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Continuous Selections as Uniform Limits of δ-Continuous ε-Selections

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Abstract. We prove a selection theorem which unifies the nonconvex-valued and the zero-dimensional selection theorems. Our proof is based on a new method which unifies the methods of outside and inside approximations of a given lower semicontinuous multivalued mapping. As an application, we obtain a unified selection theorem ('modulo' a countable subset of the domain) and a theorem on density of selections.

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0. Introduction

Continuous singlevalued selections f of a given multivalued mapping F are usually constructed as uniform limits of sequences of certain approximations $\{f_n\}_{n\in N}$ of F. Practically all known selection results are obtained by using one of the following two approaches to the construction of $\{f_n\}_{n\in N}$. In the first (and the most popular) one, the method of outside approximations, mappings f_n are continuous ε_n-selections of F, i.e., $f_n(x)$ all lie *near* the set $F(x)$ and all mappings f_n are *continuous*. In the second one, the method of inside approximations, f_n are δ_n-continuous selections of F, i.e., $f_n(x)$ all lie *in* the set $F(x)$, however f_n are *discontinuous*.

Roughly speaking, for infinite dimensional domains the key ingredient of the method of outside approximations is a consideration of the convex hull $\mathrm{conv}\{D(f_n(x), \varepsilon_n) \cap F(x)\}_{x\in X}$ and the construction (using these hulls) of a more precise ε_{n+1}-selection f_{n+1}, for some $\varepsilon_{n+1} < \varepsilon_n$. The sets $\mathrm{conv}\{D(f_n(x), \varepsilon_n) \cap F(x)\}_{x\in X}$ lie inside $F(x)$ whenever $F(x)$ is convex, and this is the key point in the proofs of various convex-valued selection theorems [11, Part B, §4].

A version of such an approach is related to the situation where one has an appropriate upper estimate of the distance between such a convex hull and $F(x)$. This idea was realized by Michael [4] who introduced the notion of *paraconvex* subsets of normed spaces. In [8, 12] these authors considered *functional paraconvexity* and a function $\alpha_P\colon (0,\infty) \to [0,2]$ associated to each nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ (cf. Definition 1.1 below).

The identity $\alpha_P \equiv 0$ is equivalent to the convexity of P and the more α_P differs from zero, the less 'convex' P is. It has been found that if such *functions of nonconvexity* $\alpha_{F(x)}$ of the values $F(x)$, $x \in X$, admit a nice common majorant, then the continuous selections of F exist. For example, we have (cf. [8]):

THEOREM A. *Let $\beta\colon (0,\infty) \to (0,1)$ be a monotone increasing function and $F\colon X \to Y$ a closed-valued lower semicontinuous mapping from a paracompact space X into a Banach space Y. Suppose that $\alpha_{F(x)}(r) < \beta(r)$, for every $x \in X$ and $r > 0$. Then F has a singlevalued continuous selection.*

But what happens if no nice information concerning nonconvexity of values $F(x)$ is available for $x \in X$? The following selection theorem was proved by Michael and Pixley [7]:

THEOREM B. *Let X be a paracompact space, Y a Banach space, $Z \subset X$ a subset such that $\dim_X Z \leqslant 0$, and $F\colon X \to Y$ a closed-valued lower semicontinuous mapping with $F(x)$ convex, for all $x \in X\backslash Z$. Then F has a singlevalued continuous selection.*

For $Z = \emptyset$ (resp., for $X \setminus Z = \emptyset$) such a 'unified' theorem coincides with the classical convex-valued (resp., zero-dimensional) Michael selection theorem [3]. In [1] an analog of Theorem B was proved for an axiomatically defined (in the sense of Horvath [2]) convexity structure in a complete metric space Y. In [7] (resp., in [1]) the method of outside approximations works successfully due to the convexity (resp., generalized convexity) of the values of the mapping F outside the zero-dimensional subset $Z \subset X$ of the domain.

However, for nontrivial functions of nonconvexity $\alpha_{F(x)}$, $x \in X\backslash Z$, this method does not work because the consideration of the convex hull $\mathrm{conv}\{D(f_n(z), \varepsilon_n) \cap F(z)\}$ at points $z \in Z$ cannot give an improvement from ε_n-precision to ε_{n+1}-precision, with $\varepsilon_{n+1} < \varepsilon_n$.

The second approach, i.e., the method of inside approximations is frequently used for constructing various compact-valued selections (cf. [11, Parts A, §4 and B, §1.2]), and briefly, states the existence of a sequence $\{F_n\}_{n\in N}$ of appropriate finite-valued selections F_n of F. Applying to the convex-valued situation, the values $f_n(x)$ of single-valued (but δ_n-continuous) selections f_n are here certain convex combinations (with respect to a suitable partition of unity) of elements of $F_n(x)$. Clearly, in a nonconvex setting the mappings f_n, being δ_n-continuous as above, will automatically lie outside of F.

In the present paper, we deal with situations where various steps of both methods of outside and inside approximations work simultaneously. On one hand, we generalize Theorem A by replacing the monotonicity restriction for the function $\beta(\cdot)$ by a purely analytical property that $\beta(\cdot)$ is geometrically summable (g-*summable*). Moreover, in comparison with [14] we use weak g-summability (cf. Definition 1.2 below). On the other hand, the main goal of this paper is the proof of the following selection theorem which unifies the zero-dimensional selection theorem and Theorem A, in the spirit of Theorem B. For a given function $\alpha\colon (0,\infty) \to \mathbb{R}$, we shall denote by $\alpha^+\colon (0,\infty) \to \mathbb{R}$ the function of the upper right limits of α, i.e., $\alpha^+(t) = \limsup_{s\to t, s>t} \alpha(s)$.

THEOREM C. *Let $\beta\colon (0,\infty) \to (0,\infty)$ be a weakly g-summable function and $F\colon X \to Y$ a closed-valued lower semicontinuous mapping from a paracompact space X into a Banach space Y. Suppose that $Z \subset X$ is a subset such that* $\dim_X Z \leqslant 0$. *Then F has a singlevalued continuous selection, whenever $\beta(\cdot)$ is a pointwise strong majorant of the function* $(\sup\{\alpha_{F(x)}(\cdot) \mid x \in X \setminus Z\})^+$.

We repeat that the hypotheses of Theorem C allow one to construct a selection f of F as a uniform limit of a sequence $\{f_n\}_{n\in N}$ of δ_n-continuous ε_n-selections of F. So we must control the behavior of *two* parameters, namely, $\delta_n \to 0$ and $\varepsilon_n \to 0$, as $n \to \infty$.

As an application, we prove that in Theorem C one can additionally assume that the values $F(x)$ are nonclosed for all x from some at most countable subset $C \subset X$. Such a triple union exploits the techniques of [6]:

THEOREM D. *Let $\beta\colon (0,\infty) \to (0,\infty)$ be a weakly g-summable function and $F\colon X \to Y$ a lower semicontinuous mapping from a paracompact space X into a Banach space Y. Suppose that $C \subset X$ is a countable subset of the domain such that values $F(x)$ are closed for all $x \in X \setminus C$ and that $Z \subset X$ with* $\dim_X Z \leqslant 0$. *Then F has a single-valued continuous selection, whenever $\beta(\cdot)$ is a pointwise strong majorant of the function* $(\sup\{\alpha_{\mathrm{Cl}(F(x))}(\cdot) \mid x \in X \setminus Z\})^+$.

Clearly, Theorems A, B and C are special cases of Theorem D, as well as Michael's zero-dimensional, convex-valued, paraconvex-valued [4] and countable selection theorems (see [6, Theorem 7.1]). We want to emphasize that, in comparison with [4, 8, 10, 12], we do not use the classical convex-valued selection theorem as an ingredient of the proofs.

For finite-dimensional domains X and for $Z = \emptyset$ such an approach was proposed in [14]. A generalization of Theorems C and D to axiomatically defined H-convex structures (in the spirit of [1, 2]) can be obtained in the same manner.

Finally, we prove the following 'density' version of Theorem D, however, for majorants $\beta(\cdot)$ with the additional property that $\limsup_{t\to 0} \beta(t) < 1$ (compare with Proposition 1.4 below):

THEOREM E. *Assume that in the hypotheses of Theorem* D*, the domain X is additionally perfectly normal, the range Y is separable, and the upper limit of the majorant β: $(0, \infty) \to (0, \infty)$ at zero is less than one. Then there exists a countable family $\{f_w\}_{w \in W}$, $|W| = \aleph_0$, of continuous selections of F such that the set $\{f_w(x)\}_{w \in W}$ is dense in $F(x)$, for each $x \in X$.*

1. Preliminaries

For a nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ and for an open ball $D \subset Y$ of radius r, we define

$$\delta(P, D) = \sup\{\mathrm{dist}(q, P)/r \mid q \in \mathrm{conv}(P \cap D)\},$$

where for the empty intersection $P \cap D$ we put $\delta(P, D) = 0$. Clearly, for a convex set P with nonempty intersection $P \cap D$, the equality $\delta(P, D) = 0$ means that the intersection $P \cap D$ is a convex subset of D.

DEFINITION 1.1. For any nonempty closed subset $P \subset Y$ of a Banach space $(Y, \|\cdot\|)$ the value of its *function of nonconvexity* α_P at a point $r > 0$ is defined as $\alpha_P(r) = \sup\{\delta(P, D) \mid D$ is an open ball of radius $r\}$.

For any function α: $\mathbb{R} \to \mathbb{R}$ we define the function $\alpha^{[1]}(\cdot)$ as the product of the independent real variable t and the value of the function α at t: $\alpha^{[1]}(t) = t \cdot \alpha(t)$, $t \in \mathbb{R}$. We also set $\alpha^{[0]}(t) = t$, and for all $n \in \mathbb{N}$ we set

$$\alpha^{[n+1]}(t) = \alpha^{[1]}(\alpha^{[n]}(t)).$$

Clearly, for a *constant* α, the sequence $\{\alpha^{[n]}(t)\}_{n=0}^{\infty}$ is the usual geometric progression with the coefficient $\alpha \in \mathbb{R}$. Notice that, if α: $(0, \infty) \to (0, 1)$, then $\alpha^{[1]}(t) < t$ and $\alpha^{[n+1]}(t) < \alpha^{[n]}(t)$, for all $t > 0$, i.e., the sequence $\{\alpha^{[n]}(t)\}_{n=0}^{\infty}$ is monotonely decreasing at each $t > 0$.

DEFINITION 1.2. (a) A function α: $(0, \infty) \to (0, \infty)$ is said to be *geometrically summable at the point* $t > 0$ if the series $\sum_{n=0}^{\infty} \alpha^{[n]}(t)$ is convergent. The sum of this series is denoted by $\alpha^{\infty}(t)$;

(b) A function $\alpha(\cdot)$ is said to be *geometrically summable* if it is geometrically summable at each point $t > 0$;

(c) A function $\alpha(\cdot)$ is said to be *weakly geometrically summable* if it is geometrically summable at some sequence of points t_n, converging to infinity, $n \in \mathbb{N}$.

A straightforward verification shows that the following holds:

LEMMA 1.3. *For each function $\alpha(\cdot)$, each positive t and each $N \in \mathbb{N}$ the following holds*:

$$\sum_{n=N}^{\infty} \alpha^{[n]}(t) = \sum_{k=0}^{\infty} \alpha^{[k]}(\alpha^{[N]}(t)) = \alpha^{\infty}(\alpha^{[N]}(t)).$$

Hence, g-summability of the function $\alpha(\cdot)$ at the point t implies its g-summability at the sequence of points $\{\alpha^{[n]}(t)\}_{n\geqslant 0}$. So, in order to illustrate the difference between g-summability and weak g-summability suppose, for example, that for a function $\alpha\colon (0,\infty) \to (0,1)$ the corresponding function $\alpha^{[1]}$ maps $(0,\infty)$ onto itself.

Then weak g-summability of α follows by g-summability of α at points of some *discrete subset* of the ray $(0,\infty)$. Namely, at the double-sided sequence $\{t_n\}_{n=-\infty}^{\infty}$ with the property that $\alpha^{[1]}(t_{n+1}) = t_n$. Clearly, the subsequence $\{t_{-n}\}_{n=0}^{\infty}$ monotonely decreases to zero and subsequence $\{t_n\}_{n=0}^{\infty}$ monotonely increases to infinity. In comparison with this, for g-summability one must check the g-summability at *each* positive t.

It is easy to check (see [12]) the following sufficient condition for g-summability of a function $\alpha(\cdot)$ over the ray $(0,\infty)$.

PROPOSITION 1.4. *A function* $\alpha\colon (0,\infty) \to (0,1)$ *is* g*-summable whenever all upper right limits of* α *over* $[0,\infty)$ *are less than* 1.

Note that the possibility of considering the upper right limits over the open ray $(0,\infty)$ in Proposition 1.4 is still an open problem. It seems that the character of convergence of $\alpha(t)$ to 1, as $t \to 0$, reflects differently on g-summability of $\alpha(\cdot)$. An interesting testing example is the following: is the function $\alpha(t) = (1+t^p)^{-1}$ g-summable?

Recall that a multivalued mapping $F\colon X \to Y$ is said to be *lower semicontinuous* if the set

$$F^{-1}(U) = \{x \in X \mid F(x) \cap U \neq \emptyset\}$$

is open in X, whenever U is open in Y. A singlevalued mapping $f\colon X \to Y$ is called a *selection* (resp., ε*-selection*) of F if $f(x) \in F(x)$ (resp., $\operatorname{dist}(f(x), F(x)) < \varepsilon$), for all $x \in X$.

A singlevalued mapping $f\colon X \to Y$ into a metric space Y is said to be δ*-continuous* if for every $x \in X$ and every $r > 0$, there exists a neighborhood $U(x)$ such that $\operatorname{dist}(f(x'), f(x)) < r + \delta$, for all $x' \in U(x)$. Finally, if $Z \subset X$, then $\dim_X Z \leqslant 0$ means that $\dim E \leqslant 0$, for every $E \subset Z$ which is closed in X. Recall that $D(y, r)$ denotes the open ball of radius r centered at the point y.

2. Proof of Theorem C: The Inductive Step

We shall present below a procedure for improving δ-continuous ε-selections.

THEOREM 2.1. *Let* X, Y, F, *and* Z *be as in Theorem* C *and let* $\alpha(\cdot)$ *be a pointwise strong majorant of each function of nonconvexity* $\alpha_{F(x)}(\cdot)$, $x \in X \setminus Z$. *Suppose that positive numbers* ε, δ, *and* σ *are given and let* $f\colon X \to Y$ *be a* δ*-continuous* ε*-selection of the mapping* F. *Then for* $\varepsilon^* = \varepsilon + 2\delta + \sigma$ *there exists a mapping* $f^*\colon X \to Y$ *such that*:

(a) $\|f^*(x) - f(x)\| < \varepsilon^*$, *for all* $x \in X$;
(b) f^* *is an* $(\alpha(\varepsilon^*) \cdot \varepsilon^*)$*-selection of* F; *and*
(c) f^* *is* σ*-continuous.*

We make some remarks before we begin the proof. First, we note that in this theorem, there are no iteration type restrictions for the majorant α. Second, note that by setting $\delta \to 0$ and $\sigma \to 0$, the assertion 2.1(b) above states that f^* is a closer approximation of F than f, whenever the upper right limit of function α at the point ε is less than 1. This fact explains the appearance of the upper right limits in our technique. Finally, we wish to emphasize that the degree σ of discontinuity of f^* does not depend on the degree δ of discontinuity of f.

Proof. We describe the construction which includes the following 12 steps.

(1) For every $x \in X$, choose a point $y(x) \in F(x) \cap D(f(x), \varepsilon)$.
(2) Using the δ-continuity of f, for every $x \in X$ fix a neighborhood $V(x)$ of x in which $\|f(x') - f(x)\| < 2\delta$, $x \in X$.
(3) Consider the open covering $\omega = \{W(x)\}_{x \in X}$ of the paracompact space X, where

$$W(x) = V(x) \cap F^{-1}(D(y(x), \sigma/3) \neq \emptyset.$$

(4) Let $\nu = \{U_\gamma\}_{\gamma \in \Gamma}$ be a locally finite open covering of X such that the closed covering $\{\mathrm{Cl}(U_\gamma)\}_{\gamma \in \Gamma}$ is inscribed into the covering ω.
(5) For every $\gamma \in \Gamma$, fix a point $x_\gamma \in X$ such that $\mathrm{Cl}(U_\gamma) \subset W(x_\gamma)$.
(6) For every $\gamma \in \Gamma$ and every $x \in \mathrm{Cl}(U_\gamma)$, pick a point

$$s_\gamma(x) \in F(x) \cap D(y(x_\gamma), \sigma/3)).$$

(The nonemptiness of the last intersection follows from (3) and (5).)
(7) For every point $x \in X$, denote by $\Gamma(x)$ the finite subset $\{\gamma \in \Gamma \mid x \in \mathrm{Cl}(U_\gamma)\}$ of the index set Γ, and by $K(x)$ the finite subset $\{y(x_\gamma) \mid \gamma \in \Gamma(x)\}$ of the Banach space Y. Clearly, all elements of $K(x)$ are $(\sigma/3)$-close to the set $F(x)$, $x \in X$.
(8) For every $x \in X \setminus Z$, consider the finite-dimensional simplex $\Delta(x) = \mathrm{conv}\{s_\gamma(x) \mid \gamma \in \Gamma(x)\}$ and in this compact set choose a finite $\eta(x)$-net $N(x)$, where

$$\eta(x) = (\alpha(\varepsilon^*) - \alpha_{F(x)}(\varepsilon^*)) \cdot \varepsilon^*/3 > 0.$$

(9) For each $x \in X \setminus Z$, set $G(x) = \bigcap\{F^{-1}(D(y, \mathrm{dist}(y, F(x)) + \eta(x))) \mid y \in N(x)\} \setminus \bigcup\{\mathrm{Cl}(U_\gamma) \mid x \notin \mathrm{Cl}(U_\gamma)\}$. Then, $G(x)$ is an open neighborhood of x, due to the lower semicontinuity of F and the local finiteness of the covering ν. Moreover, if $x' \in G(x)$ and $x \notin \mathrm{Cl}(U_\gamma)$ then $x' \notin \mathrm{Cl}(U_\gamma)$, i.e., $K(x') \subset K(x)$, for all $x' \in G(x)$.
(10) If $G = \bigcup\{G(x) \mid x \in X \setminus Z\}$, then G is open in X and $X \setminus G \subset Z$. Hence $\dim(X \setminus G) \leqslant 0$.

(11) Inscribe into the covering ν a locally finite open covering $\tau = \{T_\gamma\}_{\gamma\in\Gamma}$ such that any two of the T_γ's have intersection disjoint from $X \setminus G$ and pick a locally finite partition of unity $\{e_\gamma\}_{\gamma\in\Gamma}$ which refines the covering τ.

(12) Finally, put

$$f^*(x) = \sum_{\gamma\in\Gamma} e_\gamma(x) \cdot s_\gamma(x) = \sum_{\gamma\in\Gamma(x)} e_\gamma(x) \cdot s_\gamma(x).$$

Let us now check the properties (a)–(c). By (6), we have that:

$$\begin{aligned}\|s_\gamma(x) - f(x)\| \;&\leqslant\; \|s_\gamma(x) - y(x_\gamma)\| + \|y(x_\gamma) - f(x_\gamma)\| + \\ &\quad + \|f(x_\gamma) - f(x)\| < \sigma/3 + \varepsilon + 2\delta < \varepsilon^*.\end{aligned}$$

Hence $s_\gamma(x) \in D(f(x), \varepsilon^*)$, for every $x \in X$ and every $\gamma \in \Gamma(x)$. So, $f^*(x)$ is ε^*-close to $f(x)$, because $f^*(x)$ lies in the convex hull of points of all which are ε^*-close to $f(x)$. Therefore (a) is proved.

Let us verify (b). If $x \in X \setminus G$, then there exists a single index $\gamma \in \Gamma$ such that $x \in T_\gamma$. Hence

$$f^*(x) = s_\gamma(x) \in F(x), \quad \operatorname{dist}(f^*(x), F(x)) = 0.$$

As in the proof of (a) above, we see that $f^*(x) \in \operatorname{conv}\{D(f(x), \varepsilon^*) \cap F(x)\}$. So, for each $x \in X \setminus Z$ we obtain

$$\operatorname{dist}(f^*(x), F(x)) \leqslant \alpha_{F(x)}(\varepsilon^*) \cdot \varepsilon^* < \alpha(\varepsilon^*) \cdot \varepsilon^*,$$

due to the definition of the nonconvexity function.

So, in the third possible case, where $x \in Z \cap G$, we fix $x_0 \in X \setminus Z$ such that $x \in G(x_0)$, i.e., $f^*(x) \in \Delta(x) \subset \Delta(x_0)$, by (9). Choose an element y_0 from the $\eta(x_0)$-net $N(x_0)$ of the simplex $\Delta(x_0)$ such that

$$\|y_0 - f^*(x)\| < \eta(x_0) = \eta_0.$$

By the construction of the neighborhood $G(x_0)$ it follows that

$$D(y_0, \operatorname{dist}(y_0, F(x_0)) + \eta_0) \cap F(x) \neq \emptyset.$$

Therefore, the ball $D(f^*(x), \operatorname{dist}(y_0, F(x_0)) + 2\eta_0)$ intersects $F(x)$. For the distance $\operatorname{dist}(y_0, F(x_0))$ we have the upper estimate $\alpha_{F(x_0)}(\varepsilon^*) \cdot \varepsilon^*$, because $x_0 \in X \setminus Z$ and $y_0 \in \Delta(x_0) \subset \operatorname{conv}\{F(x_0) \cap D(f(x_0), \varepsilon^*)\}$. Hence

$$\operatorname{dist}(f^*(x), F(x)) \leqslant \alpha_{F(x_0)}(\varepsilon^*) \cdot \varepsilon^* + 2\eta_0 < \alpha(\varepsilon^*) \cdot \varepsilon^*,$$

which completes the verification of (b).

In checking (c), our argument reminds one of the techniques from [5] (see, also [11, Part A, §1.3]). So, having the partition of unity $\{e_\gamma\}$ from (11), we fix $x \in X$ and let

$$A(x) = \{\gamma \in \Gamma \mid e_\gamma(x) > 0\}, \qquad B(x) = \{\gamma \in \Gamma \mid x \in \operatorname{supp}(e_\gamma)\},$$

$$G_0(x) = \Big(\bigcap\{\mathrm{Int}(\mathrm{supp}(e_\gamma)) \mid \gamma \in A(x)\}\Big) \setminus \Big(\bigcup\{\mathrm{supp}(e_\gamma) \mid \gamma \notin B(x)\}\Big).$$

It is well known that for all points x' from the neighborhood $G_0(x)$, the following inclusions hold:

$$A(x) \subset A(x') \subset B(x') \subset B(x).$$

Hence, for $x' \in G_0(x)$ we have

$$\begin{aligned} f^*(x') - f^*(x) &= \sum_{\gamma \in A(x')} e_\gamma(x') \cdot s_\gamma(x') - \sum_{\gamma \in A(x)} e_\gamma(x) \cdot s_\gamma(x) \\ &= \sum_{\gamma \in A(x)} [e_\gamma(x') - e_\gamma(x)] \cdot s_\gamma(x) + \\ &\quad + \sum_{\gamma \in A(x)} e_\gamma(x') \cdot [s_\gamma(x') - s_\gamma(x)] + \\ &\quad + \sum_{\gamma \in A(x') \setminus A(x)} e_\gamma(x') \cdot s_\gamma(x'). \end{aligned}$$

The first item is the sum of a fixed finite number of mappings which are continuous on $G_0(x)$ and each is zero at the point x. Hence, in some smaller neighborhood of x, the norm of this item is less than $\sigma/6$. The norm of the second item is less than $2\sigma/3$ because points $s_\gamma(x')$ and $s_\gamma(x)$ are $\sigma/3$-close to points $y(x_\gamma)$ (see (5)). Finally, the norm of the third item is less than or equal to $\sum_{\gamma \in B(x) \setminus A(x)} e_\gamma(x') \cdot \|s_\gamma(x')\|$.

But in the last sum, the index set $B(x) \setminus A(x)$ is a fixed finite set, all real-valued functions e_γ are continuous and equal to zero at the point x, and all real-valued functions $\|s_\gamma\|$ are bounded at x. Hence, in some neighborhood of x, the norm of the third item is less than $\sigma/6$. Finally, for all points x' sufficiently close to x, we obtain that $\|f^*(x') - f^*(x)\| < \sigma$. Theorem 2.1 is thus proved. □

Below we need the following slight generalization of Theorem 2.1

THEOREM 2.1′. *If we add to the assumptions of Theorem* 2.1 *the property that over some closed* $A \subset X$ *there exists a continuous selection* s: $A \to Y$ *of the restriction* $F|_A$, *then the conclusions of Theorem* 2.1 *can be expanded by the property*

(d) $\|f^*(a) - s(a)\| < \alpha(\varepsilon^*)\varepsilon^*$, $a \in A$.

Proof. If F_s: $X \to Y$ coincides with F over $X \setminus A$ and $F_s(a) = \{s(a)\}$, $a \in A$, then Theorem 2.1 is applicable to the lower semicontinuous mapping F_s, due to the fact that functions of nonconvexity of singletons are identically equal to zero. □

3. Proof of Theorem C: From ε-Selections to Selections

Starting from a continuous ε-selection g_0, we obtain a continuous selection g and estimate the distance between g_0 and g. The proof consists of the inductive repetitions of previous Theorem 2.1 and in the steps of this procedure we use *discontinuous* singlevalued mappings.

THEOREM 3.1. *Let X, Y, F, Z be as in Theorem* C *and let ε_0 and τ be positive numbers. Suppose that the function β: $(0,\infty) \to (0,\infty)$ is geometrically summable at the point ε_0 and that $\beta(\cdot)$ strongly majorates the function* $(\sup\{\alpha_{F(x)}(\cdot) \mid x \in X \setminus Z\})^+$ *at the sequence of points $\{\varepsilon_n = \beta^{[n]}(\varepsilon_0)\}_{n=0}^{\infty}$. Then, for each continuous ε_0-selection g_0 of the mapping F there exists a continuous selection g of F such that $\|g(x) - g_0(x)\| < \tau + \beta^{\infty}(\varepsilon_0)$.*

Proof. Let $\gamma(\cdot)$ be the pointwise supremum of the set $\{\alpha_{F(x)}(\cdot) \mid x \in X \setminus Z\}$ of functions of nonconvexity. By the assumptions of Theorem 3.1, $(\beta - \gamma^+)$ is positive at every point $\varepsilon_n = \beta^{[n]}(\varepsilon_0)$, $n \in \mathbb{N}$. The following lemma is clear – it suffices to observe that ε_n converges to zero and that $(\beta - \gamma^+)(\varepsilon_n)$ are fixed positive numbers.

LEMMA 3.2. *There exists a positive function ν: $(0,\infty) \to (0,\infty)$ such that*

$$\nu^+(\varepsilon_n) < (\beta - \gamma^+)(\varepsilon_n), \quad n \in \mathbb{N}.$$

So, the function $\alpha = \gamma + \nu$ is a pointwise strong majorant of each function of nonconvexity $\{\alpha_{F(x)}(\cdot) \mid x \in X \setminus Z\}$) and Theorem 2.1 applies to this function. Moreover, by Lemma 3.2, we have that

$$\alpha^+(\varepsilon_n) = \gamma^+(\varepsilon_n) + \nu^+(\varepsilon_n) < \beta(\varepsilon_n), \quad n \in \mathbb{N}.$$

Step 1. Function β majorates the upper right limit of the function α at the point ε_0. So, one can choose $\tau_0 > 0$ such that the inclusion $\varepsilon \in (\varepsilon_0, \varepsilon_0 + \tau_0)$ implies the inequality

$$\alpha(\varepsilon) \cdot \varepsilon < \beta(\varepsilon_0) \cdot \varepsilon_0 = \beta^{[1]}(\varepsilon_0).$$

Pick positive numbers δ_0 and δ_1 such that $2\delta_0 + \delta_1 < \tau_0$. Note that g_0 is continuous and hence δ_0-continuous, too. So Theorem 2.1 applies to the mapping F in the case where:

$$f = g_0, \quad \varepsilon = \varepsilon_0, \quad \delta = \delta_0, \quad \sigma = \delta_1,$$
$$\varepsilon^* = \varepsilon_0 + 2\delta_0 + \delta_1 \in (\varepsilon_0,\ \varepsilon_0 + \tau_0).$$

Therefore, there exists a δ_1-continuous $(\alpha(\varepsilon^*) \cdot \varepsilon^*)$-selection (hence $(\beta^{[1]}(\varepsilon_0))$-selection) g_1 of the mapping F such that

$$\|g_1(x) - g_0(x)\| < \varepsilon^* < \varepsilon_0 + \tau_0, \quad x \in X.$$

Step 2. The function β majorates the upper right limit of the function α at the point $\varepsilon_1 = \beta^{[1]}(\varepsilon_0)$. So one can choose $\tau_1 > 0$ such that the inclusion $\varepsilon \in (\varepsilon_1, \varepsilon_1 + \tau_1)$ implies the inequality

$$\alpha(\varepsilon) \cdot \varepsilon < \beta(\varepsilon_1) \cdot \varepsilon_1 = \beta^{[2]}(\varepsilon_0).$$

If $2\delta_1 \geqslant \tau_1$, then we simply decrease δ_1 in Step 1 in such a manner that $2\delta_1 < \tau_1$ (this is always possible, due to the independence of δ_1 of δ_0). Next, we pick a positive number δ_2 such that $2\delta_1 + \delta_2 < \tau_1$. The ε_1-selection g_1 is δ_1-continuous. So Theorem 2.1 applies to the mapping F in the case where:

$$f = g_1, \quad \varepsilon = \varepsilon_1, \quad \delta = \delta_1, \quad \sigma = \delta_2,$$
$$\varepsilon^* = \varepsilon_1 + 2\delta_1 + \delta_2 \in (\varepsilon_1,\ \varepsilon_1 + \tau_1).$$

Therefore, there exists a δ_2-continuous $(\alpha(\varepsilon^*) \cdot \varepsilon^*)$-selection (hence, $(\beta^{[2]}(\varepsilon_0))$-selection) g_2 of the mapping F such that

$$\|g_2(x) - g_1(x)\| < \varepsilon^* < \varepsilon_1 + \tau_1, \quad x \in X.$$

The continuation of such a procedure gives a sequence of δ_n-continuous $\beta^{[n]}(\varepsilon_0)$-selections $g_n\colon X \to Y$ of the mapping F such that $2\delta_n < \tau_n$ and

$$\|g_{n+1}(x) - g_n(x)\| < \beta^{[n]}(\varepsilon_0) + \tau_n, \quad x \in X.$$

To complete the proof, it suffices to choose a sequence $\tau_0, \tau_1, \tau_2, \ldots$ with the additional property that $\sum_{n=0}^{\infty} \tau_n < \tau$ and set $g(x) = \lim_{n\to\infty} g_n(x)$. Theorem 3.1 is thus proved. □

Analogously to Section 2, we have the following generalization of Theorem 3.1.

THEOREM 3.1′. *If we add to the assumptions of Theorem* 3.1 *the property that over some closed $A \subset X$ there exists a continuous selection $s\colon A \to Y$ of the restriction $F|_A$, then the conclusions of Theorem* 3.1 *can be strengthened by the property that selection g of F also extends s.*

Proof. As in the proof of Theorem 2.1′, we apply Theorem 3.1 to the lower semicontinuous mapping $F_s\colon X \to Y$ which coincides with F over $X \setminus A$ and $F_s(a) = \{s(a)\}$, $a \in A$. By Step 1 in the proof of Theorem 3.1 we have that:

$$\|g_1(a) - s(a)\| < \alpha(\varepsilon^*) \cdot \varepsilon^* < \beta^{[1]}(t), \quad a \in A,$$

because g_1 is a $(\alpha(\varepsilon^*) \cdot \varepsilon^*)$-selection (hence a $(\beta^{[1]}(\varepsilon_0))$-selection) of the mapping F_s. Similary, in Step 2 of the proof we additionally obtain that

$$\|g_2(a) - s(a)\| < \beta^{[2]}(t), \qquad a \in A.$$

Hence, for the final selection g of G we have $g(a) = \lim_{n\to\infty} g_n(a) = s(a)$, $a \in A$, i.e., g is an extension of s. □

4. End of Proof of Theorem C

In view of Theorems 2.1 and 3.1 it suffices to find an arbitrary continuous ε-selection of F for some $\varepsilon > 0$ with $\beta^{\infty}(\varepsilon) < \infty$ and then improve it to a genuine selection. We begin by performing such an improvement over a proper subset A of the domain X. Using Theorem 3.1 it is possible to repeat such an improvement over some larger subset A', $A \subset A' \subset X$, for a bigger $\varepsilon' > \varepsilon$. So, we prove the theorem by performing such extensions for a suitable sequence $A_0 \subset A_1 \subset A_2 \subset \cdots$ which cover the whole domain X. Here we exploit an idea from [4].

Proof. We fix an increasing sequence of positive numbers $\{t_n\}_{n=0}^{\infty}$ such that $\beta^{\infty}(t_n) < \infty$ and $t_{n+1} > 1 + \beta^{\infty}(t_n)$ for all $n \in \mathbb{N}$. The existence of such a sequence follows directly from weak g-summability of the function $\beta(\cdot)$ (cf. Definition 1.2(c)).

Consider an arbitrary continuous singlevalued mapping $h\colon X \to Y$. Clearly, one can additionally assume that

$$G_0 = \{x \in X \mid D(h(x), t_0) \cap F(x) \neq \emptyset\} \neq \emptyset.$$

Inequalities $t_n > n$ imply that, in the domain X the sets

$$G_n = \{x \in X \mid D(h(x), t_n) \cap F(x) \neq \emptyset\}$$

constitute a sequence of open subsets $\{G_n\}_{n=0}^{\infty}$ such that

$$\emptyset \neq G_0 \subset G_1 \subset \cdots \subset G_n \subset \cdots, \quad \bigcup_{n=0}^{\infty} G_n = X.$$

Paracompactness (in fact, countable paracompactness) of X guarantees in X the existence of a sequence of closed subsets $\{A_n\}_{n=0}^{\infty}$ such that

$$\emptyset \neq A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots, \quad \bigcup_{n=0}^{\infty} A_n = X,\ A_n \subset G_n.$$

Moreover, we can assume that the family $\{A_n\}$ is locally finite.

We now apply Theorem 3.1 to the paracompact space A_0, the lower semicontinuous mapping $F_0 = F|_{A_0}$, the above chosen number t_0, the continuous t_0-selection $h_0 = h|_{A_0}$ of the mapping F_0, and the strong majorant $\beta(\cdot)$ of the function $(\sup\{\alpha_{F(x)}(\cdot) \mid x \in X \setminus Z\})^{+}$. So, setting $\tau = 1$ in Theorem 3.1, we can find a continuous selection $f_0\colon A_0 \to Y$ of the mapping F_0 such that

$$\|f_0(x) - h_0(x)\| < 1 + \beta^{\infty}(t_0) < t_1, \quad x \in X.$$

Next we pass to the closed subset $A_1 \subset X$. Let us consider the lower semicontinuous multivalued mapping, say F_1, over this paracompact space, which coincides with f_0 over the closed subset $A_0 \subset A_1$, and with F over the complement

$A_1 \setminus A_0$. We claim that the restriction $h_1 = h|_{A_1}$ is a t_1-selection of F_1. In fact, for $x \in A_0$ we have from the previous inequality that

$$\operatorname{dist}(h_1(x), F_1(x)) = \|h_0(x) - f_0(x)\| < t_1.$$

In the case $x \in A_1 \setminus A_0$, it is clear that

$$\operatorname{dist}(h_1(x), F_1(x)) = \operatorname{dist}(h(x), F(x)) < t_1$$

because

$$A_1 \subset G_1 = \{x \in X \mid D(h(x), t_1) \cap F(x) \neq \emptyset\}.$$

Hence, Theorem 3.1 applies to the paracompact space A_1, the lower semicontinuous mapping F_1, the number t_1, and the continuous t_1-selection h_1 of the mapping F_1. Thus, setting $\tau = 1$ in this theorem, we can find a continuous selection $f_1\colon A_1 \to Y$ of the mapping F_1 which extends f_0 and for which

$$\|f_1(x) - h_1(x)\| < 1 + \beta^{\infty}(t_1) < t_2, \quad x \in X.$$

Inductive repetition gives a sequence $\{f_n\}_{n=0}^{\infty}$ of continuous mappings into Y such that the domain of f_n equals to A_n, each f_{n+1} extends the previous f_n, and each f_n is a selection of F over A_n. For every $x \in X$ there exists $\min\{n \mid x \in A_n\} = n_x$ and for this minimal index we have

$$f_{n_x}(x) = f_{n_x+1}(x) = \cdots = f(x) \in F(x).$$

So, the pointwise limit f of the sequence $\{f_n\}_{n=0}^{\infty}$ is a selection of F. The local finiteness of the family $\{A_n\}_{n=0}^{\infty}$ and the continuity of each f_n imply the continuity of f. Theorem C is thus proved. □

In contrast to Theorem 3.1 note that in the last step we did not estimate the distance between the initial mapping h and the final selection f.

5. Proofs of Theorems D and E

By virtue of Proposition 5.1 from [6], in order to prove Theorem D it suffices to check the so-called *selection extension property* and *selection approximation property* for the mapping $G = \mathrm{Cl}(F)$, which associates the closure $\mathrm{Cl}(F(x))$ of the set $F(x)$ with each $x \in X$.

DEFINITION 5.1. *Selection extension property* (*SEP*) of a mapping $G\colon X \to Y$ means that, for every closed $A \subset X$ each partial continuous selection $s\colon A \to Y$ of the restriction $G|_A$ admits a continuous extension $\tilde{s}\colon X \to Y$ which, in turn, is a selection of G.

In order to check the SEP it suffices (as in Theorems 2.1′ and 3.1′) to consider the mapping G_s. The functions of nonconvexity of singletons are identically zero. Hence Theorem C applies to the mapping G_s. So a selection of G_s, say $\tilde{s}$, is the desired extension of s. Therefore SEP has been proved for the mapping $G = \mathrm{Cl}(F)$.

DEFINITION 5.2. *Selection approximation property* (*SAP*) of a mappping F: $X \to 2^Y$, with Y a metric space, means that for every $\varepsilon > 0$ there corresponds $\delta = \delta_F(\varepsilon) > 0$ satisfying the following condition: If h: $X \to Y$ is continuous with $d(h, F) < \delta$ and if $A \subset X$ is closed, then every selection g for $F|_A$ with $d(g, h|_A) < \delta$ extends to a selection f of F with $d(f, h) < \varepsilon$. Here, d is the metric on Y and $d(g, F) < \varepsilon$ means that $d(g(x), F(x)) < \varepsilon$ for all $x \in X$. For more on this property see Section 5 in [6].

In order to check the SAP for $G = \mathrm{Cl}(F)$, let s be a selection of $G|_A$, where A is a closed subset of X, and let a positive ε be given. Pick a number $t > 0$ such that $\beta^\infty(t) = \sum_{n=0}^\infty \beta^{[n]}(t) < \infty$. By Lemma 1.3, $\sum_{n=N}^\infty \beta^{[n]}(t) = \beta^\infty(\beta^{[N]}(t))$ and therefore $\beta^\infty(\beta^{[N]}(t)) < \varepsilon/2$ for some $N \in \mathbb{N}$.

The number $\delta = \beta^{[N]}(t)$ is positive due to the positivity of the function β. Now, let g_0 be a continuous δ-selection of the mapping G_s. Then $\|g_0(a) - s(a)\| < \delta$, $a \in A$. Apply Theorem 3.1′ to the multivalued mapping G_s and its continuous ε_0-selection g_0, where $\varepsilon_0 = \delta$, and $\tau = \varepsilon/2$. Then we obtain a continuous selection g of G_s (and hence an extension of s) such that

$$\|g(x) - g_0(x)\| < \tau + \beta^\infty(\delta) < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ we find $\delta > 0$ with the property that for every closed subset $A \subset X$ and every selection s of $G|_A$, from each δ-selection g_0 of the mapping G_s one can obtain a selection g of G_s such that $\mathrm{dist}(g, g_0) < \varepsilon$. The latter means that by definition [6] the mapping $G = \mathrm{Cl}(F)$ has SAP. Theorem D is thus proved.

In a similar way one can prove the analog of Theorem 3.1 ‘modulo’ a countable subset of the domain.

THEOREM D′. *Theorem* 3.1 *holds for any mapping F whose values are closed except for at most countably many points of the domain.*

In order to prove Theorem E, we introduce another, very natural, selection-type property of multivalued mappings. First, observe that SAP for an empty subset $A \subset X$ simply means that for each $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for each continuous δ-selection g of F there exists a continuous selection f of F such that $\mathrm{dist}(f(x), g(x)) < \varepsilon$, $x \in X$. In other words, the mapping F has the *selection improvement property* (*SIP*).

DEFINITION 5.3. A multivalued mapping F: $X \to Y$ into a metric space Y is said to have the *hereditary selection improvement property* (*HSIP*) if for each $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that for each closed $A \subset X$ and for each continuous δ-selection g of $F|_A$ there exists a continuous selection f of $F|_A$ such that $\mathrm{dist}(f(a), g(a)) < \varepsilon$, $a \in A$.

We wish to emphasize that, in comparison with SAP, we do not fix here values of f over some prescribed subset $B \subset A$. So, we derive Theorem E from the following ‘conditional’ selection theorem:

THEOREM 5.4. *Suppose that a multivalued mapping $F\colon X \to Y$ from a perfectly normal space X into a separable metric space Y has SEP and HSIP. Then there exists a countable family $\{f_w\}_{w\in W}$, $|W| = \aleph_0$, of continuous selections of F such that the set $\{f_w(x)\}_{w\in W}$ is dense in $F(x)$, for each $x \in X$.*

Proof. The desired countable index set W for a desired family of selections will consist of certain triples (i, j, k) of natural numbers. The first argument i is the index of a dense countable subset $\{y_i\}$ of Y. The second argument j is the index of a sequence $\{r_j\}$ of positive numbers, converging to zero.

Denote by $\delta_j = \delta(r_j)$ the positive numbers chosen with respect to HSIP. Due to the perfect normality of the domain X the open set $G_{ij} = F^{-1}(D(y_i, \delta_j))$ can be represented for each pair (i, j) as a union $G_{ij} = \bigcup_{k=1}^{\infty} A_{ijk}$ of closed (in X) subsets. So we set

$$W = \{(i, j, k) \mid A_{ijk} \neq \emptyset\} \subset \mathbb{N}^3.$$

Pick an element $w = (i, j, k) \in W$ and denote A_{ijk} by A_w. Consider the restriction $F|_{A_w} = F_w$. By construction, the constant mapping from A_w to the point y_i is a continuous δ_j-selection of F_w. By HSIP we can improve this selection to a genuine selection, say h_w, of the mapping F_w such that $\mathrm{dist}(h_w(x), y_j) < r_j$, $x \in A_w$. By SEP we can extend h_w over the whole domain, i.e., we can find a selection f_w of F such that $f_w|_{A_w} \equiv h_w$.

We claim that $\{f_w\}_{w\in W}$ is the desired countable family of selections of F. Indeed, let us fix $x \in X$, $y \in F(x)$ and the second natural argument j. Then there exists an element y_i from the above chosen dense countable subset of Y such that $\mathrm{dist}(y, y_i) < \delta_j$. Hence $x \in G_{ij}$ and $x \in A_{ijk} = A_w$ for some natural k. We now estimate the distance between points y and $f_w(x)$. So, having $x \in A_w$ we obtain that

$$\mathrm{dist}(f_w(x), y) = \mathrm{dist}(h_w(x), y) \leqslant \mathrm{dist}(h_w(x), y_i) + \mathrm{dist}(y_i, y) < r_j + \delta_j.$$

Hence $\mathrm{dist}(f_w(x), y) < 2r_j \to 0$, $j \to \infty$. Theorem 5.4 is thus proved. □

Observe that in fact we used a weak version of HSIP because in the proof we worked only with the *constant* δ-selections.

Proof of Theorem E. As it was pointed out in [6], SEP and SAP for $\mathrm{Cl}(F)$ imply SEP for F, whenever the values of F are closed except for at most countably many points of a paracompact domain X. We checked SEP and SAP for $\mathrm{Cl}(F)$ in proving Theorem D. Hence we conclude that F has SEP.

By hypotheses of the theorem we have that $\beta(r) \leqslant q < 1$, for some constant q and for all sufficiently small $r > 0$. Hence, for all such r the series $\sum_{k=0}^{\infty} \beta^{[k]}(r) = \beta^{\infty}(r)$ is majorated by the usual geometric series $\sum_{k=0}^{\infty} q^k r = r\frac{1}{1-q}$. Therefore $\beta^{\infty}(r) \to 0$, whenever $r \to 0$. Thus for a fixed $\varepsilon > 0$ we can find a positive number δ such that the majorant $\beta(\cdot)$ is geometrically summable at the point $t = \delta$ and

$$\beta^{\infty}(r) < \frac{\varepsilon}{2}, \quad r \in (0, \delta].$$

So let A be a closed subset of X. Hence A is a paracompact space. Then $A \cap C \subset C$ and hence is at most countable. Clearly, $\dim_A(A \cap Z) \leqslant 0$, by closedness of A. Thus, if g is a continuous δ-selection of $F|_A$, then Theorem D$'$ is applicable to the restriction $F|_A$ with $\varepsilon_0 = \delta$ and $\tau = \varepsilon/2$, i.e., one can find a continuous selection f of $F|_A$ such that

$$\|f(a) - g(a)\| < \tau + \beta^{\infty}(\delta) < \varepsilon, \quad a \in A.$$

Therefore we have checked HSIP for F and Theorem E now follows Theorem 5.4. $\square$

Observe that in order to prove Theorem E for perfectly normal (not necessarily paracompact) domains, one must prove that Theorem C and Theorem D hold for normal countably paracompact domains and verify that for such domains SEP and SAP for $\mathrm{Cl}(F)$ imply SEP for F. Then, in the proof of Theorem E the closed subset A of perfectly normal space X will also be perfectly normal and hence it will be normal and countably paracompact and one can use generalizations of Theorems C and D. Similar changes must be included for normal τ-paracompact domains.

We believe that this way differs from the above case of paracompact domains only in technical details. We finish by stating two problems concerning the relations between SEP, SAP, and HSIP.

QUESTION 5.5. *Does SAP imply HSIP?*

QUESTION 5.6. *Do SEP and HSIP together imply SAP?*

For using SAP in (a) we must know that a given δ-selection over the closed subset $A \subset X$ is extendable to a δ-selection over the whole domain X. But it is not clear why such an extension should exist. In (5.6), SEP gives no information about upper estimates of the distance between the initial δ-selection over X and the resulting selection. On the other hand, HSIP guarantees such an upper estimate, but, in general, HSIP does not preserve values of partial selections over $A \subset X$. Therefore we suspect that the answers to both questions are in fact negative.

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References

1. Ben-El-Mechaiekh, H. and Oudadess, M.: Some selection theorems without convexity, *J. Math. Anal. Appl.* **195** (1995), 614–618.
2. Horvath, C. D.: Contractibility and generalized convexity, *J. Math. Anal. Appl.* **156** (1991), 341–357.
3. Michael, E.: Selected selection theorems, *Amer. Math. Monthly* **63** (1956), 233–238.
4. Michael, E.: Paraconvex sets, *Scand. Math.* **7** (1959), 372–376.
5. Michael, E.: Convex structures and continuous selections, *Canad. J. Math.* **11** (1959), 556–575.
6. Michael, E.: Continuous selections and countable sets, *Fund. Mat.* **111** (1981), 1–10.
7. Michael, E. and Pixley, C. P.: A unified theorem on continuous selections, *Pacific J. Math.* **87** (1980), 187–188.
8. Repovš, D. and Semenov, P. V.: On functions of nonconvexity for graphs of continuous functions, *J. Math. Anal. Appl.* **196** (1995), 1021–1029.
9. Repovš, D. and Semenov, P. V.: On paraconvexity of graphs of continuous functions, *Set-Valued Anal.* **3** (1995), 23–32.
10. Repovš, D. and Semenov, P. V.: On nonconvexity of graphs of polynomials of several real variables, *Set-Valued Anal.* **6** (1998), 39–60.
11. Repovš, D. and Semenov, P. V.: *Continuous Selections of Multivalued Mappings*, Math. Appl. 455, Kluwer Acad. Publ., Dordrecht, 1998.
12. Semenov, P. V.: Functionally paraconvex sets, *Mat. Zametki* **54**(6) (1993), 74–91 (in Russian); Engl. transl. in *Math. Notes* **54**(6) (1993), 1236–1240.
13. Semenov, P. V.: On paraconvexity of star-like sets, *Sibirsk. Mat. Zh.* **37**(2) (1996), 399–405 (in Russian); Engl. transl. in *Siberian Math. J.* **37**(2) (1996), 344–349.
14. Semenov, P. V.: Fixed-points theorems for a controlled withdrawal of convexity of values of a set-valued map, *Mat. Sb.* **189**(3) (1998), 141–160 (in Russian); Engl. transl. in *Russian Acad. Sci. Sb. Math.* **189**(3) (1998), 461–480.