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ON CONTRACTIBLE n-DIMENSIONAL COMPACTA, NON-EMBEDDABLE INTO $\mathbb{R}^{2n}$

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ABSTRACT. We present a very short proof of a well-known result, that for each n there exists a contractible n-dimensional compactum, non-embeddable into $\mathbb{R}^{2n}$.

We present a very short proof of the following well-known result, which answers a question from [DD] and was first proved in [RSS, Corollary 1.5] (later an alternative proof appeared in [KR]).

Theorem. *For each $n \geq 1$ there exists a contractible n-dimensional compactum which does not embed into $\mathbb{R}^{2n}$.*

We shall use a construction and an idea from [RSS] (see also [CRS, §4], [RS1] and [RSSp]). However, instead of using the main result of [We], we shall apply its corollary, to the effect that for every n there exists a contractible n-polyhedron X, for which there is no equivariant map $\widetilde{X} \to S^{2n-1}$. A simple proof of this corollary was presented in [Sc, p. 223]. Our proof also makes it possible to avoid referring to a (not difficult) result in [CF, Theorem 2.5] and [Hu].

Proof of Theorem. There exist a contractible n-polyhedron X and a map $\varphi : S^{2n-1} \to X$ which does not identify antipodal points [Sc, p. 223]. (Notice that the map $\varphi_n = p^n|_{\partial(D^2)^n} : \partial(D^2)^n \to T^n$ also has this property, where T is the triod and p is the map defined in [KR, §2]. Indeed, φ_1 does not identify antipodal points [KR, §2], hence neither does φ_n.) Let $X' = X \times (0 \cup \{\frac{1}{k}\}) \cup x \times [0,1]$, where $x \in X$. Clearly, X' is contractible.

Suppose that there existed an embedding $f : X' \to \mathbb{R}^{2n}$. Then we could define a map $\psi : S^{2n-1} \to X \times X$ by $\psi(s) = (\varphi(s), \varphi(-s))$. Since φ does not identify antipodal points, it would follow that $\psi(S^{2n-1}) \cap \operatorname{diag} X = \emptyset$. Hence the maps $g_0 : \psi(S^{2n-1}) \to S^{2n-1}$ and $g_k : X \times X \to S^{2n-1}$ given by

$$g_0(x,y) = \frac{f(x,0) - f(y,0)}{|f(x,0) - f(y,0)|} \qquad \text{and} \qquad g_k(x,y) = \frac{f(x,0) - f(y,\frac{1}{k})}{|f(x,0) - f(y,\frac{1}{k})|}$$

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would be well-defined. The maps ψ, g_0 and g_k would be equivariant with respect to involutions on $\psi(S^{2n-1}) \subset X \times X$ and S^{2n-1}, exchanging factors and antipodal points, respectively.

Since $dist(\psi(S^{2n-1}), \operatorname{diag} X) > 0$, it would follow that for sufficiently large k and any point $(x, y) \in \psi(S^{2n-1})$, the points $g_0(x, y)$ and $g_k(x, y)$ would be close and hence could not be antipodal. Therefore $g_0 \simeq_{eq} g_k|_{\psi(S^{2n-1})}$. But $g_k|_{\psi(S^{2n-1})}$ extends to a contractible space $X \times X$ and therefore is null-homotopic. Hence $g_0 : \psi(S^{2n-1}) \to S^{2n-1}$ is null-homotopic. Thus the map $g_0 \circ \psi : S^{2n-1} \to S^{2n-1}$ is equivariant and null-homotopic, which contradicts the Borsuk-Ulam Theorem. So X' cannot embed into $\mathbb{R}^{2n}$. □

By attaching k-dimensional cells to X' we can make X' locally $(k-1)$-connected, hence our compactum can even be made to be locally $(n-1)$-connected. This observation (due to R. J. Daverman) is interesting because the Borsuk Conjecture states that every contractible locally n-connected n-dimensional compactum embeds into $\mathbb{R}^{2n}$.

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