \underset{(5)}{\geq} \dim_{\mathbb{Z}}(M/G) = n+2.$$

Here (1) follows by [12] and [28]. Since the action is Lipschitz, the metrics ρ and ρ_G are equivalent, so (2) follows. Since $p : M \to M/G$ does not increase the distance between points, (3) follows. (4) follows by [19, Theorem 7.3], (5) is a classical result [2], and for (6) see [6,36].

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References

[1] S. M. Ageev, *Classifying spaces for free actions and the Hilbert-Smith conjecture*, Mat. Sb., **183:1**(1992), 143–152 (in Russian).

[2] P. S. Aleksandrov, *Introduction to Homological Dimension Theory*, Nauka, Moscow, 1975 (in Russian).

[3] L. Antoine, *Sur l'homéomorphie de deux figures et de leur voisinages*, J. Math. Pures Appl., **86**(1921), 221–325.

[4] M. Bestvina and R. D. Edvards, *Around the Hilbert-Smith conjecture*, Proc. 6th Workshop in Geom. Topol., Provo, 1989.

[5] S. Bochner and D Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. (2), **47**(1946), 639–653.

[6] G. E. Bredon, F. Raymond and R. F. Williams, *p-Adic transformation groups*, Trans. Amer. Math. Soc., **99**(1961), 488–498.

[7] R. J. Daverman, *Embedding phenomena based upon decomposition theory: wild Cantor sets satisfying strong homogeneity properties*, Proc. Amer. Math. Soc., **75**(1979), 177–182.

[8] R. J. Daverman and L. D. Loveland, *Wildness and flatness of codimension one spheres having double tangent balls*, Rocky Mount. J., **11**(1981), 113–121.

[9] A. N. Dranišnikov, *Homological dimensional theory*, Uspehi Mat. Nauk **43:4** (1988), 11–55 (in Russian); English transl. in: Russian Math. Surveys **43:4**(1988), 11–63.

[10] A. N. Dranišnikov, *On free actions of zero-dimensional compact groups*, Izv. Akad. Nauk SSSR Ser. Mat. **52:1**(1988), 212–228 (in Russian); English translation in: Math. USSR Izvestiya, **32:1**(1989), 217–232.

[11] R. D. Edwards, *Some remarks on the Hilbert-Smith conjecture*, Proc. 4th West Workshop Geom. Topol., Corvalis, 1987.

[12] K. Falconer, *Fractal Geometry*, John Wiley, New York. 1990.

[13] H. Federer, *Geometric Measure Theory*, Grund. Math. Wisch., **153**, Springer-Verlag, Berlin, 1969.

[14] P. J. Giblin and D. B. O'Shea, *The bitangents sphere problem*, Amer. Math. Monthly, **97**(1990), 5–23.

[15] A. Gleason, *Groups without small subgroups*, Ann. of Math. (2), **56**(1932), 193–212.

[16] H. C. Griffith, *Spheres uniformly wedged between balls are tame in E^3*, Amer. Math. Monthly, **75**(1968), 767.

[17] P. R. Halmos, *Measure Theory*, Van Nostrand, Amsterdam, 1969.

[18] D. Hilbert, *Mathematical problems*, Bull. Amer Math. Soc., **8**(1901-1902), 437–439.

[19] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, 1941.

[20] L. D. Loveland, *A surface in E^3 is tame if it has round tangent balls*, Trans. Amer. Math. Soc., **152**(1970), 389-397.

[21] L. D. Loveland, *Double tangent ball embeddings of curves in E^3*, Pacif. J. Math., **104**(1983), 391–399.

[22] L. D. Loveland, *Tangent ball embeddings of sets in E^3*, Rocky Mount. J., **17**(1987), 141–150.

[23] L. D. Loveland, *Spheres with continuous tangent planes*, Rocky Mount. J., **17**(1987), 829–844.

[24] L. D. Loveland and D. G. Wright, *Codimension one spheres in $\mathbb{R}^n$ with double tangent balls*, Topol. Appl., **13**(1982), 311-320.

[25] J. Malešič, *Toroidal decompositions of the 3-dimensional sphere*, Ph. D. thesis, Univ. of Ljubljana, Ljubljana, 1995.

[26] D. Montgomery and L. Zippin, *Small groups of finite-dimensional groups*, Ann. of Math. (2), **56**(1952), 213–241.

[27] D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience Tracts in Pure and Appl. Math. **1**, Interscience Publ., New York, 1955.

[28] J. Nash, *The imbedding problem for Riemannian manifold*, Ann. of Math. (2), **63**(1956), 20–63.

[29] D. Repovš and E. V. Ščepin, *On smoothness of compacta, preprint*, University of Ljubljana, Ljubljana, 1995.

[30] D. Repovš and E. V. Ščepin, *A proof of the Hilbert-Smith conjecture for actions by Lipschitz homeomorphisms*, preprint, Univ. of Ljubljana, Ljubljana, 1995.

[31] D. Repovš and A. B. Skopenkov and E. V. Ščepin, *C^1-homogeneous compacta in $\mathbb{R}^n$ are C^1-submanifolds of $\mathbb{R}^n$*, Proc. Amer. Math. Soc., **124**(1996), 1219–1226.

[32] A. Shilepsky, *Homogeneity by isotopy for simple closed curves*, Duke Math. J., **40**(1973), 463–472.

[33] A. Shilepsky, *Homogeneity and extension properties of embeddings of S^1 in $\mathbb{R}^3$*, Trans. Amer. Math. Soc., **195**(1974), 265–276.

[34] P. A. Smith, *Periodic and nearly periodic transformations*, Lectures in Topology, Univ. of Michigan Press, Ann Arbor, 1941.

[35] D. G. Wright, *Bing-Whitehead Cantor sets*, Fund. Math., **132**(1989), 105–116.

[36] C. T. Yang, *p-Adic transformation groups*, Michigan Math. J., **7**(1960), 201–218.