On zero-dimensional Milutin maps and Michael selection theorems

D. Repovš *, P.V. Semenov and E.V. Ščepin

Steklov Mathematical Institute, 42 Vavilova Street, 117966 Moscow GSP-1, Russia

Abstract

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We study the relationship among the three best known Michael's theorems on the existence of selections of lower semicontinuous multi-valued maps: (A) the theorem for zero-dimensional paracompact spaces; (B) the theorem for convex-valued maps on paracompact spaces; and (C) the theorem for compact-valued selections.
    We prove that the theorems (B) and (C) follow from theorem (A). This is a corollary of our main theorem that every paracompact space is the image of a zero-dimensional paracompact space under a Milutin map.

Keywords: Multi-valued map; Milutin map; Michael selection theorems; Regular averaging operator; Pull-back operation; Probability measure.

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1. Introduction

A map $f: X \to Y$ (possibly multi-valued) is called a *selection* of a multi-valued map $F: X \to Y$ if for every $x \in X$, $f(x) \subset F(x)$. A multi-valued map $F: X \to Y$ is said to be *lower* (respectively *upper*) *semicontinuous* if for every open subset $G \subset Y$ of Y, the set $\{x \in X \mid F(x) \cap G \neq \emptyset\}$ (respectively $\{x \in X \mid F(x) \subset G\}$) is open in X.

We begin by stating the three best known Michael's theorems about the existence of continuous single-valued selections of multi-valued lower semicontinuous maps [1,3–7].

Theorem 1.1 (The zero-dimensional theorem). *Let X be a paracompact space with* dim $X = 0$, *M a complete metric space and $F : X \to M$ a lower semicontinuous map with closed values. Then F admits a continuous single-valued selection.*

Theorem 1.2 (The convex theorem). *Let X be a paracompact space, B a Banach space and $F : X \to B$ a lower semicontinuous map with closed convex values. Then F admits a continuous single-valued selection.*

Theorem 1.3 (The compact theorem). *Let X be a paracompact space, M a complete metric space and $F : X \to M$ be a lower semicontinuous map with closed values. Then F admits an upper semicontinuous compact-valued selection, which admits a lower semicontinuous compact-valued selection.*

The main purpose of our paper is to show that Theorem 1.1 implies both Theorems 1.2 and 1.3. This follows from our main result:

Theorem 1.4. *Every paracompact space is the image of some zero-dimensional paracompact space under some Milutin map.*

Corollary 1.5. *Theorem 1.1 implies both Theorems 1.2 and 1.3.*

Corollary 1.6. *Let X be a paracompact space, G an open subset of some Banach space and $F : X \to G$ a lower semicontinuous map with convex closed (in G) values. Then F admits a continuous selection.*

Corollary 1.7. *Let $G = \bigcap G_n$ where G_n are open convex subsets of some Banach space. Then any lower semicontinuous map from a paracompact space X into G with convex closed (in G) values admits a continuous single-valued selection.*

Remark. Nedev and Valov have announced an alternative proof of Corollary 1.6 [10]. For an alternative proof of Corollary 1.7 see [8].
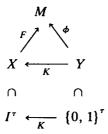
In the special case, when the space X is compact, the proof is relatively simple and it uses the so-called *Milutin maps* [9]. In the general case some additional construction is needed. For metrizable spaces X, Theorem 1.4 was proved by Čoban [13]. (For related work see also [14].)

2. The compact case

Step 1. Let X be a compact space, M a complete metric space and $F: X \to M$ a lower semicontinuous map with closed values. We may assume that X lies in the cube I^τ for some cardinal τ, where I denotes the interval $I = [0, 1]$. Let $K : \{0, 1\}^\tau \to I^\tau$ be any continuous map of the zero-dimensional space $\{0, 1\}^\tau$ onto the cube I^τ. Then K is a closed map and K^{-1} is an upper semicontinuous map with compact values. Let $Y = K^{-1}(X)$.

Then Y is a zero-dimensional compactum and by Theorem 1.1, the lower semicontinuous map $(F \circ K)|_Y : Y \to M$ admits a continuous selection ϕ. The formula $f(x) = \phi(K^{-1}(x))$ defines an upper semicontinuous compact-valued selection $f: f(x) \subset F(x)$, for every $x \in X$.

$$
\begin{array}{ccc}
 & M & \\
{}^F\nearrow & & \nwarrow{}^\phi \\
X & \xleftarrow{\hspace{1em}K\hspace{1em}} & Y \\
\cap & & \cap \\
I^\tau & \xleftarrow{\hspace{1em}K\hspace{1em}} & \{0, 1\}^\tau
\end{array}
$$

Step 2. We begin by providing some preliminary information about averaging operators. For a compact space X we denote by $C(X)$ the Banach space of all continuous functions on X with the sup-norm topology and by $P(X)$ we denote the space of all regular probability measures on X, equipped with the weak*-topology; every $\mu \in P(X)$ is a continuous positive linear functional on $C(X)$ with $\mu(1_X) = 1$.

A continuous map $f: X \to Y$ of a compact space X onto a compact space Y is called a *Milutin map* if there exists a continuous map $\nu: Y \to P(X)$ such that supp $\nu_y \subset f^{-1}(y)$, for every $y \in Y$.

If $f: X \to Y$ is a Milutin map and $\nu: Y \to P(X)$ is the map associated with f, we can define the so-called *regular averaging operator* $A: C(X) \to C(Y)$ by the formula $(A\phi)(y) = \nu_y(\phi)$, for every $\phi \in C(X)$. Obviously, $A(\psi \circ f) = \psi$, for every $\psi \in C(Y)$. As a consequence, $C(X)$ is isomorphic to $C(Y) \oplus \mathrm{Ker}\, A$.

It is well known that: (1) there exists a Milutin map of a Cantor set C onto the interval $I = [0,1]$ (see [11, Lemma (5.5)]); (2) every product of Milutin maps is again a Milutin map (see [11, Proposition (4.7)]); and (3) if X and Y are compact metric spaces and $f: X \to Y$ is a Milutin map then f^{-1} admits a lower semicontinuous compact-valued selection (see [2, Theorem (3.4)]).

We now use Step 1 for a Milutin map of the Cantor set C onto I and consider the restriction of the τ-degree of a lower semicontinuous compact-valued selection onto a compact $X \subset I^\tau$ then we obtain the desired lower semicontinuous compact-valued selection. Next, we use Step 1 for this lower semicontinuous map and

obtain an upper semicontinuous compact-valued selection. In this way, we have proved that Theorem 1.1 implies Theorem 1.3, however, with the interchange of lower and upper semicontinuity of selections. In order to prove precisely Theorem 1.3 we need more sophisticated techniques (see Section 3).

Step 3. Let X be a compact space, B a Banach space and $F: X \to B$ a lower semicontinuous map with closed convex values. We may assume that X lies in I^τ for some cardinal τ, where $I = [0, 1]$, as before. Let $M: C^\tau \to I^\tau$ be any Milutin map of the zero-dimensional space C^τ onto the cube I^τ, ν be the map associated with M, from I^τ into $P(C^\tau)$, and let $Y = M^{-1}(X)$.

Then Y is a compact space, dim $Y = 0$ and the composition $(F \circ M)|_Y: Y \to B$ is a lower semicontinuous map with closed convex values. By Theorem 1.1, the map $F \circ M$ admits a continuous selection $\phi: Y \to B$, $\phi(y) \in F(M(y))$, for every $y \in Y$.

We define a continuous map $f: X \to B$ by the following formula

$$f(x) = \int_{M^{-1}(x)} \phi \, \mathrm{d}\nu_x.$$

Consider the diagram

Recall that supp $\nu_x \subset M^{-1}(x)$. By the definition of f, we have that

$$f(x) \in \overline{\mathrm{conv}}\{\phi(y) \mid y \in M^{-1}(x)\} \subset F(x),$$

i.e., f is a selection of the lower semicontinuous map F. So, we have proved that Theorem 1.1 implies Theorem 1.2.

If one uses a technique of Valov (see [12, Corollary (3.7)]) then the above constructions can also be applied to the class of p-paracompact spaces X.

3. The general case

Step 1. Let ω be a locally finite open covering of X, $\omega = \{G_\alpha\}_{\alpha \in A(\omega)}$, and let $e = \{e_\alpha\}_{\alpha \in A(\omega)}$, be a locally finite partition of the unity, inscribed into the cover ω, i.e., for every $\alpha \in A(\omega)$, supp $e_\alpha \subset G_\alpha$. In the direct product of X and the discrete space $A(\omega)$ we consider the following subspace $X_{\omega,e}$

$$X_{\omega,e} = \{(x, \alpha) \in X \times A(\omega) \mid x \in \text{supp } e_\alpha\}.$$

$X_{\omega,e}$ is called the *closed graph* of the covering $\{\text{supp } e_\alpha\}_{\alpha \in A(\omega)}$.

Next we consider the natural projection $p_{\omega,e} : X_{\omega,e} \to X$, given by $p_{\omega,e}(x, \alpha) = x$, for every $(x, \alpha) \in X_{\omega,e}$. It's easy to see that $p_{\omega,e}$ is a closed map. Because ω is locally finite, the preimages $p_\omega^{-1}(x)$ are compact (in fact, they are finite sets). So, the map $p_{\omega,e}$ is perfect and the space $X_{\omega,e}$ is paracompact.

We construct a *regular averaging operator* L for the map $p = p_{\omega,e}$ in the following natural way. Let $f : X_{\omega,e} \to \mathbb{R}$ be a bounded continuous function. Define a function $L_f : X \to \mathbb{R}$ by $(L_f)(x) = \sum_\alpha f(x, \alpha) e_\alpha(x)$, for every $x \in X$, where the sum is taken over all α such that $(x, \alpha) \in p^{-1}(x)$. Then L_f is bounded and continuous, if $f \geqslant 0$ then also $L_f \geqslant 0$, and for any bounded continuous function $g : X \to \mathbb{R}$ we have that

$$L_{g \circ p}(x) = \sum_\alpha g(p(x, \alpha)) e_\alpha(x) = g(x) \sum_\alpha e_\alpha(x) = g(x).$$

Therefore, L_f is a regular averaging operator, i.e., $p = p_{\omega,e}$ is a Milutin map.

Step 2. Next we introduce the so-called *pull-back operation* for the map $p_{\omega,e} : X_{\omega,e} \to X$, over all pairs (ω, e). More precisely, let Ω be the discrete set of all locally finite coverings of X. For any $\omega \in \Omega$ we pick a locally finite partition of unity e inscribed into a cover ω. In the Cartesian product of X and the discrete space $\prod_{\omega \in \Omega} A(\omega)$ we consider the following subspace

$$X_{\text{LFC}} = \Big\{ (x, \{\alpha(\omega)\}_{\omega \in \Omega}) \in X \times \prod_{\omega \in \Omega} A(\omega) \mid x \in \text{supp } e_{\alpha(\omega)},$$

$$\text{for every } \omega \in \Omega \Big\}$$

and consider the natural projection $p : X_{\text{LFC}} \to X$, given by $p(x, \{\alpha(\omega)\}_{\omega \in \Omega}) = x$, for every $(x, \{\alpha(\omega)\}_{\omega \in \Omega})$.

We omit the routine verification of the fact that the pull-back operation of a perfect (Milutin) map gives a perfect (Milutin) map. So, we have that X_{LFC} is paracompact and that the map $p : X_{\text{LFC}} \to X$ is a Milutin map.

Step 3. It remains to prove that dim $X_{\text{LFC}} = 0$. We prove that for any open covering of the space X_{LFC} there exists an inscribed open pairwise disjoint covering. We may assume that the original open covering consists of the basic open sets (with respect to the topology induced by the topology of the Cartesian product $X \times \prod_{\omega \in \Omega} A(\omega)$).

First, we consider the local situation. Pick $x \in X$ and choose a finite collection $V_1, \ldots, V_n$ of basic open sets such that the collection $\{V_i \cap p^{-1}(x)\}_{i=1}^n$ is the covering of the compact space $p^{-1}(x)$. We have that for every $i = 1, 2, \ldots, n$,

$$V_i = \Big(U_i \times \prod_{\omega \in F_i} B(\omega) \times \prod_{\omega \notin F_i} A(\omega) \Big) \cap X_{\text{LFC}},$$

for some finite set $F_i \subset \Omega$, for some subsets $B(\omega) \subset A(\omega)$, $\omega \in F_i$, and for some neighbourhood U_i of the point x.

We should remark that, in fact $\omega = \{G_\alpha\}_{\alpha \in A(\omega)}$, is a locally finite covering of X and that we may consider the index α as a closed set supp $e_\alpha \subset G_\alpha$.

For every $\omega \in F_i$, there exists only a finite set of functions $e_1(\omega), \ldots, e_{k(i)}(\omega)$ such that $e_j(\omega) \in B(\omega)$ and $x \in \operatorname{supp} e_j(\omega) \subset G_j(\omega)$, $1 \leqslant j \leqslant k(i)$.

We define "very small" neighbourhood of the point $x \in X$:

$$W(x) = \bigcap_{i=1}^{n} \left\{ U_i \bigcap_{\omega \in F_i} \left(\bigcap_{j=1}^{k(i)} G_j(\omega) \right) \right\}$$

Now, we consider the first basic open set V_i. For every $\omega \in F_1$, we pick some element $h(\omega) \in \{e_1(\omega), \ldots, e_{k(1)}(\omega)\}$ and define the following basic open set $V_{1,h} \subset V_1$

$$V_{1,h} = \left(W(x) \times \prod_{\omega \in F_1} \{h(\omega)\} \times \prod_{\omega \notin F_1} A(\omega) \right) \cap X_{\text{LFC}}.$$

Then the sets $\{V_{1,h}\}_h$ are pairwise disjoint.

In the similar manner we define the sets $\{V_{2,h}\}, \ldots, \{V_{n,h}\}$ which are inscribed into the sets $V_2, \ldots, V_n$. The union of these sets makes up the pairwise disjoint open covering of $p^{-1}(x)$, inscribed into the covering $V_1, \ldots, V_n$.

Next, we consider the open covering $\{W(x)\}_{x \in X}$ of X by the "small" neighbourhoods $W(x)$. Let ω_0 be any locally finite open covering which refines the covering $\{W(x)\}_{x \in X}$. Then $\omega_0 \in \Omega$ and we may distinguish the above sets $\{V_{i,h}(x)\}$ by using the ω_0-coordinate, i.e., let $e_1(\omega_0), \ldots, e_{k(0)}(\omega_0)$ be all indexes from $A(\omega_0)$ such that $x \in \operatorname{supp} e_j(\omega_0) \subset G_j(\omega_0)$, for every $1 \leqslant j \leqslant k(0)$.

Then we may consider the following basic open subsets of set $V_{i,h}(x)$:

$$V_{i,h,j} = \left(W(x) \times \prod_{\omega \in F_i} \{h(\omega)\} \times \{j(\omega_0)\} \times \prod_{\omega \notin F_i \cup \omega_0} A(\omega) \right) \cap X_{\text{LFC}}.$$

So the sets $\{V_{i,h,j}(x)\}$ over all i, h, j, x form an open, pairwise disjoint covering of X_{LFC}.

We remark that the map p constructed above is in fact an *inductively* open map, i.e., the upper semicontinuous compact-valued map p^{-1} admits a lower semicontinuous compact-valued selection S. To construct such a selection S we consider the class of all coverings of the paracompact space X by interiors of the sets $\operatorname{supp} e_\alpha$, $\alpha \in A(\omega)$, $\omega \in \Omega$, and afterwards we repeat the construction above for this class. This proves that Theorem 1.1 indeed implies Theorem 1.3.

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