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On projected embeddings and desuspending the α-invariant

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Abstract

A map $f: K \to L$ is called a *projected embedding from $L \times B^s$* if there is an embedding $F: K \to L \times B^s$ such that $f = \pi \circ F$, where $\pi: L \times B^s \to L$ is the projection. A map $f: S^p \sqcup S^q \to S^m$ is a *link map* if $fS^p \cap fS^q = \emptyset$. We apply projected embeddings to desuspending the α-invariant of link maps and to embeddings of double covers into Euclidean space.

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A map $f: K \to L$ is called a *projected embedding from $L \times B^s$* if there is an embedding $F: K \to L \times B^s$ such that $f = \pi \circ F$, where $\pi: L \times B^s \to L$ is the projection. A map $f: X \sqcup Y \to Z$ is a *link map* if $f(X) \cap f(Y) = \emptyset$. In this paper we apply projected embeddings to desuspending the α-invariant of link maps (Theorem 1) and to embeddings of double covers into Euclidean space (Theorem 3). For an introduction and motivation see [9,14,12], [16, Question on p. 152], [17, §6], [22,2].

We shall work in the smooth category. Let EM^m_{pq} be the set of link maps $S^p \sqcup S^q \to S^m$ which embed S^p standardly in the PL category (note that *any* embedding $S^p \to S^m$ is

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PL standard for $m \geqslant p+3$ [7]). Let $\lambda : EM^m_{pq} \to \pi_q(S^{m-p-1})$ be the linking coefficient. A *link concordance* between link maps $f_0, f_1 : S^p \sqcup S^q \to S^m$ is a link map

$$F : S^p \times I \sqcup S^q \times I \to S^m \times I$$

such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. The link concordance does not necessarily embed $S^p \times I$.

Theorem 1. *Denote $k = 2p+1-m$. The mapping $a = \Sigma^k \lambda : EM^m_{pq} \to \pi_{k+q}(S^p)$ is a link concordance invariant, provided $\frac{3p}{2}+1 \leqslant m \leqslant 2p$ and the binomial coefficient $\binom{m-p}{k}$ is odd.*

Clearly, $\operatorname{im} a = \Sigma^k \pi_q(S^{m-p-1})$ and $\Sigma^{q+3-m} a = \Sigma^\infty \lambda$ is the well-known α-invariant [8,10], see also [19,21]. Thus Theorem 1 for $q \geqslant m-1$ together with examples of non-surjectivity and non-injectivity of a non-stable suspension homomorphism gives examples of non-surjectivity and non-injectivity of the α-invariant. Theorem 1 is not interesting for $q \leqslant m-2$: for $q \leqslant m-3$ the a-invariant is a suspension of the α-invariant, and for $q = m-2$ we have

$$\operatorname{im} a = \ker\bigl(h : \pi_{2p-1}(S^p) \to \mathbb{Z}_{(p)}\bigr) \cong \pi_{2p}\bigl(S^{p+1}\bigr) = \pi^S_{p-1},$$

and a gives no more information than α.

Denote by LM^m_{pq} the set of link maps $S^p \sqcup S^q \to S^m$, up to the link concordance. In [9, 14] an invariant $a' : LM^m_{pq} \to \pi_{k+q+1}(S^{p+1})$ was constructed such that $\Sigma^{q+2-m} a' = \alpha$ (note that the concordance invariance of $\Sigma a = a'$ follows analogously to Lemma 2 below, since $S^p \times I$ embeds into $S^m \times I \times \mathbb{R}^{k+1}$ by general position).

The desuspension of α given by Theorem 1 is stronger in the sense that $a' = \Sigma a$ but weaker in the sense that a is defined only on EM^m_{pq} not on LM^m_{pq}. It would be interesting to know if EM^m_{pq} in Theorem 1 can be replaced by LM^m_{pq} (we can approximate the composition $S^p \to S^m \to S^m \times \mathbb{R}^k$ by embeddings, but it remains to prove that our invariant will not depend on this approximation).

For $m = 2p \geqslant 6$ and $q \leqslant 3p-6$ Theorem 1 (with the invariant defined even on LM^m_{pq}) follows from [27, Proposition F], and was also stated without proof in [15]. Nezhinskij outlined a geometric proof of this simplest case of Theorem 1 (without the restriction $q \leqslant 3p-6$ at the Alexandrov Session in 1999, but with the invariant defined on EM^{2p}_{pq} not on LM^{2p}_{pq}). Our proof of Theorem 1 extends his ideas.

Proof of Theorem 1. Suppose that

$$F : S^p \times I \sqcup S^q \times I \to S^m \times I$$

is a link concordance between $F_0, F_1 : S^p \sqcup S^q \to S^m$ such that $F|_{S^p \times \{0,1\}}$ is an embedding. Since there exists *some* proper framed immersion $S^p \times I \sqcup S^q \times I \to S^m \times I$, we may assume by [6, 1.2.2], [1, Lemma 2] that $F_{S^p \times I}$ is a general position framed immersion.

By general position, $F|_{S^p \times I}$ has no triple points. Therefore by Lemma 2 below for $n = p+1$, there is an embedding

$$\overline{F} : S^p \times I \to S^m \times I \times \mathbb{R}^k$$

such that $\pi \circ \overline{F} = F|_{S^p \times I}$, where

$$\pi : S^m \times I \times \mathbb{R}^k \to S^m \times I$$

is the projection.

We may assume that $S^m \times I \times \mathbb{R}^k \subset \Sigma^k(S^m \times I)$ close to the base $S^m \times I \subset \Sigma^k(S^m \times I)$. Let $\overline{F}|_{\Sigma^k(S^q \times I)} = \Sigma^k F|_{S^q \times I}$. Since $F(S^p \times I) \cap F(S^q \times I) = \emptyset$, it follows that

$$\overline{F} : S^p \times I \sqcup \Sigma^k\big(S^q \times I\big) \to \Sigma^k\big(S^m \times I\big)$$

is a link concordance, which embeds $S^p \times I$, between $\overline{F_0} = \Sigma^k F_0$ and $\overline{F_1} = \Sigma^k F_1$. Therefore $\Sigma^k \lambda(F_0) = \lambda(\overline{F_0}) = \lambda(\overline{F_1}) = \Sigma^k \lambda(F_1)$. □

Lemma 2. *If the binomial coefficient $\binom{n-k}{k}$ is odd, N is an n-manifold and $f : N \to B^{2n-k}$ is a proper general position framed immersion without triple points and such that $f|_{\partial N}$ is an embedding, then f is a projected embedding from $B^{2n-k} \times B^k$.*

Proof. Let

$$\Delta = \big\{x \in B^{2n-k}\colon \big|f^{-1}x\big| \geqslant 2\big\} \quad \text{and} \quad \widetilde{\Delta} = \big\{x \in N\colon \big|f^{-1}fx\big| \geqslant 2\big\}.$$

Then $\hat{f} = f|_{\widetilde{\Delta}} : \widetilde{\Delta} \to \Delta$ is a double covering. Denote by $\hat{f}$ the line bundle associated with the double cover $\hat{f}$ and let $w_1(\hat{f}) \in H^k(\Delta, \mathbb{Z}_2)$ be the first Stiefel–Whitney class of this line bundle.

The normal bundle of Δ in B^{2n-k} is isomorphic to $(n-k) \oplus (n-k)\hat{f}$. Hence

$$\bar{w}(\Delta) = \big(1 + w_1\big(\hat{f}\big)\big)^{n-k}, \quad \text{so} \quad 0 = \bar{w}_k(\Delta) = \binom{n-k}{k}\big(w_1\big(\hat{f}\big)\big)^k = \big(w_1\big(\hat{f}\big)\big)^k$$

cf. [3, proof of proposition].

By general position $\dim \Delta = k$. Hence it follows by Theorem 3(a) below that $\hat{f}$ *is a projected embedding from* $\Delta \times B^k$. This implies that f *is a projected embedding from* $B^{2n-k} \times B^k$.

Indeed, take a map $\hat{g} : \widetilde{\Delta} \to B^k$ such that $\hat{f} \times \hat{g} : \widetilde{\Delta} \to \Delta \times B^k$ is an embedding. Take a Riemannian metric on N such that 1-neighborhood U of $\widetilde{\Delta}(f)$ in N is a tubular neighborhood of $\widetilde{\Delta}$ in N. Let $r : U \to \widetilde{\Delta}$ be the projection of the normal bundle. Define a map $g : N \to B^k$ by $g(x) = 0$ for $x \notin U$ and $g(x) = (1 - \operatorname{dist}(x, \widetilde{\Delta}))\hat{g}(r(x))$ for $x \in U$. Then $f \times g : N \to N \times B^k$ is an embedding. □

Theorem 3. *Let Δ be a k-manifold* (*closed or with boundary*), *$\widetilde{\Delta}$ its double cover and* $\mathrm{pr} : \widetilde{\Delta} \to \Delta$ *the projection. Consider the following conditions*:

(E) *there exists an equivariant map $g : \widetilde{\Delta} \to S^{s-1}$*;

(P) pr *is a projected embedding from $\Delta \times B^s$*;

(A) *the composition $\widetilde{\Delta} \overset{\mathrm{pr}}{\to} \Delta \subset \Delta \times B^s$ is approximable by embeddings*;

(W) $(w_1(\mathrm{pr}))^s = 0 \in H^s(\Delta, \mathbb{Z}_2)$.

Then (E) $\Leftrightarrow$ (P) $\Rightarrow$ (A) $\Rightarrow$ (W). *Moreover,*

(a) *if* $s = k$, *then* (E) $\Leftrightarrow$ (P) $\Leftrightarrow$ (A) $\Leftrightarrow$ (W);
(b) *if* $2s \geqslant k + 3$ *and both* Δ *and* $\widetilde{\Delta}$ *are parallelizable*, *then* (E) $\Leftrightarrow$ (P) $\Leftrightarrow$ (A).

Note that in Theorem 3 (and below) $\widetilde{\Delta}$ and Δ are arbitrary manifolds, not necessarily double point sets. By general position, all conditions of Theorem 3 hold for $s > k$.

The implications (P) $\Rightarrow$ (A) and (E) $\Rightarrow$ (W) are obvious and well known. To prove (E) $\Rightarrow$ (P), it suffices to observe that the map $\mathrm{pr} \times g : \widetilde{\Delta} \to \Delta \times S^{s-1}$ is an embedding. Note that the embedding $\mathrm{pr} \times g$ has a trivial normal bundle.

To prove (P) $\Rightarrow$ (E), take an embedding $F = F_1 \times F_2 : \widetilde{\Delta} \to \Delta \times B^s$ such that $\pi \circ F = \mathrm{pr}$ and define an equivariant map $g : \widetilde{\Delta} \to S^{s-1}$ by

$$g(x) = \frac{F_2(x) - F_2(-x)}{|F_2(x) - F_2(-x)|}.$$

To prove Theorem 3(a) it suffices to prove either (W) $\Rightarrow$ (E) or (W) $\Rightarrow$ (P). The implication (W) $\Rightarrow$ (E) is a folklore result from obstruction theory. For completeness, we present below its proof which was kindly communicated to us by A. Volovikov. We also sketch a geometric proof of the implication (W) $\Rightarrow$ (P). The proofs of (A) $\Rightarrow$ (W), (W) $\Rightarrow$ (P) and 5(b) below are based on the ideas of [26], [7, §11], [13], [1, proof of Lemma 3], [18, §5]. Theorem 3 should be compared to [5,24].

The following remark improves [16, Theorem 2], [17, Hacon's remark in §6], see also [11,25].

Remark 4. The group $\mathrm{Spin}(r)$ embeds into Euclidean space with a trivial normal bundle in codimension

$$s = \begin{cases} l^2 - l + 2, & r = 2l \ (\dim \mathrm{Spin}(r) = 2l^2 - l), \\ l^2 + l + 2, & r = 2l + 1 \ (\dim \mathrm{Spin}(r) = 2l^2 + l). \end{cases}$$

Proof. Let $\Delta = SO(r)$ and $\widetilde{\Delta} = \mathrm{Spin}(r)$. By [16, Theorem 1 and table on p. 154], Δ embeds with trivial normal bundle in codimension $[\frac{r+1}{2}]$, and hence in any greater codimension.

By [16, lemma on p. 166], there is an equivariant map $g : \widetilde{\Delta} \to S^{s-1}$. Now Remark 4 follows from the implication (E) $\Rightarrow$ (P) of Theorem 3 (since the embedding obtained there has a trivial normal bundle). $\square$

Proof of (A) $\Rightarrow$ (W) **in Theorem 3.** We need the following two facts. For a general position immersion $F : \widetilde{\Delta} \to \Delta \times B^s$, ε-close to $i \circ \mathrm{pr}$, let

$$\Sigma(F) = \{x \in \Delta \times B^s \mid \text{there are } y, z \in \widetilde{\Delta} \text{ such that } |y, z| > 5\varepsilon,\ Fy = Fz = x\}$$

be the 'far away double points' immersed submanifold.

It is proved analogously to [26], [7, §11] that the class $[\Sigma(F)] \in H_{k-s}(\Delta, \mathbb{Z}_2)$ does not depend on homotopy of F through maps, ε-close to $i \circ \mathrm{pr}$. It is proved analogously to [13] that this class is dual to $(w_1(\mathrm{pr}))^s$ (it suffices to prove this for the case when $\pi \circ F = \mathrm{pr}$). This implies (A) $\Rightarrow$ (W). $\square$

Sketch of the proof of (W) $\Rightarrow$ (P) **in Theorem 3(a).** For $s=1$ the proof is obvious so assume that $s \geqslant 2$. We may assume that Δ is connected. If $w_1(\mathrm{pr})=0$, then there exists an equivariant map $\widetilde{\Delta} \to S^0$, hence (E) and (P) are true.

If $w_1(\mathrm{pr}) \neq 0$, then $\widetilde{\Delta}$ is connected. Take a general position immersion $F:\widetilde{\Delta} \to \Delta \times B^s$ such that $\pi \circ F = \mathrm{pr}$. Since $[\Sigma(F)] = (w_1(\mathrm{pr}))^k = 0$, it follows that the number of double points of F is even. If k is even and Δ is orientable, then the algebraic number of double points of F is zero by [20, Lemma 5]. Therefore, as in [1, proof of Lemma 3], we can apply 'projected version' of the Whitney trick to eliminate double points of F and obtain an embedding $F':\widetilde{\Delta} \to \Delta \times B^s$ such that $\pi \circ F = \mathrm{pr}$. $\square$

Proof of (W) $\Rightarrow$ (E) **in Theorem 3(a).** (*A. Volovikov*) We can assume without loss of generality that $\widetilde{\Delta}$ is connected. Let $\mathbb{Z}_2$ act on $\mathbb{R}^k$ by multiplication with -1. An equivariant map $\widetilde{\Delta} \to S^{k-1}$ exists if and only if there exists a non-zero section of the bundle $\widetilde{\Delta} \times_{\mathbb{Z}_2} \mathbb{R}^k \to \Delta$. We will show that the unique obstruction class to defining a non-zero section of this bundle is trivial and hence this bundle has a non-zero section.

If $\widetilde{\Delta}$ has nonempty boundary, then it is easy to see that the obstruction class lies in the zero group. Suppose further that $\widetilde{\Delta}$ is closed.

First case: k is even. The unique obstruction class to defining a non-zero section lies in $H^k(\Delta;\mathbb{Z})$ (coefficients in cohomology are not twisted since k is even). This obstruction class reduced mod 2 equals to $(w_1(\mathrm{pr}))^k = 0 \in H^k(\Delta,\mathbb{Z}_2)$, i.e., vanishes. If Δ is nonorientable, then $H^k(\Delta;\mathbb{Z}) = \mathbb{Z}_2$ and the reduction is an isomorphism, hence the obstruction class vanishes.

If Δ is orientable, then $H^k(\Delta;\mathbb{Z}) = \mathbb{Z}$ and we obtain that the obstruction class is represented by an even number (since its reduction mod 2 equals to zero). On the other hand a non-zero obstruction class in any case (for k odd or even) has order 2 by [20, Lemma 5]. Hence the obstruction class also vanishes.

Second case: k is odd. In this case coefficients are twisted and we have the following Smith–Richardson sequence

$$\cdots \to H^k(\Delta;\mathbb{Z}) \to H^k\big(\widetilde{\Delta};\mathbb{Z}\big) \to H^k\big(\Delta;\widehat{\mathbb{Z}}\big) \to 0.$$

This Smith–Richardson sequence (one of the two Smith–Richardson sequences) is induced by the short coefficient sequence $0 \to \mathbb{Z} \to \mathrm{pr}_*\mathbb{Z} \to \widehat{\mathbb{Z}} \to 0$ of sheaves over Δ. Here $\mathbb{Z}$ is the constant sheaf over Δ (with $\mathbb{Z}$ as a fiber), $\mathrm{pr}_*\mathbb{Z}$ is the direct image of the constant sheaf $\mathbb{Z}$ over $\widetilde{\Delta}$ and $\widehat{\mathbb{Z}}$ is a subsequent factor sheaf where the inclusion is defined on a fiber as $m \to (m,m)$, $m \in \mathbb{Z}$. Note that $H^i(\Delta;\mathrm{pr}_*\mathbb{Z}) = H^i(\widetilde{\Delta};\mathbb{Z})$.

It follows from this sequence that $H^k(\Delta;\widehat{\mathbb{Z}})$ can be one of 0, $\mathbb{Z}_2$ or $\mathbb{Z}$. Indeed, if $\widetilde{\Delta}$ is not orientable, then $H^k(\widetilde{\Delta};\mathbb{Z}) = \mathbb{Z}_2$, hence $H^k(\Delta;\widehat{\mathbb{Z}})$ is either 0 or $\mathbb{Z}_2$. If $\widetilde{\Delta}$ and Δ are orientable, then $H^k(\Delta;\widehat{\mathbb{Z}}) = \mathbb{Z}_2$ because $\widetilde{\Delta} \to \Delta$ is a double cover. In the remaining case when $\widetilde{\Delta}$ is orientable and Δ is not orientable we have $H^k(\Delta;\widehat{\mathbb{Z}}) = \mathbb{Z}$.

The obstruction class obviously vanishes if $H^k(\Delta;\widehat{\mathbb{Z}}) = 0$. If $H^k(\Delta;\widehat{\mathbb{Z}}) = \mathbb{Z}_2$, then $H^k(\Delta;\widehat{\mathbb{Z}}) \to H^k(\Delta;\mathbb{Z}_2)$ is an isomorphism and we see that the obstruction class also vanishes. Finally, if $H^k(\Delta;\widehat{\mathbb{Z}}) = \mathbb{Z}$, then the obstruction class again vanishes since the nonzero obstruction class has order 2 by [20, Lemma 5]. $\square$

Proof Theorem 3(b). It suffices to prove (A) ⇒ (E). We shall construct an equivariant map $\Sigma^k\widetilde{\Delta} \to S^{k+s-1}$. If $k \leqslant 2(s-1)-1$, then Theorem 2.5 of [4] implies (E). Consider the natural action of $\mathbb{Z}_2$ on $\widetilde{\Delta}$ and denote it by $x \mapsto -x$. Since both Δ and $\widetilde{\Delta}$ are parallelizable, there is a continuous family $\{h_x : D^k \to \Delta\}_{x\in\Delta}$ of embeddings such that $h_x 0 = x$ and $h_{-x} \equiv -h_x$.

Denote by $i : \Delta \to \Delta \times B^s$ the inclusion. Let $F = F_1 \times F_2 : \widetilde{\Delta} \to \Delta \times B^s$ be an embedding sufficiently close to $i \circ \mathrm{pr}$. Since F is close to $i \circ \mathrm{pr}$, we may assume that $F_1 h_x(\frac{D^k}{2}) \subset h_x(D^k)$. Therefore a map $\phi : \widetilde{\Delta} \times \frac{D^k}{2} \to D^k \times B^s$ is well-defined by the formula $\phi(x,t) = (h_x^{-1} F_1 h_x(t), F_2 h_x(t))$ see [18, Fig. 4]. Since F is an embedding, it follows that ϕ does not identify antipodes (x,t) and $(-x,-t)$. Extend ϕ to

$$\Sigma^k \Delta \cong \frac{\Delta \times D^k}{\{\Delta \times t \mid t \in \partial D^k\}}$$

by

$$\phi[x,t] = \begin{cases} \phi(x,t), & |t| \leqslant \frac{1}{2}, \\ \left(2 - 2\dfrac{t}{|t|}\right)\phi\left(x, \dfrac{t}{2|t|}\right) + \left(2\dfrac{t}{|t|} - 1\right)(t,0), & |t| \geqslant \frac{1}{2}. \end{cases}$$

Since $\phi(x, \frac{t}{2|t|})$ is close to $(\frac{t}{2|t|}, 0)$, it follows that the new map ϕ does not identify antipodes. Hence we can obtain an equivariant map $\Sigma^k\widetilde{\Delta} \to S^{k+s-1}$. □

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