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Embeddability and Isotopy of Polyhedra in Euclidean Spaces

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1. THE PROBLEMS OF EMBEDDABILITY AND ISOTOPY

Many theorems in mathematics state that an abstractly defined space is necessarily a subspace of some 'concrete' one. Such is e.g. Caley's theorem on finite groups, theorem on compact Lie groups (stating that they are virtual subgroups of GL(V) for some linear space V), Urysohn's theorem on normal spaces with countable basis, general position theorem for polyhedra, Menger–Nöbeling–Pontryagin's theorem on compact spaces of finite dimension, Whitney's theorem on smooth manifolds, Nash's theorem on Riemann manifolds etc. For such embedding problem we may even go a step further and study figures in a Euclidean space of fixed dimension.

Definition 1.1. A polyhedron X is said to be *PL-embeddable* in $\mathbb{R}^m$ if there is a PL homeomorphism $f: X \to \mathbb{R}^m$ (which is called an *embedding* of X into $\mathbb{R}^m$).

Problem 1.2. Find conditions for a polyhedron P to be PL-embeddable in $\mathbb{R}^m$, for a given m.

Similar problem can be stated in the Top- or Diff-category. In this survey we shall work in the PL-category, in particular, all maps will be assumed to be PL unless stated otherwise. We shall also mention analogous results for the Diff- and Top-category. For the survey on Diff-embeddings, immersions and isotopy see [1]. When $m \geq 2\dim P + 1$, the polyhedron P is embeddable in $\mathbb{R}^m$, by general position. In fact, $P \times I$ is embeddable in $\mathbb{R}^m$ [6]. We shall limit ourselves to describing partial results on the problem 1.2 which is one of the goals of the present survey.

We can relax the injectivity condition in the definition of embedding in two (dual) ways. First, a polyhedron X is said to be *immersible* in $\mathbb{R}^m$ if there are $\varepsilon > 0$ and a map $f: X \to \mathbb{R}^m$ such that $f(x) \neq f(y)$, whenever $\operatorname{dist}(x,y) < \varepsilon$. Second, a polyhedron X is said to be *quasi-embeddable* in $\mathbb{R}^m$ if for every $\varepsilon > 0$ there is a map $f: X \to \mathbb{R}^m$ such that $f(x) \neq f(y)$, whenever $\operatorname{dist}(x,y) > \varepsilon$. These two concepts are used mostly in the Diff- and Top-category, respectively. Many results of this survey have their parallels for immersability and quasi-embeddability but we shall not state them.

Another interesting problem (and perhaps central to geometric topology) is how to determine whether two given subpolyhedra of $\mathbb{R}^m$ are the same. Best known is the problem of classification of knots in $\mathbb{R}^3$. More precisely:

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Definition 1.3. Two embeddings $f, g: X \hookrightarrow \mathbb{R}^m$ of a polyhedron X are said to be *isotopic* if there exists an embedding $F: X \times I \to \mathbb{R}^m \times I$ (which is called an *isotopy*) such that $F(X \times \{t\}) \subset \subset \mathbb{R}^m \times \{t\}$ for every $t \in I$ and $F(x,0) = f(x)$, $F(x,1) = g(x)$.

Problem 1.4. Find conditions for embeddings f, g of a polyhedron X into $\mathbb{R}^m$ to be isotopic.

When $m \geq 2\dim P + 2$, every two embeddings of a polyhedron P in $\mathbb{R}^m$ are isotopic by general position. The second aim of this survey is to describe partial results concerning the problem of isotopy.

Evidently, isotopy is an equivalence relation on the set of embeddings of X in $\mathbb{R}^m$. It is the strongest among known equivalence relations between embeddings such as isoposition, concordance, bordance, etc. Two embeddings $f, g: X \to \mathbb{R}^m$ of a polyhedron X are said to be (orientation preserving) *isopositioned* if there is an (orientation preserving) homeomorphism $h: \mathbb{R}^m \to \mathbb{R}^m$ such that $h \circ f = g$. By Alexander–Guggenheim theorem [7], orientation preserving isoposition is equivalent to an isotopy. Two embeddings $f, g: X \to \mathbb{R}^m$ of a polyhedron X are said to be *concordant* if there is an embedding $F: X \times I \to \mathbb{R}^m \times I$ (which is called a concordance) such that $F(X \times \{t\}) \subset \subset \mathbb{R}^m \times \{t\}$ for $t = 0$ and $t = 1$. A surprising result of Lickorish and Hudson says that for $m - \dim X \geq 3$ concordance implies isotopy (in PL- and Diff-category) [5, 4]. Thus the problem of isotopy can be reduced to the relativized problem of embeddability. Note that this is not the case in codimension one or two.

Problems of embeddability and isotopy can be generalized from $\mathbb{R}^m$ to an arbitrary space Y. Cases when Y is a manifold or is a product of trees have been studied most widely [8, th. 4.6 and remark; 3, 9, 2]. In Secs. 2–6 of this survey we present *necessary* conditions for embeddability and isotopy and formulate *sufficency* theorems for these conditions. In Secs. 7–8 we briefly describe two important ideas of geometric topology, which can be applied to prove these sufficiency theorems. In Sec. 9 we give *controlled* and *mapping* versions of this theory, which are motivated by studies of embeddability of compacta.

2. PROHIBITED SUBPOLYHEDRA

If some subpolyhedron of a polyhedron X is not embeddable in $\mathbb{R}^m$ then X is not embeddable in $\mathbb{R}^m$. Also, if f and g are embeddings of a polyhedron X in $\mathbb{R}^m$ such that their restrictions onto some subpolyhedron of X are not isotopic, then f and g are not isotopic. A natural idea is to try to put together a list of 'prohibited' polyhedra (or a list of 'prohibited' pairs of non-isotopic embeddings) such that X is embeddable in $\mathbb{R}^m$ if X does not contain any of these 'prohibited' subpolyhedra (respectively, two embeddings are isotopic if they do not contain any 'prohibited' pair).

Theorem 2.1 [15]. *A graph (i.e., a 1-dimensional polyhedron) is embeddable in $\mathbb{R}^2$ if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$* (Fig. 1).

Theorem 2.2 [17, 14]. *A 2-dimensional polyhedron is embeddable in S^2 if an only if it does not contain K_5, $K_{3,3}$ or T^2* (Fig. 1).

Theorem 2.3 [12]. *A Peano continuum is embeddable in S^2 if and only if it does not contain K_5, $K_{3,3}$, C_1 or C_2* (Fig. 1).

Theorem 2.4 [18]. *Two embeddings $f, g: K \to \mathbb{R}^2$ of a graph (or even Peanian continuum) K are isotopic if and only if K does not contain T^1 or S^1 such that the pairs $\{f|_{T^1}, g|_{T^1}\}$ or*

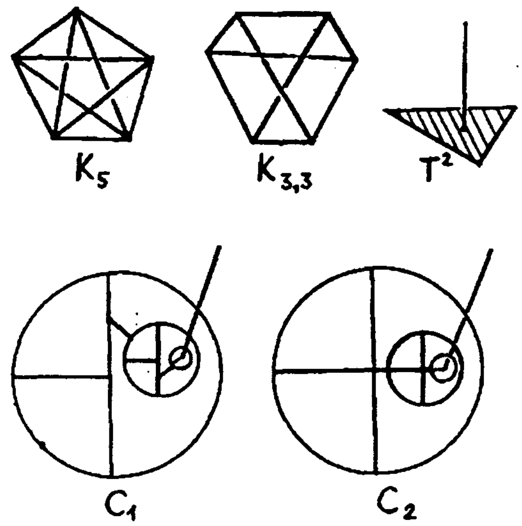

Fig. 1

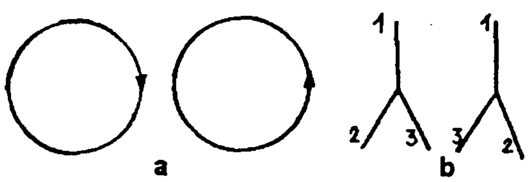

Fig. 2

$\{f|_{S^1}, g|_{S^1}\}$ *are as shown on* Fig. 2.

Simple proofs of Kuratowski criterion appear even nowadays [22, 16]. For the description of continua, basically embeddable in $\mathbb{R}^2$, in terms of prohibited subpolyhedra, see [21]. There are many other graph planarity criteria. Some of them are lower dimensional analogues of criteria of embeddability to be discussed below. A list of prohibited subgraphs for embeddability of K in $\mathbb{R}P^2$ contains 103 graphs [13]. Even the existence of such a finite list for arbitrary surface has a very long proof [10, 19]. Such a list is infinite for embeddability into $\mathbb{R}^{2m}$ when $m \geq 2$ [20]. Thus we have to consider another necessary conditions to embeddability.

3. COMPLEMENTS

Suppose that a polyhedron X is embedded in $\mathbb{R}^m$. The study of properties of $\mathbb{R}^m \setminus X$ in terms of those of X will then give necessary conditions for X to be embeddable in such an $\mathbb{R}^m$. This method can be traced back to early works of Alexander. Let us, for example, verify the nonplanarity of $K_{3,3}$. The Alexander duality for graphs in $\mathbb{R}^2$ is just the Euler formula $V - E + F = 2$. For $K_{3,3}$ we have $V = 6$, $E = 9$. If $K_{3,3} \subset \mathbb{R}^2$ then $4F \leq 2E$ (since every face has at least four edges in its boundary). Hence $V - E + F \leq 1.5$, which is a contradiction.

In general, it follows from the Alexander duality theorem that Betti numbers satisfy $b^m(X) = b^{-1}(\mathbb{R}^m \setminus X) = 0$. This gives a necessary condition for X to be embeddable in $\mathbb{R}^m$. Using this method Hantzsche obtained conditions in terms of Betti numbers and torsion numbers for a closed

$(m-1)$-manifold X to be embeddable in $\mathbb{R}^m$, by studying duality between homology groups of X and $\mathbb{R}^m \setminus X$. Similarly, by studying the duality between the cohomology rings of X and $\mathbb{R}^m \setminus X$ Hopf proved that $X = \mathbb{R}P^{m-1}$ is not embeddable in $\mathbb{R}^m$ [28]. In the same manner Thom obtained conditions in terms of cohomology ring of X for a closed $(m-1)$-manifold X to be embeddable in $\mathbb{R}^m$ [35]. Peterson studied the duality between cohomology operations in X and $S^m \setminus X$ and gave some interesting embeddability theorems.

The idea of complement is also applicable to the problem of isotopy. If $f, g: X \hookrightarrow \mathbb{R}^m$ are isotopic embeddings then $\mathbb{R}^m \setminus f(X)$ and $\mathbb{R}^m \setminus g(X)$ are homeomorphic. This idea was first applied by Alexander to knots $S^1 \subset S^3$ in 1910's. For example, using Van Kampen theorem on the fundamental group of the union, we obtain that $\pi_1(S^3 \setminus (\text{trefoil knot})) = \langle x, y \mid xyx = yxy \rangle = \pi$. Evidently, $\pi_1(S^3 \setminus (\text{trivial knot})) = \mathbb{Z}$. The first idea to distinguish between π and $\mathbb{Z}$ is to look on $\pi/[\pi,\pi]$. But it turns out that $\pi/[\pi,\pi] = \mathbb{Z}$. Perhaps Alexander, trying so to distinguish knots, observed that $\pi/[\pi,\pi] = \mathbb{Z}$ for fundamental group π of the complement to a knot, which lead him to discovery of his duality theorem. To distinguish between trefoil knot and trivial knot one can construct a non-trivial homomorphism $\pi \to S_3$, defined by $x \to (12)$, $y \to (23)$. Hence π is not abelian and not-isomorphic to $\mathbb{Z}$. The theory of knots $S^1 \subset S^3$ (or more generally, $S^n \subset S^{n+2}$) is too extensive to be included in this survey. This theory is in fact based on the following criterion:

Theorem 3.1 [32, 29, 92] (see also [27]). *A* (Diff, *or* PL-*locally flat, or* Top-*locally flat*) *embedding* $S^n \subset S^{n+2}$ *is* (Diff, *or* PL, *or* Top) *unknotted if and only if* $S^{n+2} \setminus S^n$ *is homotopically equivalent to* S^1. *Here* $n \neq 2$ (*for* PL- *and* Diff-*category even* $n \neq 3$). *For* $n = 1$, $(S^{n+2} \setminus S^n) \sim S^1$ *is equivalent to* $\pi_1(S^{n+2} \setminus S^n) = \mathbb{Z}$.

A (PL or Top)-embedding $S^n \subset S^m$ is said to be (PL or Top)-*locally flat*, if every point of S^n has a neighborhood U in S^m such that $(U, U \cap S^n) \cong (B^n \times B^{m-n}, B^n \times 0)$ (PL- or Top-homeomorphism, resp.). Local flatness assumption in Top category is necessary in order to rule out wild embeddings, which were first constructed by Antoine in 1920 and Alexander in 1923 [24], using the same complement idea. For references see [33]. Alexander constructed his example while studying the knots in codimension 1. The well-known Jordan theorem, first proved by Brouwer, states that every S^n, contained in S^{n+1}, splits S^{n+1} into two components. It is easy to prove the 'analogue' of criterion 3.1: $S^n \subset S^{n+1}$ is unknotted if and only if the closures of these components are balls. In 1912 Schönflies proved that every $S^1 \subset S^2$ is unknotted. Thus, unknottedness $S^n \subset S^{n+1}$ is called 'Schönflies theorem' or 'problem'. In 1921 Alexander announced that he has proved Schönflies theorem for arbitrary n. However, in 1923 he found a counterexample — the celebrated Horned sphere. The conjecture was then modified by adding a local flatness condition to the embedding $S^n \subset S^{n+1}$.

Theorem 3.2 [26, 30, 31, 34, 25, 7]. *Every* Top-*locally flat or* PL-*locally flat or* Diff-*embedding* $S^n \subset S^{n+1}$ *is unknotted in respective category* ($n \neq 3$ *for* Diff- *and* PL-*category*).

Note that Brown's elegant short proof was a beginning of theory of 'cellular sets', which is now an important branch of geometric topology. The PL-Shöenflies theorem is true for $n \in \{1, 2\}$ [23] and is an outstanding unsolved problem for $n \geq 3$ [7].

4. NEIGHBORHOODS

The local flatness assumption in Sec. 3 leads to another idea in studying embeddings. Instead of considering relations between X and $\mathbb{R}^m \setminus X$ we can consider relations between X and its neighborhoods in $\mathbb{R}^m$. This method seems to have been first introduced by Whitney in Diff-category. Whitney created a theory of sphere-bundles and introduced the so-called Stiefel–Whitney classes $w^k \in H^k(X, \mathbb{Z}_2)$ and the dual Stiefel–Whitney classes $\bar{w}^k \in H^{n-k}(M, \mathbb{Z}_2)$ of a differential manifold X which have played an important role in topology and differential geometry. The notion of Stiefel–Whitney classes was generalized by Pontryagin who introduced characteristic classes of differential manifolds among which, besides the Stiefel–Whitney classes, the most important ones are so-called Pontryagin classes $p^{4k} \in H^{4k}(X, \mathbb{Z})$ and their dual $\bar{p}^{4k} \in H^{n-4k}(X, \mathbb{Z})$. As far as the embedding problem is concerned, we have the following classical theorem:

Theorem 4.1 [42, 39]. *If an n-dimensional* Diff*-manifold M is embeddable in $\mathbb{R}^m$, then $\bar{w}^k(M) = 0$ for $k \geq m - n$ and $\bar{p}^{4k}(M) = 0$ for $2k > m - n$.*

Taking the same principle at a basis, Thom derived the following from the study of Steenrod squares in X and those in its neighborhood in $\mathbb{R}^m$:

Theorem 4.2 [40]. *For a locally contractible compactum X, to be embeddable in $\mathbb{R}^m$, it is necessary that* $\mathrm{Sm}^i H^r(X, \mathbb{Z}_2) = 0$ *for $r + 2i \geq m$, where* $\mathrm{Sm}^i \colon H^r(X, \mathbb{Z}_2) \to H^{r+i}(X, \mathbb{Z}_2)$ *are certain homomorphisms determined by the Steenrod squares by* $\sum_{i+j=k} \mathrm{Sm}^i \mathrm{Sq}^j = \delta_{k0}$.

The same principle has also been applied in Diff-geometry, e.g. by Massey who studied the cohomological rings of the tubes around a manifold M embedded in an Euclidean space and also by Atiyah who studied these tubes considered as elements in $K(M)$. Note that this idea of 'neighborhood' is also applicable in studying of immersions. Since normal bundles for different embeddings of manifold X in $\mathbb{R}^m$ are stably equivalent, this idea is hardly applicable to studying of isotopy. For a closely related concept of thickability and thickenings see [41, 36–38].

5. VAN KAMPEN'S OBSTRUCTION

Van Kampen took up the problem of the embeddability of an n-dimensional polyhedron K in $\mathbb{R}^{2n}$. Consider a general position map $f \colon K \to \mathbb{R}^{2n}$. Such an f has only finitely many double points, all contained in the interiors of n-simplices of K. From the study of such double points we can derive certain obstruction to embeddability of K in $\mathbb{R}^{2n}$, independent on f. Let us prove as an example that K_5 is not planar. Take a general position map $f \colon K_5 \to \mathbb{R}^2$, as above. Let $\vartheta(f)$ be the sum mod 2 of the numbers $|f(\sigma) \cap f(\tau)|$ of intersection points of f-images $f(\sigma)$ and $f(\tau)$, for all non-ordered pairs $\{\sigma, \tau\}$ of disjoint edges of K_5. For the map f, shown on Fig. 1, $\vartheta(f) = 1$. Every general position map $f \colon K_5 \to \mathbb{R}^2$ can be transformed to any other such map $g \colon K_5 \to \mathbb{R}^2$ through isotopies of $\mathbb{R}^2$ and operations, shown on Fig. 3. Since for each edge of K_5 with vertices a, b, the graph $K_5 \setminus \{a, b\}$, obtained by deleting from K_5 vertices a, b and interiors of incident to their edges, is a circle, it follows that $\vartheta(f)$ is invariant under these transformations. Therefore $\vartheta(f) = 1$ for *each* general position map $f \colon K_5 \to \mathbb{R}^2$ and hence K_5 is not planar.

This construction can be generalized to arbitrary graphs as follows. For any general position map $f \colon K \to \mathbb{R}^2$ and disjoint edges σ, τ of K let $\vartheta_f(\sigma, \tau) = |f(\sigma) \cap f(\tau)| \bmod 2$. Then $\vartheta_f \in C^2_s(\widetilde{K}, \mathbb{Z}_2)$, where $\widetilde{K} = \cup\{\sigma \times \tau \in K \times K \mid \sigma \cap \tau = \varnothing\}$. This ϑ_f is invariant under isotopy of $\mathbb{R}^2$

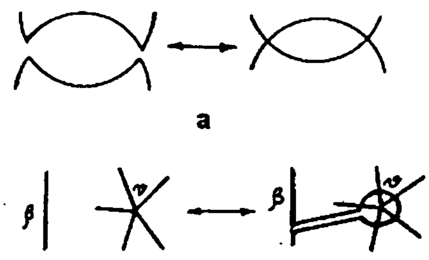

Fig. 3

and perturbation of f shown on Fig. 3a. Perturbation of f, shown on Fig. 3b, adds to ϑ_f the coboundary of the elementary symmetric cochain, which is 1 on $v \times \beta$ and $\beta \times v$, and 0 elsewhere. Therefore its equivalence class $\vartheta(K) \in H^2_s(\widetilde{K}, \mathbb{Z}_2)$ does not depend on f. This is Van Kampen obstruction for embeddability of K in $\mathbb{R}^2$: it is clear that for all planar graphs K one has $\vartheta(K) = 0$. A $\mathbb{Z}$-analogue of $\vartheta(K)$ is constructed as follows. Fix an orientation of $\mathbb{R}^2$ and for any general position map $f: K \to \mathbb{R}^2$ and any two disjoint oriented σ, τ of K, count an intersection as $+1$, when the orientation of $f(\sigma)$ followed by that of $f(\tau)$ agrees with that of $\mathbb{R}^2$, and as -1 otherwise. Then $\tilde{\vartheta}(K) \in H^2_s(\widetilde{K}, \mathbb{Z})$ is the class of the cocycle $\tilde{\vartheta}_f(\widetilde{K})$ which counts the intersection of $f(\sigma)$ and $f(\tau)$ algebraically in this fashion. Analogously one defines Van Kampen obstruction $\tilde{\vartheta}(K) \in H^{2n}_s(\widetilde{K}, \mathbb{Z})$ to embeddability of an n-dimensional polyhedron K in $\mathbb{R}^{2n}$. From its definition we see that the cochain $\tilde{\vartheta}_f(K)$ associated to a general position map $f: K \to \mathbb{R}^{2n}$ serves somewhat as a measure of the deviation of f from embedding. So it is natural to suggest that the vanishing of $\tilde{\vartheta}(K)$ implies the possibility of removing the singularities so that condition $\tilde{\vartheta}(K) = 0$ will be not only necessary, but also sufficient for K to be embeddable in $\mathbb{R}^{2n}$.

Theorem 5.1 [46, 49, 52, 48, 44]. *For a finite n-dimensional polyhedron K to be embeddable in $\mathbb{R}^{2n}$ it is necessary that $\tilde{\vartheta}(K) = 0$. For $n \neq 2$ it is also sufficient, for $n = 2$ it is not.*

Theorem 5.2 [46, 43]. *Any n-dimensional* PL- *or* Diff-*manifold is* PL- *or* Diff-*embeddable in $\mathbb{R}^{2n}$.*

Theorem 5.3 [52, 47], see also [45]. *Every acyclic n-polyhedron is* PL (*if $n = 2$* Top)-*embeddable in $\mathbb{R}^{2n}$.*

Whitney and Pontryagin considered the problem of embeddability in Diff-category, based on somewhat the same method of 'obstructions'. If $f: M \to \mathbb{R}^m$ is a general position differentiable map of a differentiable n-dimensional manifold M, then, in general, singularities will occur. These singularities carry certain cycles whose dual cohomology classes are independent of f and will hereafter be called characteristic classes of the manifold. Among them we have the Stiefel–Whitney classes and Pontryagin classes already mentioned in Sec. 4. As it was pointed out by Shapiro, one can construct 'second obstruction' to embeddability of K in $\mathbb{R}^{2n-1}$, when $\tilde{\vartheta}(K) = 0$ (and hence K is embeddable in $\mathbb{R}^{2n}$, provided $n > 2$), etc. Analogously, one can construct difference element $u(f) \in H^{2n}_s(\widetilde{K}, \mathbb{Z})$ of an embedding $f: K \hookrightarrow \mathbb{R}^{2n+1}$, and prove:

Theorem 5.4. (a) ([73]) *Two embeddings $f, g: K \to \mathbb{R}^{2n+1}$ of an n-dimensional polyhedron K* *are isotopic if and only if* $u(f) = u(g)$, *provided* $n \geq 2$.

(b) ([50]) *Two embeddings* $f, g: K \to \mathbb{R}^3$ *of a graph* K *are homologous if and only if* $u(f) = u(g)$.

6. THE DELETED PRODUCT CONDITION

The deleted product necessary condition for embeddability and isotopy is application of general mathematical idea of complements of diagonals. It was introduced by Lefschetz and Borsuk, and has played pervasive role in several branches of mathematics ever since [70]. Useful and interesting consequences of this method arise through an analysis of the complements of diagonals viewed through the eyes of cohomology groups. In particular, obstructions of Secs. 2–5 can be deduced from the deleted product necessary condition in a purely algebraic way [73].

To illustrate the main idea let us prove that S^n is not embeddable into $\mathbb{R}^n$. If, on the contrary, $f: S^n \to \mathbb{R}^n$ is an embedding, define a map $\tilde{f}: S^n \to S^{n-1}$ by $\tilde{f}(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$. Here by $-x$ we denote the antipode of the point $x \in S^n$. Since f is embedding, $\tilde{f}$ is well-defined. Evidently, $\tilde{f}$ is equivariant with respect to antipodal involutions on S^n and on S^{n-1}. This is a contradiciton (since $\tilde{f}|_{S^{n-1}}$ is then an inessential equivariant map), hence S^n is not embeddable in $\mathbb{R}^n$.

To generalize this proof, let us introduce $\tilde{X} = \{(x, y) \in X \times X | \, x \neq y\}$ — *the deleted product* of X. An embedding $f: X \to \mathbb{R}^m$ then induces a natural map $\tilde{f}: \tilde{X} \to S^{m-1}$, defined by $\tilde{f}(x, y) = \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$. This map is equivariant with respect to the involution $(x, y) \leftrightarrow (y, x)$ on X and the antipodal involution on S^{m-1}. The existence of an equivariant map $\tilde{X} \to S^{m-1}$ is called *the deleted product* necessary *condition* for embeddability of X in $\mathbb{R}^m$. If X is a polyhedron with a triangulation T, let $\tilde{T} = \cup\{\sigma \times \tau \times T \times T | \, \sigma \cap \tau = \varnothing\}$ be *the simplicial deleted product* of X. Since $\tilde{T}$ is equivariant retract of $\tilde{X}$ [63], the deleted product condition is equivalent to the existence of an equivariant map $\tilde{T} \to S^{m-1}$.

Taking into account that the injectivity of the map f has already been applied in defining the map $\tilde{f}$, we are naturally led to conjecture that the continuity of the map $\tilde{f}$ would imply something more than the mere continuity of the map f. Then the 'homotopic' methods applied to $\tilde{f}$ would give essential informations about the embeddability of X in $\mathbb{R}^m$. It turns out that this is indeed the case. As it was pointed out by Haefliger, the existence of an equivariant map $\tilde{X} \to S^{m-1}$ is equivalent to the existence of a cross-section of the bundle $\tilde{X} \times S^{m-1}/(t \times a) \xrightarrow[g]{S^{m-1}} \tilde{X}/t$. Here t is the involution $(x, y) \leftrightarrow (y, x)$ on X and a is the antipodal involution on S^{m-1}, the map g is defined by $[(x, y), \alpha] \to [(x, y)]$. So if X is a polyhedron or differentiable manifold, then methods of obstruction theory can be applied. In particular, Van Kampen obstruction is just the first obstruction to the existence of such a cross-section.

The deleted product condition to isotopy is constructed as follows. If $f, g: X \to \mathbb{R}^m$ are isotopic embeddings and $F: X \times I \to \mathbb{R}^m$ is an isotopy between them, then the map $\Phi: \tilde{X} \times I \to S^{m-1}$, defined by $\Phi(x, y, t) = \frac{F(x,t)-F(y,t)}{\|F(x,t)-F(y,t)\|}$ is an equivariant homotopy between $\tilde{f}$ and $\tilde{g}$. So, the deleted product condition for isotopness of f and g is homotopness of $\tilde{f}$ and $\tilde{g}$. This condition is equivalent to equivalence of Haefliger's cross-sections above.

Theorem 6.1 [58, 72]. (a) *An n-dimensional polyhedron (resp.* Diff-*manifold) X is* PL *(resp.* Diff)*-embeddable in* $\mathbb{R}^m$ *if and only if there exists an equivariant map* $\Phi: \widetilde{X} \to S^{m-1}$, *provided* $m \geq \frac{3(n+1)}{2}$. *Moreover, for every such* Φ *there exists an embedding* $f: X \hookrightarrow \mathbb{R}^m$ *such that* $\widetilde{f} \simeq \varphi$.

(b) *Two* PL *(resp.* Diff)*-embeddings* $f, g: X \hookrightarrow \mathbb{R}^m$ *of an n-dimensional polyhedron (resp.* Diff-*manifold) X are* PL *(resp.* Diff)*-isotopic if and only if* $\widetilde{f}$ *is equivariantly homotopic to* $\widetilde{g}$, *provided* $m \geq \frac{3n}{2} + 2$.

Theorem 6.2 [68]. *A closed* PL- *or* Diff-*n-manifold X is embeddable in* $\mathbb{R}^m$ *if and only if there exists an equivariant map* $\Phi: \widetilde{X} \to S^{m-1}$, *provided either*

(a) $m = \frac{3n}{2} + 1$; *or*

(b) X *is simply connected and* $m = \frac{3n+1}{2}$; *or*

(c) X *is 2-connected and* $m = \frac{3n}{2}$.

Theorem 6.3. (a) ([58, 72, 68]) *Suppose that* $m \geq n + 3$, $m \geq \frac{3n}{2}$ *and M is homologically (for* $m \leq \frac{3n+1}{2}$, *homotopically)* $(2n - m - 1)$*-connected* PL- *or* Diff-*n-manifold. Then M is* PL- *or* Diff-*embeddable in* $\mathbb{R}^m$ *if and only if* $\bar{w}_{m-n}(M) = 0$.

(b) (corollary of [64]) *Every n-dimensional homological sphere is embeddable in* $\mathbb{R}^{n+1}$ *(if* $n \neq 3$, *then* PL *or* Diff, *if* $n = 3$ *only* Top).

(c) ([66]) *Let* $n \geq 5$ *be an integer,* $n \not\equiv 1(8)$. *Let M be a closed differential* $(n-1)$*-connected* $2n$*-manifold. If n is even and M is* π*-manifold, or n is odd and* $\operatorname{Arf} M = 0$, *then M is* Top-*embeddable into* $\mathbb{R}^{2n+1}$.

Theorem 6.4 [58, 72]. (a) *If* $m > \frac{3(n+1)}{2}$ *and M is a closed orientable homologically* $(2n-m)$*-connected n-manifold* (PL *or* Diff), *then* PL *(or* Diff)*-isotopic classes of M in* $\mathbb{R}^m$ *are in 1-1 correspondence to:*

$$\begin{cases} H^{m-n-1}(M, \mathbb{Z}), & m - n \text{ even}, \\ H^{m-n-1}(M, \mathbb{Z}_2), & m - n \text{ odd}. \end{cases}$$

(b) *If* $n \neq 2$ *and M is a closed non-orientable connected n-manifold* (PL *or* Diff), *then* (PL *or* Diff)*-isotopic classes of M in* $\mathbb{R}^{2n}$ *are in 1-1 correspondence to:*

$$\begin{cases} H^{n-1}(M - \mathrm{pt}, \mathbb{Z})/2H^{n-1}(M, \mathbb{Z}), & n \text{ even}, \\ H^{n-1}(M - \mathrm{pt}, \mathbb{Z}_2), & n \text{ odd}. \end{cases}$$

Theorem 6.5 [72, 71, 60, 61]. *Every closed (if* $n = 2^k$, *orientable) n-manifold* (PL *or* Diff) *is* (PL *or* Diff)*-embeddable in* $\mathbb{R}^{2n-1}$.

A well known conjecture is that every closed n-manifold is embeddable into $\mathbb{R}^{2n-\alpha(n)+1}$, where $n = 2^{k_1} + \ldots + 2^{k_{\alpha(n)}}$ and $k_1 < \ldots < k_{\alpha(n)}$ (cf. [65, 53]).

Note that from Haefliger's example of Diff-knots $S^{4k-1} \subset S^{6k}$ it follows that the relative version of Theorem 6.1(a) does not hold for $(m, n) = (6k + 1, 4k)$. It follows from Theorem 6.1(a) that PL embeddability of an n-polyhedron X in $\mathbb{R}^m$ does not depend on PL-structures on X, when $m \geq \frac{3(n+1)}{2}$. Theorem 6.1(a) can be applied to calculate the minimal dimension m, such that a polyhedron, which is the product of graphs, is embeddable in $\mathbb{R}^m$ [55]. For another corollary see [6, Theorem 1.3] (there is a direct proof [62] under weaker assumptions on X and a stronger assumption that $X \times I$ embeds in $\mathbb{R}^{m+1}$).

For embeddings of compacta in $\mathbb{R}^2$ we have the following corollaries of theorems 2.3, 2.4:

Theorem 6.6. (a) ([67]) *A peanian continuum K is embeddable in $\mathbb{R}^2$ if and only if there exists an equivariant map $\widetilde{K} \to S^1$;*

(b) ([73]) *Two embeddings $f, g: K \to \mathbb{R}^2$ of a peanian compactum K are isotopic if and only if maps $\tilde{f}$ and $\tilde{g}$ are equivariantly homotopic.*

7. WHITNEY TRICK

Van Kampen published a proof of sufficiency of Theorem 5.1 in the case $n > 2$ but a fatal mistake destroyed his argument and the question remained open until much later. However, his technique suffices to prove Theorem 5.2 in the PL-case.

Whitney proved that the singularities of a general position Diff-map $f: K \to \mathbb{R}^{2n}$ from a Diff n-manifold M will consist of an even number of isolated points. The device to remove them pair by pair has led Whitney to a proof of Theorem 5.2 in Diff-category. Since then, Whitney trick has become an important toolin several branches of geometric topology. To illustrate the idea, let us complete Whitney's proof of Theorem 5.2. Take points x_1, y_1, x_2, y_2 in M such that $f(x_1) = f(x_2)$, $f(y_1) = f(y_2)$ and that these double points have 'opposite signs' (Fig. 4). Join x_1 to y_1 and x_2 to y_2 by arcs l_1 and l_2. By general position ($n \geq 2$), we may assume that $f|_{l_1}$ and $f|_{l_2}$ are embeddings and that l_1 and l_2 do not contain other double points of f. Since $2n \geq 4$, we can embed a 2-disk D in $\mathbb{R}^{2n}$ so that $\partial D = f(l_1) \cup f(l_2)$. Since $n \geq 3$, then, by general position, we may assume that $D \cap f(M) = \partial D$. Such a disk D is called Whitney's disk. We can move f-image of a regular neighborhood of l_1 in M along D so as to 'cancel' double points $f(x_1) = f(x_2)$ and $f(y_1) = f(y_2)$. Whitney trick may also be used to remove the singularities occuring in a general position map $f: K \to \mathbb{R}^{2n}$ of an n-polyhedron K for $n \neq 2$ thus filling the gap in the original proof of Van Kampen.

Note that for $n = 2$ one cannot made such a construction [75]. The proof of Theorem 6.1(a) in the PL-case is based on generalizations of the Whitney trick and Van Kampen's construction. For a shorter proof of the latter without application of the Freudenthal theorem see [67] (for controlled version see [109]). The dimension restriction $m \geq \frac{3(n+1)}{2}$ is due to the use of the Freudenthal suspension theorem and general position arguments. Using Whitehead's generalization ('hard part') of the Freudenthal theorem and higher-dimensional finger moves (cf. [74, 76]), it can be showed that the restriction due to the former is essential.

Theorem 7.1 [77, 44, 67]. *For every pair (m, n) of integers such that $m < \frac{3(n+1)}{2}$ and $(m, n) \notin \{(3,2), (3,3), (10,6), (11,7), (12,8), (22,14), (23,15), (24,16)\}$, there exist an n-dimensional polyhedron P, non-embeddable (even topologically!) in $\mathbb{R}^m$, and an equivariant map $\Phi: \widetilde{P} \to S^{m-1}$.*

8. ENGULFING

To illustrate the idea of engulfing and its application to embeddings, let us prove that for $n \geq 3$, an n-dimensional connected PL-manifold M is embeddable into $\mathbb{R}^{2n}$. Take a general position map $f: M \to \mathbb{R}^{2n}$. Then f has only double points. Denote them by $x_1, y_1, \dots, x_n, y_n$. We have that

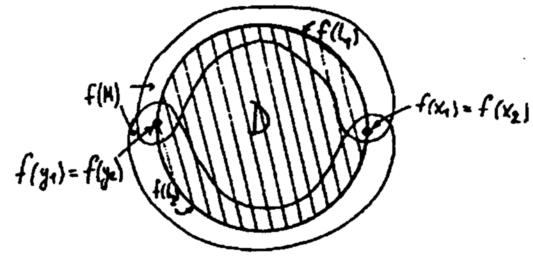

Fig. 4

$f(x_i) = f(y_i)$ and f is an embedding outside $\{x_1, y_1, \dots, x_n, y_n\}$. To 'kill' x_1 and y_1, take an arc $l \subset M$, joining x_1 to y_1 and non-intersecting others x_i, y_i. Then $f(l)$ is a circle in $\mathbb{R}^{2n}$. Take a 2-disk $D \subset \mathbb{R}^{2n}$ such that $\partial D = f(l)$. By general position, $D \cap f(M) = f(l)$. Take a regular neighborhood of l in M. It is an n-ball. The embedding $f: \partial(f^{-1}(B)) \to \partial B$ can be extended conically to an embedding $f_1: f^{-1}(B) \to B$, agreeing with f on $\partial(f^{-1}(B))$. Making such modifications for each $i = 1, \dots, n$, we 'kill' all x_i, y_i and obtain an embedding of M into $\mathbb{R}^{2n}$.

Theorem 8.1. (a) ([93, 92, 79]) *Any* PL (Top, *locally flat*)*-embedding* $S^n \subset S^m$ *is* PL (Top)*-unknotted if* $m - n \geq 3$.

(b) ([57, 58]) *Any* Diff*-embedding* $S^n \subset S^m$ *is* Diff*-unknotted if* $m > \frac{3(n+1)}{2}$. *There exist* Diff*-knots* $S^{4k-1} \subset S^{6k}$.

Theorem 8.2 [90, 88]. (a) *If* $m - n \geq 3$, *then every* $(2n - m)$*-connected closed* PL*-n-manifold is* PL*-embeddable in* $\mathbb{R}^m$.

(b) *If* $m - n \geq 3$, *then every two embeddings of a* $(2n - m + 1)$*-connected closed* PL*-n-manifold into* $\mathbb{R}^m$ *are isotopic.*

PL-case of Theorem 8.1(a) is a corollary of (8.2)(b). The transition from $m \geq \frac{3n+3}{2}$ to $m \geq \frac{3n+2}{2}$ in the proof of Theorem 8.2(a) in [90] was in fact the first step of induction. This induction was accomplished to $m \geq n + 3$ by Irwin. Note that if $m < \frac{3n+2}{2}$, then a $(2n - m)$-connected closed n-manifold is a homotopy sphere, hence a PL-sphere when $n \geq 5$. Next generalizations were made in [86, 87, 80]. In fact, Irwin, Hudson and Gordon proved more general theorems: they considered embeddings of bounded manifold into a bounded manifold. Therefore the existence of an embedding, homotopic to given map f, is the stronger property than mere embeddability. This property is used in surgery. The obstruction theory, arising from Hudson theorem, was developed in [89].

9. APPROXIMABILITY BY EMBEDDINGS

A possible method of studying embeddability of compacta is by decomposing them into inverse limits [111, 113, 106]. Roughly speaking, the embeddability of compacta is reduced to the embeddability of PL maps between polyhedra. A map $f: K \to M$ is said to be *embeddable* in $\mathbb{R}^m$ if there exists an embedding $\psi: M \to \mathbb{R}^m$ for which $\psi \circ f$ is aproximable by embeddings (this notion differs slightly from [111, 113]). Examples [113] show that this notion is rather geometric and is also interesting in itself. The following theorem is a *controlled* version of Theorem 6.1(a) and the *polyhedral* version of [103, 116]. Its corollary is a criterion for embeddability of maps in $\mathbb{R}^m$ (which is the *mapping* version of Theorem 6.1(a)).

Theorem 9.1 [109]. *Suppose that $f\colon K \to \mathbb{R}^m$ is a PL-map of an n-dimensional finite polyhedron K into $\mathbb{R}^m$. If f is aproximable by (PL or Top)-embeddings then there exists an equivariant homotopical extension $\Phi\colon \widetilde{K} \to S^{m-1}$ of the map $\widetilde{f}\colon \widetilde{K}^f \to S^{m-1}$, where $\widetilde{f}(x,y) = \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$ and $\widetilde{K}^f = \{(x,y) \in K \times K |\ f(x) \neq f(y)\}$. For $m \geq \frac{3(n+1)}{2}$ this condition is also sufficient.*

Corollary 9.2 [109]. *Suppose that $f\colon K \to M$ is a PL-map between finite polyhedra K and M such that $\dim K \leq n$ and $\dim M \leq n$. If f is embeddable in $\mathbb{R}^m$ then there exist equivariant maps $\Phi\colon \widetilde{K} \to S^{m-1}$ and $\Psi\colon \widetilde{M} \to S^{m-1}$ such that $\Psi \circ f$ is equivariantly homotopic to $\Phi|_{\widetilde{K}^f}$, where $\widetilde{f}(x,y) = (f(x), f(y))$. For $m \geq \frac{3(n+1)}{2}$ this condition is also sufficient.*

This necessary condition is equivalent to the one actually stated in [115]: $o \in \mathbb{R}^m$ must be an inessential point of the map $\bar{f}\colon K^2 \to \mathbb{R}^m$, defined by $\bar{f}(x,y) = f(x) - f(y)$. For another (simple) criterion for aproximability of maps by embeddings see [114].

For a triangulation T of K, let $\widetilde{T} = \cup\{\sigma \times \tau \in T \times T |\ \sigma \cap \tau = \varnothing\}$ and $\widetilde{T}^f = \cup\{\sigma \times \tau \in \in T^2 |\ f(\sigma) \cap f(\tau) = \varnothing\}$. Since $(\widetilde{T}, \widetilde{T}^f)$ is an equivariant retract of $(\widetilde{K}, \widetilde{K}^f)$ we can replace in Theorem 9.1 $\widetilde{K}$ by $\widetilde{T}$ and $\widetilde{K}^f$ by $\widetilde{T}^f$, for sufficiently small T [63]. This is convenient in applications.

The proof of necessity of Theorem 9.1 is easy. Take a triangulation T of K such that $f|_\sigma$ is linear for each $\sigma \in T$. Take $\varepsilon < \frac{1}{2}\min\{\operatorname{dist}(f(\sigma), f(\tau))|\ f(\sigma) \cap f(\tau) = \varnothing\}$ and any embedd $\varphi\colon K \to \mathbb{R}^m$, ε-close to f. Then for $(x,y) \in \widetilde{T}^f$, $\widetilde{\varphi}(x,y)$ and $\widetilde{f}(x,y)$ are not antipodal points of S^{m-1}. Hence $\widetilde{\varphi}|_{\widetilde{T}^f} \simeq \widetilde{f}$ and so $\widetilde{\varphi}$ is the required homotopical extension.

Let us construct for $m = 2n$ a cohomological obstruction $\vartheta(f) \in H^{2n}_S(\tilde{T}, \tilde{T}^f)$ to approximability of an arbitrary PL-map $f\colon K \to \mathbb{R}^{2n}$ by embeddings. Take a general position map $g\colon K \to \mathbb{R}^{2n}$, sufficiently close to f. Fix an orientation of $\mathbb{R}^{2n}$ for any two disjoint oriented edges σ and τ of T, count an intersection where the orientation of $g(\sigma)$ followed from that of $g(\tau)$ agrees with that of $\mathbb{R}^{2n}$ as $+1$, and -1 otherwise. Then $\vartheta(f)$ is the class of the cocycle $\vartheta_g(f)(\sigma,\tau)$ which counts in this fasion algebraically the intersection of $g(\sigma)$ and $g(\tau)$. If f maps all K to a point, then $\vartheta(f)$ is the van Kampen obstruction to embeddability of K in $\mathbb{R}^{2n}$ [46, 48].

Theorem 9.3 [109]. *If f is approximable by embeddings, then $\vartheta(f) = 0$. For $n > 2$ this condition is also sufficient, for $n \leq 2$ it is not.*

Theorem 9.4. (a) [95]. *If $n \geq 4, n \neq 7$, then every map $f\colon S^n \to S^n$ is embeddable in R^{2n}. For $n = 1, 3, 7$ there is a map $f\colon S^n \to S^n$, non-embeddable in R^{2n}.*

(b) [104]. *For $n \geq 2$ every map $f\colon T^n \to T^n$ between n-dimensional tori is embeddable in R^{2n}.*

Example 9.5 [109]. (a) Let $K = S^1$ and $f\colon S^1 \to S^1 \subset \mathbb{R}^2$ be a composition of a degree 3 map and an embedding. Then f is not approximable by embeddings. However, $\vartheta(f) = 0$ and there exists an equivariant map $\Phi\colon \widetilde{K} \to S^1$ such that $\Phi|_{\widetilde{K}^f}$ is equivariantly homotopic to $\widetilde{f}$.

(b) Maps $f\colon K \to I \subset R^2$ on Fig. 5 are not approximable by embeddings ($\vartheta(f) \neq 0$). However, there exists an equivariant map $\Phi\colon \widetilde{K} \to S^1$ such that $\Phi|_{\widetilde{K}^f}$ is equivariantly homotopic to $\widetilde{f}$.

Example 9.6 [67]. The 3-adic solenoid Σ is not embeddable in $\mathbb{R}^2$, however there exists an equivariant map $\widetilde{\Sigma} \to S^1$ (Fig. 5).

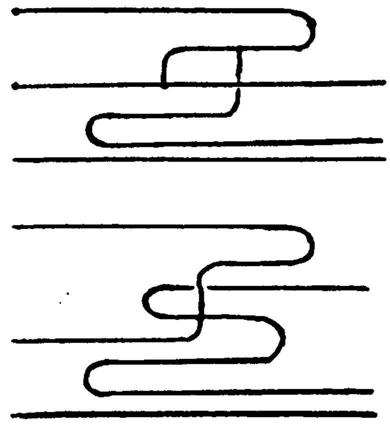

Fig. 5

It is well-known that the plane $\mathbb{R}^2$ does not contain uncountably many pairwise disjoint triods [107]. This result has been generalized in various directions [96, 98, 99, 108, 112, 117]. From Theorem 9.1 follows a strengthening of [117] and [99], which is partial answer to [100, p. 430]).

Let $\widetilde{X}_\varepsilon = \{(x, y) \in X^2 | \operatorname{dist}(x, y) \geq \varepsilon\}$ be the ε-deleted product of X. Consider the involution $t(x, y) = (y, x)$ on $\widetilde{X}_\varepsilon$. Following [105] and [102], define the *span* and the *symmetric span* of X as follows:

$$\sigma X = \sup\{\varepsilon \geq 0 | \text{ there is a subcontinuum } Z \subset \widetilde{X}_\varepsilon \text{ such that } \operatorname{pr}_1(Z) = \operatorname{pr}_2(Z)\},$$
$$sX = \sup\{\varepsilon \geq 0 | \text{ there is a subcontinuum } Z \subset \widetilde{X}_\varepsilon \text{ such that } Z = t(Z)\}.$$

Corollary 9.7 [110] (compare [97, 1.1.2; 108, I, Theorem 2.6]). *If plane contains an uncountable collection of disjoint copies of a tree-like continuum X (or even a product of X with a convergent sequence), then $sX = 0$. Moreover, if these copies are obtained by parallel transfers from one another, then $\sigma X = 0$.*

For a discussion of problems on graphs, arising from the above considerations, see [101].

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