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# On Nonconvexity of Graphs of Polynomials of Several Real Variables

DUŠAN REPOVŠ
*Institute of Mathematics, Physics and Mechanics, University of Ljubljana, PO Box 2964, Ljubljana, Slovenia 1001, e-mail: dusan.repovs@fmf.uni-lj.si*

PAVEL V. SEMENOV
*Moscow State Pedagogical University, Ul. M. Pyrogovskaya 1, Moscow, 119882 Russia e-mail: semenov.matan@mpgu.msk.su*

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**Abstract.** We consider transversal (orthogonal) perturbations of finite-dimensional convex sets and estimate the 'degree of nonconvexity' of resulting sets, i.e. we estimate the nonconvexity of graphs of continuous functions. We prove that a suitable estimate of nonconvexity of graphs over all lines induces a 'nice' estimate of the nonconvexity of graphs of the entire function. Here, the term 'nice' means that in the well-known Michael selection theorem it is possible to replace convex sets of a multivalued mapping by such nonconvex sets. As a corollary, we obtain positive results for polynomials of degree two under some restrictions on coefficients. Our previous results concerned the polynomials of degree one and Lipschitz functions. We show that for a family of polynomials of degree three such estimate of convexity in general does not exist. Moreover, for degree 9 we show that the nonconvexity of the unique polynomial $P(x, y) = x^9 + x^3 y$ realizes the worst possible case.

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## 1. Introduction

There is a principal difference between selection theorems for multivalued mappings with finite-dimensional domains compared to those whose domains are infinite-dimensional. To illustrate the point, we recall the arguments from [5]. The first one of them is, in fact, a reformulation of the selection theorem from [3]:

THEOREM 1.1 [5, Theorem 2.4]. *Let $Y$ be a completely metrizable space and $\mathcal{L}$ a hereditary family of closed nonempty subsets of $Y$ (i.e. $L \in \mathcal{L}$ and $y \in L$ imply that $\{y\} \in \mathcal{L}$). Then the following statements are equivalent:*

(a) *For every paracompact space $X$ with $\dim X \leqslant n + 1$, every lower semicontinuous mapping $F\colon\ X \to \mathcal{L}$ has a selection and*

(b) *Every* $L \in \mathcal{L}$ *is* $n$*-connected* (*i.e.* $\mathcal{L} \subset \mathcal{C}^n$) *and if* $X$ *is paracompact and* $\dim X \leqslant n+1$, *then every lower semicontinuous map* $F\colon X \to \mathcal{L}$ *has a local selection.*

Is it possible to find a purely *topological* analogue of Theorem 1.1 without any dimensional restrictions? Due to the well-known example from [6], the answer to this question is, in general, negative. In fact, Michael [5] derived from [6] the following result, related to Theorem 1.1:

THEOREM 1.2 [5, Theorem 2.8]. *There is no class* $\mathcal{C}$ *of topological spaces such that the following assertions would be equivalent for every completely metrizable space* $Y$ *and every hereditary family* $\mathcal{L}$ *of its closed nonempty sets:*

(a) *If* $X$ *is a paracompact space then every lower semicontinuous mapping* $F\colon X \to \mathcal{L}$ *has a selection and*
(b) *The inclusion* $\mathcal{L} \subset \mathcal{C}$ *holds and for every paracompact space* $X$, *every lower semicontinuous map* $F\colon X \to \mathcal{L}$ *has a local selection.*

In view of these facts, one tries to change the topological requirement in the question above by substituting some *metric* restrictions. The present paper is an attempt in this direction. The key ingredient of our work is the notion of the function of *nonconvexity*: to every nonempty closed subset $A \subset X$ of a Banach space $(X, \|\cdot\|)$ we associate some function $h_A\colon (0,\infty) \to [0,2]$. The identity $h_A \equiv 0$ is equivalent to the convexity of $A$ and the more $h_A$ differs from zero, the less 'convex' is $A$.

The notion of the function of nonconvexity was first introduced in [8] as a generalization of the concept of *paraconvexity* (cf. [4]). An intermediate notion of *functional paraconvexity* was proposed earlier in [10]. Note also that for a Hilbert space $X$, values of a function of nonconvexity always lie on the interval $[0,1]$. Moreover, the result of Klee [1] shows that if $\dim X > 2$ and if for each closed $A \subset X$, the values of $h_A$ are in $[0,1]$, then $X$ is isometric to a Hilbert space.

THEOREM 1.3 [8]. *Let* $h\colon (0,\infty) \to [0,1)$ *be a monotone increasing mapping and* $F\colon X \to Y$ *a closed-valued lower semicontinuous mapping from a paracompact space* $X$ *into a Banach space* $Y$. *Suppose that* $h_{F(x)}(r) < h(r)$, *for every* $x \in X$ *and* $r > 0$. *Then* $F$ *has a continuous selection.*

The exact evaluation of the function of nonconvexity or finding a nice majorant for it is a nontrivial task. Some positive results were obtained for the case when $A$ is the graph of a continuous function with convex domain. We consider a convex set (domain) and then perturb it along an additional direction, orthogonal to the given convex set. Of course, here we work in a Hilbert space and, moreover, in the present paper we shall only deal with finite-dimensional Euclidean spaces.

THEOREM 1.4 [7]. (a) *For every* $n \in \mathbb{N}$ *and* $C \geqslant 0$, *there exists a constant* $\alpha \in [0, 1)$ *which is a majorant for the functions of nonconvexity of graphs of an arbitrary Lipschitz* (*with constant* $C$) *functions of* $n$ *real variables with convex closed domain; and*

(b) *The assertion of* (a) *holds for graphs of functions* $f$ *with monotone restrictions* $f|_\ell$, *for every one-dimensional line* $\ell$.

THEOREM 1.5 [8]. *Let* $m \in \mathbb{N}$ *and* $C \geqslant 1$. *Then there exists a monotone increasing function* $h$: $(0, \infty) \to [0, 1)$ *which is a majorant for the function of nonconvexity of the graph of an arbitrary polynomial* $P_m(x) = \sum_{i=0}^m a_i x^i$, *with* $|a_i| \leqslant C$ *and* $|a_i/a_m| \leqslant C$.

Clearly, Theorems 1.4 and 1.5, together with Theorem 1.3, yield some special selection theorems. The purpose of the present paper is first to present some new positive results on the function of nonconvexity of graphs of continuous functions. Here, the main result remains a relation between continuity of a function and its directional continuity, when 'global' restrictions automatically imply all 'one-dimensional' restrictions and the converse holds only under some additional assumptions (see Theorem 2.4 below). As a corollary, we obtain the generalization of Theorem 1.5 for polynomials $P_2(x_1, \dots, x_n)$, $n > 1$.

On the other hand, we also present some negative results, which show that degree of $P = 2$ is the greatest possible for an existence of a nice (in the sense of Theorem 1.3) estimate for function of nonconvexity of a graph of a polynomial $P$ in more than one variable. More precisely, we prove in Theorem 2.8 that if $P_\mu = P_3(x, y) = x^3 + \mu xy$, $0 < \mu < 1$, and $A_\mu = \Gamma(P_\mu)$ is the graph of $P_\mu$, then $\sup\{h_{A_\mu} \mid 0 < \mu < 1\}$ is identically equal to the unit function. Hence, for the family $\{x^3 + \mu xy \mid 0 < \mu < 1\}$ of polynomials of the *third* degree of *two* real variables, the analogue of Theorem 1.5 is, in general, false. Moreover, for higher degrees we have a stronger counterexample, i.e. the equality $h_{\Gamma(P)} \equiv 1$ over the whole ray $(0, \infty)$ holds for a *single* polynomial, $P = P_9(x, y) = x^9 + x^3 y$ (see Theorem 2.7 below). Note that such negative results are mainly the result of the behaviour 'at infinity', since for every convex *bounded* subset $V \subset \mathbb{R}^2$, the restrictions $P|_V$ are Lipschitz functions and thus Theorems 1.3 and 1.4 hold.

## 2. Statements of Results

Let $X$ be a Banach space. We shall denote:

(a) The closed convex hull $\overline{\mathrm{conv}}\{x_1, \dots, x_k\}$ of the points $x_1, \dots, x_k$ by $[x_1, \dots, x_k]$;
(b) The infimum of the radii of all closed balls containing all points $x_1, \dots, x_k$ (respectively, bounded set $A \subset X$) by $R[x_1, \dots, x_k]$ (respectively, $R[A]$); and
(c) For a Hilbert space $X$, the center of the unique closed ball of radius $R[A]$ which contains the bounded set $A \subset X$ by $c[A]$.

Note that $R[A]$ (resp. $c[A]$) is the *Čebyšev* radius (resp. *center* of the bounded set $A$).

DEFINITION 2.1. For a nonempty closed subset $A \subset X$ of a Banach space $(X, \|\cdot\|)$ and for a convex subset $E \subset X$ with $R[E] > 0$, we define the *relative* precision of an approximation of $A$ by elements of $E$ as follows:

$$\delta(E, A) = \sup\{\mathrm{dist}(q, A)/R[E] \mid q \in \mathrm{conv}(E \cap A)\},$$

where for the empty intersection $E \cap A$ we put $\delta(E, A) = 0$.

Clearly, for a convex set $E$ with nonempty intersection $E \cap A$, the equality $\delta(E, A) = 0$ means that the intersection $E \cap A$ is a convex subset of $A$.

DEFINITION 2.2. For a nonempty closed subset $A \subset X$ of a Banach space $(X, \|\cdot\|)$, the *value* of its function of nonconvexity $h_A$ at a point $r > 0$ is defined as follows:

$$h_A(r) = \sup\{\delta(D, A) \mid D \text{ is an open ball with } R[D] = r\}.$$

Sometimes, the approximations by convex hulls of finite subsets is more suitable:

LEMMA 2.3 [11]. *Definition* 2.2 *of the function $h_A(\cdot)$ of nonconvexity of the set $A$ admits the following equivalent definition:*

$$h_A(r) = \sup\{\delta(\mathrm{conv}\, E, A) \mid E \textit{ is a finite subset of } A \textit{ with } R(E) = r\}.$$

In the sequel, we shall denote by $\mathcal{M}_{<1}$ (monotone and less than 1) the set of all strictly increasing functions from $(0, \infty)$ to $[0, 1)$.

THEOREM 2.4. *For every integer $n \in \mathbb{N}$ and every function $h \in \mathcal{M}_{<1}$, there exists a function $H \in \mathcal{M}_{<1}$ with the following property: if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function with a closed convex domain $V$ and if for every line $\ell \subset \mathbb{R}^n$, the function $h_{\Gamma(F|_{\ell\cap V})}$ of nonconvexity of the graph of the restriction $f|_{\ell\cap V}$ is less than $h$, then the function $h_{\Gamma(f)}$ of nonconvexity of the entire graph $\Gamma(f)$ is less than $H$.*

As a corollary of Theorem 2.4, we obtain the following result:

THEOREM 2.5. *For every integer $n \in \mathbb{N}$ and every constant $C > 0$, there exists a function $H \in \mathcal{M}_{<1}$ with the following property: If $P_2$ is a polynomial in $n$ real variables of degree two, i.e. $P_2(x_1, \ldots, x_n) = \sum a_{ij}x_ix_j + \sum a_kx_k + a_0$, and if $\|a_{ij}\| \leqslant C$, for every $1 \leqslant i, j \leqslant n$, then $h_{\Gamma(P_2)} < H$.*

In the next theorem, we modify the '$y = \sin 1/x$' example from [2] in order to prove the essentiality of the restriction $|a_{ij}| \leqslant C$ in Theorem 2.5 (even for $n = 1$).

THEOREM 2.6. *There exists a continuous multivalued mapping $F\colon [0,1] \to \mathbb{R}^2$ with no continuous singlevalued selection, such that every value $F(t)$ of $F$ is either an interval or a connected path on a parabola.*

THEOREM 2.7. *The function of nonconvexity $h_{\Gamma(P)}$ for the polynomial $P(x,y) = x^9 + x^3y$ is identically equal to* 1.

THEOREM 2.8. *Let $P_\mu(x,y) = x^3 + \mu xy$, where $\mu > 0$. Then for every sequence $\mu_n \to 0$, the corresponding sequence of functions of nonconvexity of graphs of $P_{\mu_n}$ has a pointwise supremum, identically equal to* 1.

We complete this section by a geometric interpretation of the equality $h_A(r) = 1$, for closed subsets $A \subset X$ of a Hilbert space $X$. We say that a closed ball $D$ with center $c$ is *inscribed* into $A$ if $c \in \operatorname{conv}(D \cap A)$ and $(\operatorname{Int} D) \cap A = \emptyset$. We also say that a fixed closed ball $D$, centered at the origin, can be *approximatively* inscribed into $A$ if for every $0 < \lambda < 1$ there exists $c \in X$ such that $c \in \operatorname{conv}((c + D) \cap A)$ and $(c + \lambda D) \cap A = \emptyset$. Observe that in a Hilbert space the equality $h_A(r) = 1$ is equivalent to the fact that the closed ball of radius $r$, centered at the origin, can be approximately inscribed into $A$. Hence, in Theorem 2.7 each closed ball $D$ can be approximately inscribed into the graph of a given unique polynomial, $P_9(x,y) = x^9 + x^3y$.

For a summary of results, let $\{P_\alpha\}_{\alpha\in A}$ be a family of polynomials of degree $\leqslant m$ in $n$ real variables and let $\mathcal{L} = \{L \subset \mathbb{R}^{n+1} \mid L = T(\Gamma(P_\alpha))$, for some $\alpha \in A$ and for some isometry $T\}$. Let $F\colon X \to \mathcal{L}$ be a lower semicontinuous mapping defined on a paracompact space $X$. So we can give some answers to the following two questions:

QUESTION 2.9. Does there exist a majorant $h \in \mathcal{M}_{<1}$ for the set $\{h_{\Gamma(P_\alpha)}\}_{\alpha\in A}$?

QUESTION 2.10. Does there exist a continuous singlevalued selection of $F$?

| $m$ | $n$ | Restrictions on coefficients of $P_\alpha$ | Question 2.9 | Question 2.10 | References |
|---|---|---|---|---|---|
| 1 | Arbitrary | None | Yes | Yes | [2] |
| 2 | 1 | None | In general, No | In general, No | Theorem 2.6 |
| Arbitrary | 1 | $\lvert a_i\rvert \leqslant c$, $\lvert a_i/a_m\rvert \leqslant c$ | Yes | Yes | Theorem 1.5 |
| 2 | Arbitrary | $\lvert a_{ij}\rvert \leqslant c$ | Yes | Yes | Theorem 2.5 |
| 3 | 2 | $\lvert a_i\rvert \leqslant 1$ | In general, No | Unknown | Theorem 2.8 |
| 9 | 2 | Unique $P$ | $h_{\Gamma(P)} \equiv 1$ possible | Unknown | Theorem 2.7 |

## 3. Proof of Theorem 2.4

Hilbert spaces have many advantages over Banach spaces. One of them, which will be useful for us, is the fact that for an estimate of the function of nonconvexity of a subset $A$ in a Hilbert space it suffices to control only the distances between $A$ and Čebyšev centers of simplices with vertices in $A$. Compare this with Definition 2.1, where supremum was taken over *all* points of $\mathrm{conv}(E \cap A)$. More precisely, we have the following lemma (see [11]). Here, we shall use the term '$h$-paraconvex set' for a set $A$ with $h_A \leqslant h$.

LEMMA 3.1. *Let $h \in \mathcal{M}_{<1}$ and suppose that for a closed subset $A \subset X$ of a Hilbert space $X$ the following property holds: for every integer $m \in \mathbb{N}$ and every $m$-tuple of points $x_1, \ldots, x_m \in A$, the distance $\mathrm{dist}(c[x_1, \ldots, x_m], A)$ is less than or equal to $h(R[x_1, \ldots, x_m])R[x_1, \ldots, x_m]$. Then $A$ is an $H$-paraconvex subset of a Hilbert space, where $H(R) = h(R) + \varepsilon(R) < 1$ and $\varepsilon(R)$ is the positive root of the equation $(h(R) + x)^2 = 1 - x^2$.*

In other words, *central* $h$-paraconvexity of a subset $A$ of a Hilbert space implies $H$-paraconvexity of $A$, for some $0 \leqslant h < H < 1$.

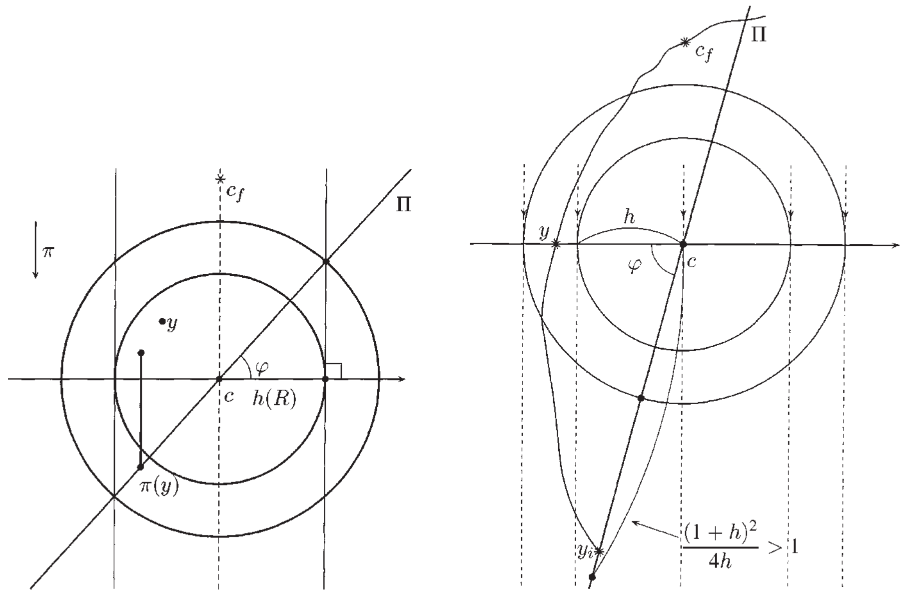
Graphs of continuous functions also have some nice properties in comparison with abstract subsets. The main one (for our purpose) is that we can always consider simplices of dimension equal to the dimension of the domain. For an arbitrary set, Carathéodory's theorem gives an upper estimate for the dimension of the simplices: it is equal to the dimension of the domain *plus* 1.

LEMMA 3.2 [9]. *Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a continuous function with a convex domain and let $y_1, y_2, \ldots, y_{k+2} \in \Gamma(f)$ and $z \in [y_1, y_2, \ldots, y_{k+2}]$ be arbitrary points. Then there exist points $p_1, \ldots, p_{k+1} \in \Gamma(f)$ such that $z \in [p_1, p_2, \ldots, p_{k+1}]$ and the simplex $[p_1, \ldots, p_{k+1}]$ can be moved into one of the faces of the simplex $[y_1, \ldots, y_{k+2}]$.*

The following property is also one of the desired ones for graphs of continuous functions:

LEMMA 3.3. *Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be a continuous function with a convex domain of dimension $k$ and let the graph $\Gamma(f)$ be an $h$-paraconvex subset of $\mathbb{R}^{k+1}$, for some function $h \in \mathcal{M}_{<1}$. Then for every $R > 0$ and every points $y_1, \ldots, y_{k+1} \in \Gamma(f)$, with $R[y_1, \ldots, y_{k+1}] = R$ and $\dim[y_1, \ldots, y_{k+1}] = k$, there exists a point $y \in \Gamma(f)$ such that:*

(1) $\|y - c[y_1, \ldots, y_{k+1}]\| \leqslant \frac{1+h(R)}{2} \cdot R$; *and*
(2) *One of the following two properties holds with respect to the plane $\Pi$, passing through the points $y_1, \ldots, y_{k+1}$:*
    (L) $\|\pi(y) - c[y_1, \ldots, y_{k+1}]\| \leqslant \frac{1+h(R)}{2} \cdot R$; *or*




(a) (b)

*Figure 1.*

(M) *The points* $y \in \Gamma(f)$ *and* $c_f \in \Gamma(f)$ *are on the same side, with respect to* $\Pi$*, where* $\pi\colon \mathbb{R}^{k+1} \to \Pi$ *is the projection, orthogonal to the domain and* $c_f = \pi^{-1}(c) \cap \Gamma(f)$.

*Proof.* Let

$$D = D(c, h(R) \cdot R) \quad \text{and} \quad D' = D(c, ((1 + h(R))/2)\, R)$$

be two concentric open balls. Exactly two cases are possible: $\pi(D) \subset D'$ or $\pi(D) \not\subset D'$.

If $\pi(D) \subset D'$, then the angle between $\Pi$ and $c + \mathbb{R}^k$ is less than or equal to $\phi = \arccos(2h(R)/(1 + h(R)))$ (see Figure 1a). So, due to $h$-paraconvexity of $\Gamma(f)$, we can find $y \in \Gamma(f) \cap D$ and, hence, (1) and (L) hold for the point $y$. If $\pi(D) \not\subset D'$ then such an angle is greater than $\phi$ (see Figure 1b). We can assume that the point $c_f$ is above the plane $\Pi$ and, hence, is above the plane $c + \mathbb{R}^k$. (Note, that $c \in \Pi \cap (c + \mathbb{R}^k)$, $cc_f$ is orthogonal to $c + \mathbb{R}^k$ and that in $\Pi$ there are no directions orthogonal to $\mathbb{R}^k$, due to the equality $\dim[y_1, \dots, y_{k+1}] = k$.)

One of the points $y_i \in \Pi \cap \Gamma(f)$ lies below $c + \mathbb{R}^k$, because of the inclusion $c \in [y_1, y_2, \dots, y_{k+1}]$. But $\|y_i - c\| \leqslant R$ and hence $y_i \in \pi(D')$. Due to the continuity of function $f$ 'on' the segment $[c, y_i]$, we see that there exists a point $y \in (c + \mathbb{R}^k) \cap \Gamma(f) \cap D'$ which is above the plane $\Pi$. Thus, for $y$, conditions (1) and (M) are satisfied. Observe also, that notations (L) and (M) were motivated

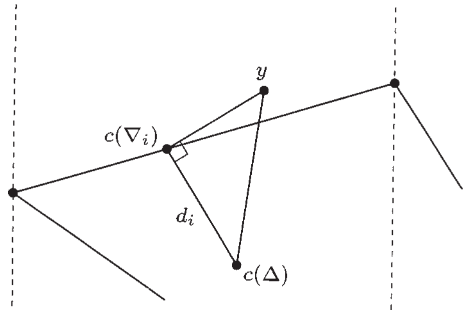



*Figure 2.*

by the terms ‘Lipschitz’ and ‘monotone’ and were originally derived exactly from these two kinds of functions. □

We now pass to the proof of Theorem 2.4. The general plan is similar to the proof of Theorem 2.4 from [8].

PROPOSITION 3.4. *For every $k \in \mathbb{N}$ and every function $h \in \mathcal{M}_{<1}$, there exists a function $H \in \mathcal{M}_{<1}$ with the following property: If $f\colon \mathbb{R}^k \to \mathbb{R}$ is a continuous function with convex closed $k$-dimensional domain $V$ and if for every $(k-1)$-dimensional hyperplane $E \subset \mathbb{R}^k$, the graph $\Gamma(f|_{E\cap V})$ is a centrally $h$-paraconvex set, then the graph $\Gamma(f)$ is centrally $H$-paraconvex set.*

Clearly, Proposition 3.4 provides an inductive step, from $k-1$ to $k$, in the proof of Theorem 2.4, and the basis of induction $(k = 1)$ is provided by the hypotheses of Theorem 2.4.

*Proof of Proposition 3.4.* We proceed by induction on $k$. Suppose that the proposition holds for $1 \leqslant m < k$. We shall verify it for $m = k$. So, we fix a map $f\colon \mathbb{R}^k \to \mathbb{R}$ and points $p_i = (x_i, f(x_i)) \in \Gamma(f)$ and we denote the simplex $[p_1, \dots, p_{k+1}]$ by $\Delta$. We may assume that $\dim(\Delta) = k$, since in the opposite case it suffices to use the previous inductive steps.

If the center $c(\Delta)$ of the simplex $\Delta$ belongs to one of its boundary simplices $\nabla$, then $c(\Delta) = c(\nabla)$, $R(\Delta) = R(\nabla)$, and we may use the inductive hypothesis for $f|_\nabla$. If the center $c(\Delta)$ of the simplex $\Delta$ is its interior point, then $\|c(\Delta) - p_i\| = R(\Delta)$ and we denote $d_i = \mathrm{dist}(c(\Delta), \nabla_i)$, where $\nabla_i$ is the boundary simplex of $\Delta$, $1 \leqslant i \leqslant k+1$. Then we choose $\varepsilon = \varepsilon(R) > 0$ so small that $h(R) + \varepsilon(R) < 1$, where $R = R(\Delta)$ (see Figure 2).

*Case A.* There exists $1 \leqslant i \leqslant k+1$ such that $d_i \leqslant \varepsilon \cdot R$.

By hypothesis, we conclude that for some $y \in \Gamma(f|_{\nabla_i})$,

$$\|c(\nabla_i) - y\| \leqslant h(R(\nabla_i)) \cdot R(\nabla_i) \leqslant h(R) \cdot R,$$

due to the monotonicity of $h$. Hence,

$$\|c(\Delta) - y\| \leqslant h(R) \cdot R + \varepsilon \cdot R = (h(R) + \varepsilon) \cdot R.$$

So, in this case we have the freedom to choose $H(R)$ as an arbitrary number from $[h(R)+\varepsilon(R), 1)$.

*Case B.* For all $1 \leqslant i \leqslant k+1$, the inequality $d_i > \varepsilon(R) \cdot R$ holds.

LEMMA 3.5 [7, Case BB of the proof of Theorem 1.4]. *There exists a monotone decreasing function* $\delta\colon (0,1) \to (0,1)$ *such that for every* $\varepsilon > 0$ *and every* $(k+1)$-*simplex* $\Delta$, *with* $R(\Delta) = R$ *and with* $\min\{d_i \mid 1 \leqslant i \leqslant k+1\} > \varepsilon \cdot R$, *the upper estimate* $\max\{d_i \mid 1 \leqslant i \leqslant k+1\} \leqslant \delta(\varepsilon) \cdot R$ *holds.*

So, we apply Lemma 3.3 to every $(k-1)$-dimensional boundary simplex $\nabla_i$ of the simplex $\Delta$ and we find a point $y_i \in \Gamma(f|_{\nabla_i})$ such that for $2\hat{h} = 1 + h$:

$(1)_i$ $\|y_i - c_i\| \leqslant \hat{h}(R_i) \cdot R_i$, where $c_i = c(\nabla_i)$ and $R_i = R(\nabla_i)$; and
$(2)_i$ for $y_i$ either $(\mathrm{L})_i$ or $(\mathrm{M})_i$ holds.

(a) Let us consider the case when all points $y_i$ lie on the same side of the plane $\Pi$ passing through the points $p_1, \ldots, p_{k+1}$, for example, above this plane. Then elementary geometry shows that one of the angles $\sphericalangle y_i c_i c$ is less or equal to 90° and, hence,

$$\begin{aligned}\|c - y_i\|^2 &\leqslant \|c - c_i\|^2 + \|c_i - y_i\|^2 \leqslant d_i^2 + (\hat{h}(R_i))^2 \cdot R_i^2 \\ &= d_i^2 + (\hat{h}(R_i))^2(R^2 - d_i^2) \\ &\leqslant R^2[(1 - (\hat{h}(R_i))^2)\delta^2(\varepsilon) + (\hat{h}(R_i))^2].\end{aligned}$$

The function $\phi(t) = (1-t^2)a^2 + t^2$ is increasing on $(0,\infty)$, for $0 < a < 1$, and $\hat{h}(\cdot)$ is also an increasing function. Hence, we obtain from $R_i \leqslant R$ that:

$$\|c - y_i\|^2 \leqslant [(1 - (\hat{h}(R))^2)\delta^2(\varepsilon) + (\hat{h}(R))^2]R^2.$$

Observe that the first factor on the right is greater than $\delta^2(\varepsilon)$ because it is a convex combination of numbers $\delta^2(\varepsilon)$ and 1 with the coefficient $(\hat{h}(R))^2 \in (0,1)$.

(b) Let us consider the case when there exist two points $y_i$ and $y_j$ lying on different sides with respect to $\Pi$ and with properties $(\mathrm{M})_i$ and $(\mathrm{M})_j$, respectively. Then the points $(c_i)_f$ and $(c_j)_f$ (i.e. the points of the graph $\Gamma(f)$ which lie 'over' the centers $c_i$ and, respectively, $c_j$) also lie on different sides with respect to $\Pi$. Due to the continuity of $f$, we see that there exists $y_{ij} \in \Gamma(f) \cap [c_i, c_j]$ and, hence, $\|c - y_{ij}\| \leqslant \max\{d_i, d_j\} \leqslant \delta(\varepsilon)R$.

(c) Let us change the property (M) in the case (b) by the property (L). By a similar argument we find $y_{ij} \in \Gamma(f) \cap [\pi(c_i), \pi(c_j)]$ and, hence, $\|c - y_{ij}\| \leqslant \max\{\|c - \pi(c_i)\|, \|c - \pi(c_j)\|\}$. But for $\|c - \pi(c_i)\|$ and $\|c - \pi(c_j)\|$ we have upper estimates *exactly* as in (a), because the (L)-property gives $\|c - \pi(c_i)\| \leqslant \hat{h}(R_i)R_i$ and $\|c - \pi(c_j)\| \leqslant \hat{h}(R_j)R_j$.

(d) The last is the case when some $y_i$ with property (L) lies above $\Pi$ and some $y_j$ with property (M) lies below $\Pi$. But then $(c_j)_f$ also lies below $\Pi$ and repetition of (b) and (c) gives the existence of a point $y_{ij} \in \Gamma(f) \cap [\pi(c_i), c_j]$. Hence,

$$\|c - y_{ij}\| \leqslant \max\{\|c - \pi(c_j)\|, \|c - c_j\|\}.$$

We can now give the answer for the function $H \in \mathcal{M}_{<1}$ from the statement of Proposition 3.4. So, for a given $h \in \mathcal{M}_{<1}$ we put $\varepsilon = (1-h)/2$ and, hence, $h + \varepsilon = \hat{h} \in \mathcal{M}_{<1}$. Next, according to Lemma 3.5 we can find a decreasing function $\delta$: $(0,1) \to (0,1)$. Note that the composition $\delta(\varepsilon(\cdot))$ is an increasing function. Finally,

$$\begin{aligned} H(R) &= \max\left\{\delta(\varepsilon(R)), [(1-(\hat{h}(R))^2)\delta^2(\varepsilon(R)) + (\hat{h}(R))^2]^{1/2}, \hat{h}(R)\right\} \\ &= \left[(1-(\hat{h}(R))^2)\delta^2(\varepsilon(R)) + (\hat{h}(R))^2\right]^{1/2} \in \mathcal{M}_{<1}. \end{aligned}$$

Proposition 3.4 is thus proved. By invoking Lemma 3.1, Theorem 2.4 is also proved. □

## 4. Proof of Theorem 2.5

In view of Theorem 2.4, it suffices to find a common majorant $h \in \mathcal{M}_{<1}$ for functions of nonconvexity of graphs of all restrictions $P|_\ell$, where $\ell$ is any line in $\mathbb{R}^n$. So, for every line $\ell \subset \mathbb{R}^n$, we pick a point $x^0 = (x_1^0, \dots, x_n^0) \in \ell$ and a unit vector $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$ parallel to $\ell$. Then $\ell = \{x^0 + t\nu \mid t \in \mathbb{R}\}$. Hence, for $x = (x_1, \dots, x_n) \in \ell$ we have $x_i = x_i^0 + t\nu_i$ and, therefore,

$$\begin{aligned} P_2(x) &= \sum_{i,j=1}^{n} a_{ij}(x_i^0 + t\nu)(x_j^0 + t\nu_j) + \sum_{k=1}^{n} a_k(x_k^0 + t\nu_k) + a_0 \\ &= at^2 + bt + c, \end{aligned}$$

where $a = \sum_{i,j=1}^n a_{ij}\nu_i\nu_j$. So, $P_2|_\ell$ is a quadratic polynomial and the hypothesis $|a_{ij}| \leqslant C$ implies that $|a| \leqslant Cn^2$. Thus, Theorem 2.5 is a corollary of Theorem 2.4 and the following lemma:

LEMMA 4.1. *For every $d > 0$, there exists $h_d \in \mathcal{M}_{<1}$ such that for every quadratic polynomial $y = ax^2 + bx + c$, with $|a| \leqslant d$, the function $h_{\Gamma(y)}$ of nonconvexity of the graph of $y$ is less than $h_d$.*

*Proof.* With respect to some orthogonal system of coordinates we have the equation $y = ax^2$, for a given quadratic polynomial for estimating of $h_{\Gamma(f)}$. For graphs of continuous functions $y = f(x)$ it suffices to find only an estimate for $\delta(\{A, B\}, \Gamma(f))$ (see [8]), i.e. we must find a suitable control for $\operatorname{dist}(M, \Gamma(f))$ for the middle points $M$ of all segments $[A, B]$ with $\operatorname{dist}(A, B) = 2R$ and $A, B \in \Gamma(f)$. So, let us fix $R > 0$ and consider an arbitrary such segment with $A(x_A, y_A)$, $B(x_B, y_B)$ and $x_A < x_B$. Exactly three cases are possible (see Figure 3):

(a) $x_A \geqslant 2R$. Then, by monotonicity of $y$ over $[2R, +\infty)$, we conclude that $\operatorname{dist}(M, \Gamma(y)) \leqslant (\sqrt{2}/2)R$; or
(b) $-2R \leqslant x_A < 2R$. Then $y$ is a Lipschitz function on $[-2R, 2R]$, with the Lipschitz constant $k = \max\{y'(x) \mid |x| \leqslant 2R\} = 4|a|R$. Hence, $\operatorname{dist}(M, \Gamma(y)) \leqslant \sin(\arctan k) \cdot R$; or

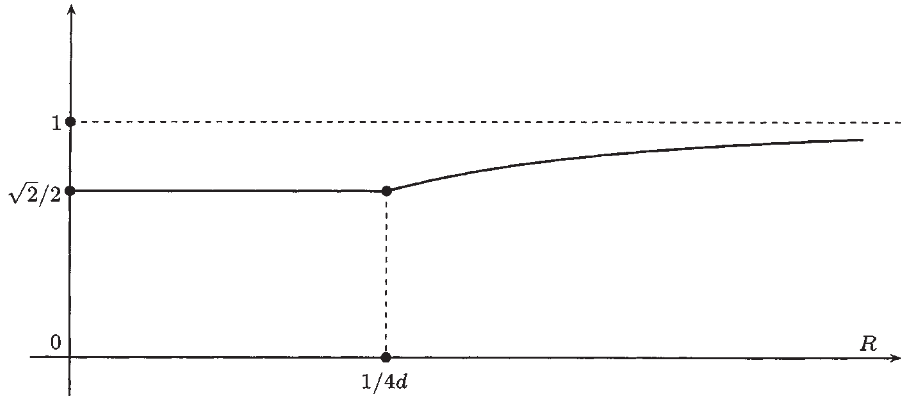



*Figure 3.*

(c) $x < -2R$. Then, by monotonicity of $y$ over $(-\infty, 0]$, we find (as in (a)) that $\operatorname{dist}(M, \Gamma(y)) \leqslant (\sqrt{2}/2)R$.

Hence, the final answer for the function $h \in \mathcal{M}_{<1}$ is

$$h(R) = \max\{\sqrt{2}/2, \sin(\arctan 4dR)\}. \qquad \square$$

This also completes the proof of Theorem 2.5. $\square$

## 5. Proof of Theorem 2.6

Pick a monotone decreasing sequence $a_n > 0$, converging to zero and let for every $n > 1$, $2b_n = a_{n+1} + a_n$. First, we denote by $p_n^-$ the quadratic polynomial with the vertex $(b_n, 1)$ and with $p_n^-(a_{n+1}) = p_n^-(a_n) = 0$. Next, we denote by $p_n^+$ the quadratic polynomial with $p_n^+(b_n) = p_n^+(b_{n+1}) = 1$ and $p_n^+(a_{n+1}) = 0$. In order to define $F\colon [1/2, 1] \to \mathbb{R}^2$, we partition the segment $[1/2, 1]$ into six congruent subsegments:

$$[1/2, 1] = [1/2, t_1] \cup [t_1, t_2] \cup [t_2, t_3] \cup [t_3, t_4] \cup [t_4, t_5] \cup [t_5, 1].$$

Next, we define $F$ at the ends of these subsegments, by letting (see Figure 4):

$$\begin{aligned}
&F(1) = \Gamma(p_1^-|_{[a_2,a_1]});\ F(t_5) = \Gamma(p_1^-|_{[a_2,b_1]});\\
&F(t_4) = \Gamma(p_1^+|_{[a_2,b_1]});\ F(t_3) = \Gamma(p_1^+|_{[b_2,b_1]});\\
&F(t_2) = \Gamma(p_1^+|_{[b_2,a_2]});\ F(t_1) = \Gamma(p_2^-|_{[b_2,a_2]}); \quad \text{and}\\
&F(1/2) = \Gamma(p_2^-|_{[a_3,a_2]});
\end{aligned}$$

When $t$ decreases from 1 to $t_5$, $F(t)$ shrinks from $F(1)$ to $F(t_5)$ over the graph of $p_1^-$. When $t$ decreases from $t_5$ to $t_4$, $F(t)$ passes from $F(t_5)$ to $F(t_4)$ as a convex combination of these parabolas. When $t$ decreases from $t_4$ to $t_3$, $F(t)$ grows from

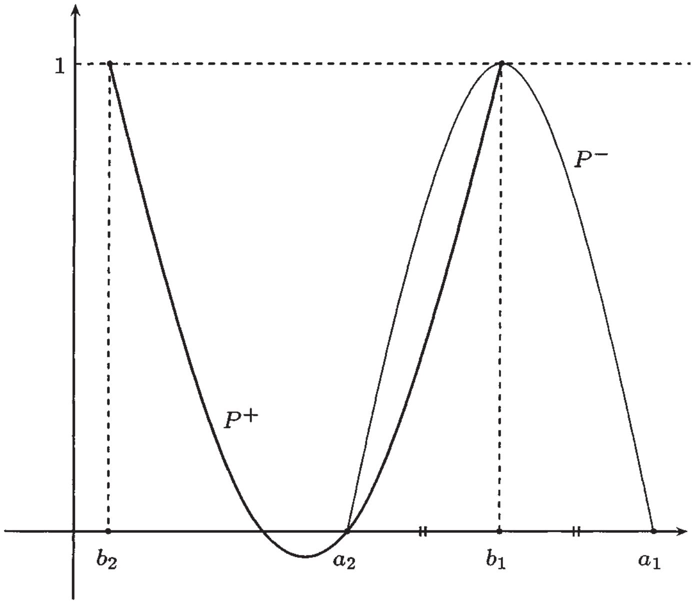



*Figure 4.*

$F(t_4)$ to $F(t_3)$ over the graph of $p_1^+$. When $t$ decreases from $t_3$ to $t_2$ then $F(t)$ shrinks from $F(t_3)$ to $F(t_2)$ over the graph of $p_1^+$. On $[t_1, t_2]$ we define a 'convex' path from $F(t_2)$ to $F(t_1)$ and on $[1/2, t_1]$ the values $F(t)$ grow from $F(t_1)$ to $F(1/2)$ over the graph of $p_2^-$. Finally, we obtain the same situation at $t = 1/2$, as when $t = 1$. Extension over segments $[1/n+1, 1/n]$, $n > 1$ is made in a similar way and $F(0) = [(0, -1); (0, 1)]$.

This completes the construction of the example $F\colon [0, 1] \to \mathbb{R}^2$. Note, that the vertices of the parabola's $p_n^+$ are below $x$-axis. Let us locate them more carefully, in order to be sure that they converge to $(0, 0)$, when $n \to \infty$. Clearly,

$$p_n^+(x) = A_n(x - a_{n+1})(x - c_n),$$

for some $A_n > 0$ and $a_{n+1} < c_n < b_n$. So, we have

$$1 = A_n(b_n - a_{n+1})(b_n - c_n) = p_n^+(b_n)$$

and

$$1 = A_n(b_{n+1} - a_{n+1})(b_{n+1} - c_n) = p_n^+(b_{n+1}).$$

We wish to find:

$$p_n^+\left(\frac{b_n + b_{n+1}}{2}\right)$$
$$= \frac{A_n}{4}(b_n - a_{n+1} + b_{n+1} - a_{n+1})(b_n - c_n + b_{n+1} - c_n)$$

$$= \frac{A_n}{4}(b_n - a_{n+1} + b_{n+1} - a_{n+1})\left(\frac{1}{A_n(b_n - a_{n+1})} + \frac{1}{A_n(b_{n+1} - a_{n+1})}\right)$$
$$= \frac{1}{4}\left(2 + \frac{b_{n+1} - a_{n+1}}{b_n - a_{n+1}} + \frac{b_n - a_{n+1}}{b_{n+1} - a_{n+1}}\right)$$
$$= \frac{1}{4}\left(2 + d_n + \frac{1}{d_n}\right),$$

where

$$d_n = \frac{b_{n+1} - a_{n+1}}{b_n - a_{n+1}} < 0$$

is completely defined by the (given) sequence $\{a_n\}$. More precisely,

$$b_n - a_{n+1} = \frac{1}{2}(a_n - a_{n+1}) \quad \text{and} \quad b_{n+1} - a_{n+1} = -\frac{1}{2}(a_{n+1} - a_{n+2}).$$

Hence,

$$d_n = \frac{a_n - a_{n+1}}{a_{n+2} - a_{n+1}}.$$

Therefore, we must choose the sequence $\{a_n\}$ so that $d_n \to -1$. Clearly, it suffices to put $a_n = 1/n$. (Note that $a_n = 2^{-n}$ would be a wrong choice.) It is then clear that $F\colon [0,1] \to \mathbb{R}^2$ is continuous.

Suppose, to the contrary, that $f\colon [0,1] \to \mathbb{R}^2$ is a continuous singlevalued selection of $\Gamma$ and $f(t) = (f_1(t), f_2(t)) \in F(t)$. Then $f_1(1) \in [a_2, a_1]$ and $f_1(1/2) \in [a_3, a_2]$. Hence, $a_2 \in \operatorname{Im}(f_1|_{[1/2,1]})$, due to continuity of $f_1$. Analogously, $a_n \in \operatorname{Im}(f_1|_{[1/n,1/(n-1)]})$ and $(0, a_2] \subset \operatorname{Im} f_1$. Hence, there exists a sequence $t_n \in [0,1]$, converging to zero, such that $f_1(t_n) = b_n$, $n > 1$. But then

$$f(t_n) = (b_n, f_2(t_n)) \in F(t_n).$$

By construction, the set $F(t)$ has at most a single common point with the vertical $x = b_n$, namely the point $(b_n, 1)$. Therefore $f_2(t_n) = 1$ and $f_2(0) = 1$, because of continuity of $f_2$. Similar arguments for the points $a_n$ show that $f_2(0) = 0$. Contradiction. Theorem 2.6 is thus proved. □

## 6. Proof of Theorems 2.7 and 2.8

We begin by some elementary facts, concerning polynomials of degree 3 in one real variable.

LEMMA 6.1. *For every $a, b \in \mathbb{R}$, the following holds:*

$$\lim_{t\to\infty} \frac{t^3 - \sqrt{t^2 - a}(t^2 - b)}{t} = \frac{a + 2b}{2}.$$

□

LEMMA 6.2. *Let* $P(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. *Then with respect to a new origin, the equation of* $P$ *has the form*

$$P(X) = aX^3 - \frac{D}{3a}\,X,$$

*where* $D = b^2 - 3ac$. □

So, if $D = b^2 - 3ac < 0$, the derivative $P' = 3a(X^2 - (D/9a^2))$ has a constant sign. Hence, $P$ is monotone and $h_{\Gamma(P)} \leqslant \sqrt{2}/2$ (see [8]). Using Lemma 6.2, we shall rename the coordinates and assume that $D \geqslant 0$.

LEMMA 6.3. *Let* $P(x) = ax^3 - (D/3a)x$ *and* $D \geqslant 0$. *Then the graph* $\Gamma(P)$ *intersects with* $x$*-axis at points* $x = 0$, $x = \pm\sqrt{D}/\sqrt{3}a$ *and has a local maximum* (*resp. local minimum*) *at the point* $(-\sqrt{D}/3a, 2D\sqrt{D}/27a^2)$ (*resp. at the point* $(\sqrt{D}/3a, -2D\sqrt{D}/27a^2)$). □

LEMMA 6.4. *Using the notations of Lemma* 6.3, *let* $0 < y_0 < 2D\sqrt{D}/27a^2$ *and let* $A(x_A, y_A)$ *and* $B(x_B, y_B)$ *be two points of intersection of the horizontal line* $y = y_0$ *with the graph* $\Gamma(P)$ *such that* $-\sqrt{D}/3a \in [x_A, x_B]$. *Let* $\mathrm{dist}(A, B) = 2R$, *then*

$$\begin{aligned} y_0 &= \frac{2\sqrt{D - 3a^2R^2}(D - 12a^2R^2)}{27a^2}; \\ x_A &= -R - \frac{\sqrt{D - 3a^2R^2}}{3a}; \quad \textit{and} \quad x_B = R - \frac{\sqrt{D - 3a^2R^2}}{3a}. \end{aligned}$$

*Proof.* Find the roots of the equation $ax^3 - (D/3a)x = y_0$ or $x^3 + px + q = 0$ with

$$p = -\frac{D}{3a^2} < 0, \qquad q = -\frac{y_0}{a} < 0.$$

Using Cardano's formula, we have

$$x_j = 2\sqrt{-\frac{p}{3}}\cos\frac{\phi + 2(j-1)\pi}{3}, \quad j \in \{1, 2, 3\},$$

where

$$\cos\varphi = -\frac{q}{2\left(\sqrt{-\frac{p}{3}}\right)^3}.$$

In our case, $\cos\varphi > 0$, i.e. $0 < \varphi/3 < \pi/6$. Hence,

$$x_A = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) \quad \text{and} \quad x_B = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{\varphi}{3} - \frac{2\pi}{3}\right).$$

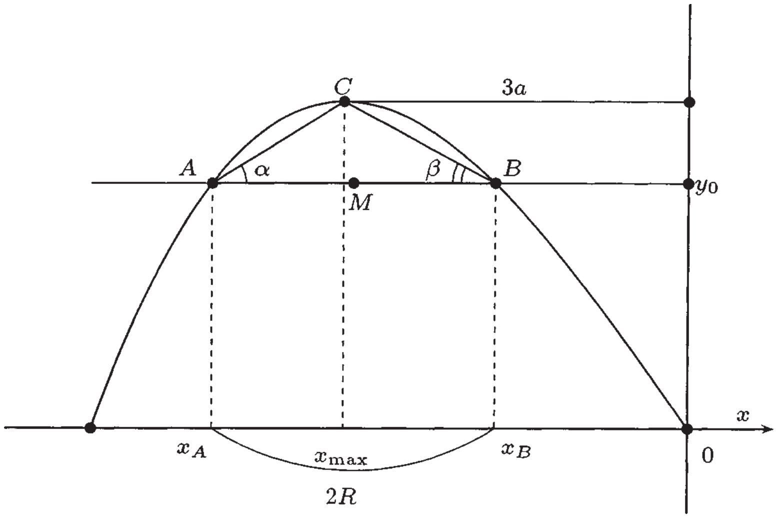



*Figure 5.*

Knowing that

$$2\sqrt{-\frac{p}{3}} = \frac{2\sqrt{D}}{a},$$

we obtain

$$2R = x_B - x_A = \frac{2\sqrt{D}}{3a}\left(\cos\frac{\varphi - 2\pi}{3} - \cos\frac{\varphi + 2\pi}{3}\right) = \frac{2\sqrt{D}}{\sqrt{3}a}\sin\frac{\varphi}{3}.$$

So, $\varphi = 3\arcsin(\sqrt{3}aR)/\sqrt{D}$ and, therefore,

$$\begin{aligned}
y_0 &= -aq = 2\left(\sqrt{-\frac{p}{3}}\right)^3 \cos\varphi = 2a\left(\frac{\sqrt{D}}{3a}\right)^3\left(4\cos^3\frac{\varphi}{3} - 3\cos\frac{\varphi}{3}\right) \\
&= \frac{2D\sqrt{D}}{27a^2}\cos\frac{\varphi}{3}\left(1 - 4\sin^2\frac{\varphi}{3}\right) \\
&= \frac{2D\sqrt{D}}{27a^2}\sqrt{1 - \frac{3a^2R^2}{D}}\left(1 - 4\cdot\frac{3a^2R^2}{D^2}\right) \\
&= \frac{2\sqrt{D - 3a^2R^2}(D - 12a^2R^2)}{27a^2}; \\
x_A &= \frac{2\sqrt{D}}{3a}\cos\frac{\varphi + 2\pi}{3} = \frac{\sqrt{D}}{3a}\left(-\cos\frac{\varphi}{3} - \sqrt{3}\sin\frac{\varphi}{3}\right) \\
&= \frac{\sqrt{D}}{3a}\left(-\sqrt{1 - \frac{3a^2R^2}{D}} - \sqrt{3}\frac{\sqrt{3}aR}{\sqrt{D}}\right) = -R - \frac{\sqrt{D - 3a^2R^2}}{3a}.
\end{aligned}$$

Finally, $x_B = x_A + 2R$. Lemma 6.4 is thus proved. □

LEMMA 6.5. *For every fixed $R > 0$ and for every $0 \leqslant \lambda < 1$, the value of the function of nonconvexity $h_{\Gamma(P)}$ of polynomial $P(x) = ax^3 + bx^2 + cx + d$ at the point $R$ is more than $\lambda$ for sufficiently large $\sqrt{D}$ and $\sqrt{D}/a$.*

*Proof.* We use the notations of Lemma 6.4. Denote by $C$ the point of the local maximum and denote by $M$ the middle point of the segment $[A, B]$ (see Figure 5).

Due to the concavity of $P$ over $[x_A, x_B]$, we have that

$$\operatorname{dist}(M, \Gamma(P)) \geqslant \min\{R \sin\alpha, R \sin\beta\},$$

where $\alpha = \sphericalangle CAB$ and $\beta = \sphericalangle CBA$. Hence,

$$h_{\Gamma(P)}(R) \geqslant \delta(\{A, B\}, \Gamma(P)) \geqslant \operatorname{dist}(M, \Gamma(P))/R \geqslant \min\{\sin\alpha, \sin\beta\}.$$

So it suffices to show that $\sin\alpha \to 1$ and $\sin\beta \to 1$, when $\sqrt{D} \to \infty$ and $\sqrt{D}/a \to \infty$. Next, we estimate $\tan\alpha$ and $\tan\beta$. Using Lemmas 6.3 and 6.4 we have that

$$\begin{aligned} \tan\alpha &= \frac{y_{\max} - y_0}{x_{\max} - x_A} > \frac{y_{\max} - y_0}{2R} \\ &= \frac{1}{2R}\,\frac{2}{27a^2}\left[D\sqrt{D} - \sqrt{D - 3a^2R^2}(D - 12a^2R^2)\right]. \end{aligned}$$

Substituting the variable $t = \sqrt{D}/a$, we see that

$$\tan\alpha > \frac{a}{27R}\left[t^3 - \sqrt{t^2 - 3R^2}(t^2 - 12R^2)\right].$$

By Lemma 6.1, the second factor is equivalent to $\frac{27}{2}R^2 t$, when $t \to \infty$. Hence, $\tan\alpha$ is more than variable equivalent to

$$\frac{a}{2}Rt = \frac{R}{2}\sqrt{D} \to \infty.$$

So, $\sin\alpha \to 1$ and, analogously, $\sin\beta \to 1$ when $\sqrt{D}/a \to \infty$ and $\sqrt{D} \to \infty$. Lemma is proved. □

We now begin the proof of Theorem 2.7. Our plan is as follows:

(I) We restrict the polynomial $z(x, y) = x^9 + x^3 y$ over lines $\ell$: $y = -\rho$ in the plane $\mathbb{R}^2$ and let $\rho \to +\infty$.

(II) Such restriction $z|_\ell$ is, in fact, a cubic parabola with respect to the variable $t = x^3$.

(III) So, for a fixed $R > 0$ and sufficiently large $\rho > 0$, we find (as in Lemmas 6.4, 6.5) points $A = A_\rho$ and $B = A_\rho$ on the graph $\Gamma(z|_\ell)$ such that $\operatorname{dist}(A, B) = 2R$, the segment $[A, B]$ is horizontal and the local maximum of $z|_\ell$ lies between $A$ and $B$. Our assertion is that the balls with the diameters $AB = A_\rho B_\rho$ are approximately inscribed into $\Gamma(P)$ when $\rho \to +\infty$.

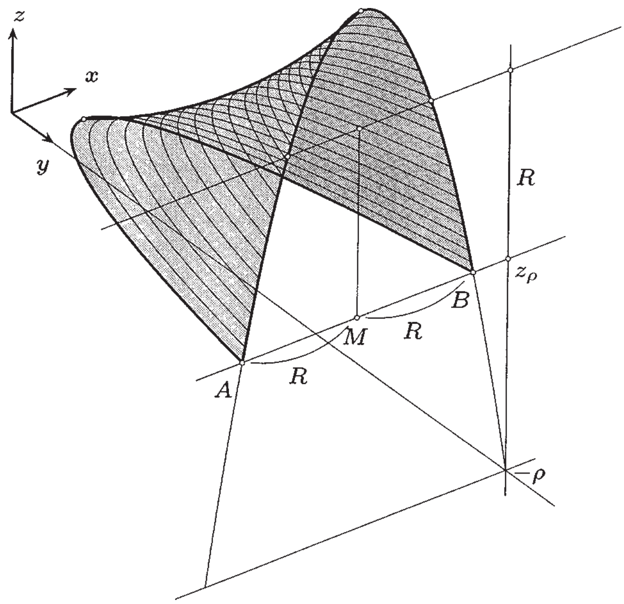



*Figure 6.*

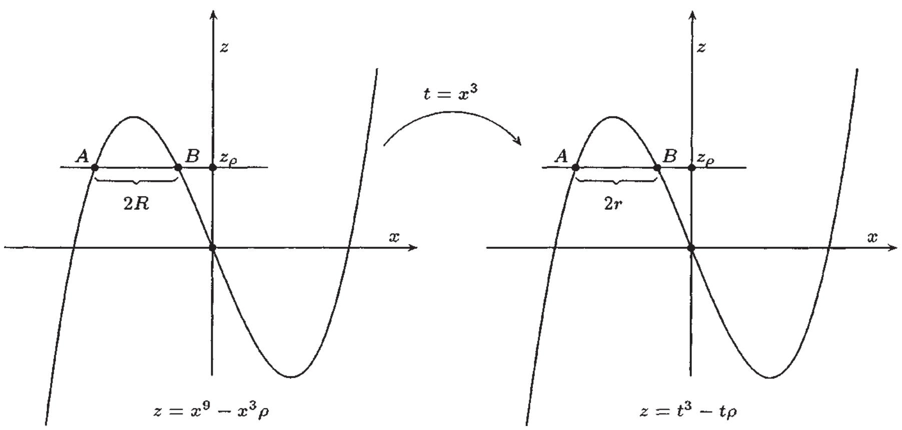



*Figure 7.*

(IV) We draw the horizontal plane $H_\rho : z = z_\rho$ through the segment $[A, B]$. The intersection $H_\rho \cap \Gamma(z)$ gives a graph of some function $y = y(x)$ which will be concave over $[A, B]$ and will be with unique point $c = c_\rho$ of maximum. Thus, we estimate the relative precision of the approximation $\delta(\{A, B\}, H_\rho \cap \Gamma(z))$ in a similar manner as in Lemma 6.5.

(V) From (I)–(IV) we get the desired estimate for the function of nonconvexity $h_{\Gamma(z)}$ in the vertical and horizontal sections. In order to finish the proof, we lift the plane $H_\rho$, up to the level $z = z_\rho + R$ and we estimate the nonconvexity of such horizontal sections, as in (IV) (see Figure 6).

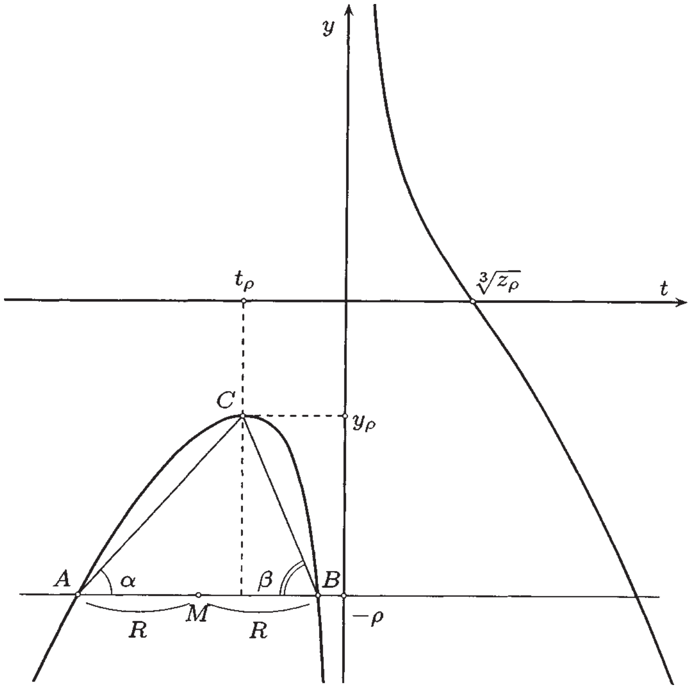



*Figure 8.*

The main technical problem here is that the restriction of $z$ over the curvilinear figure in $H_\rho$, bounded by the arc $ACB$ and by the segment $[AB]$, is not concave. So, the polynomial $z$ is given over the line $\ell$: $y = -\rho$ by the formula (see Figure 7):

$$z = x^9 - x^3\rho = t^3 - t\rho, \quad t = x^3.$$

With the notations from Lemmas 6.4, 6.5, we see that $a = 1$ and $D = b^2 - 3ac = 3\rho$. Hence, $\sqrt{D}/a \to +\infty$ and $\sqrt{D} \to +\infty$, when $\rho \to +\infty$ and for a fixed $R > 0$ and for sufficiently large $\rho$ we can always find the points $A$ and $B$, as described in (III). The intersection of the horizontal plane $H_\rho$ with $\Gamma(z)$ gives the line in $H_\rho$ defined by the equation

$$x^9 + x^3 y = z_\rho. \tag{i}$$

The equality (i) implies that

$$y = \frac{z_\rho}{t} - t^2, t = x^3 \quad \text{and} \quad y'(t) = -\frac{z_\rho}{t^2} - 2t = -\frac{1}{t^2}(z_\rho + 2t^3). \tag{ii}$$

Hence, $y$ has a local maximum at the point $t_\rho = (-z_\rho/2)^{1/3}$ and the maximal value $y_\rho$ equals to (see Figure 8)

$$y_\rho = y(t_\rho) = \frac{1}{t_\rho}(z_\rho - t_\rho^3) = \frac{3z_\rho}{2(-z_\rho/2)^{1/3}} = -\frac{3 \cdot 2^{1/3}}{2} z_\rho^{2/3}. \tag{iii}$$

LEMMA 6.6. $\lim_{\rho\to\infty}(\rho + y_\rho) = +\infty$.

*Proof.* Equation (iii) gives the expression of $y_\rho$ via $z_\rho$ and $z_\rho$ we can find as in Lemma 6.4. We must only be careful with the fact that we know the formula for $z_\rho$ via distance $\operatorname{dist}(A', B') = 2r$ between points $A'(x_A^3, z_\rho)$ and $B'(x_B^3, z_\rho)$, not from the $\operatorname{dist}(A, B) = 2R$. Clearly, $r = r(R, \rho)$.

Let us find (or estimate) this dependence. By Lemma 6.4, we know that

$$t_{A'} = -r - \frac{\sqrt{3\rho - 3r^2}}{3} \quad \text{and} \quad t'_B = r - \frac{\sqrt{3\rho - 3r^2}}{3}.$$

Hence, by temporary denoting $\sqrt{3\rho - 3r^2} = 3d$, we find that

$$2R = x_B - x_A = t_{B'}^{1/3} - t_{A'}^{1/3} = (r + d)^{1/3} + (r - d)^{1/3},$$

or

$$8R^3 = (r + d) + (r - d) + 3(r^2 - d^2)^{1/3} \cdot 2R.$$

Hence,

$$4R^3 = r + 3R\left(r^2 - \frac{\rho - r^2}{3}\right)^{1/3} = r - 3R\left(\frac{\rho - 4r^2}{3}\right)^{1/3}.$$

Finally,

$$\frac{r - 4R^3}{3R} = \left(\frac{\rho - 4r^2}{3}\right)^{1/3}. \tag{iv}$$

If for a fixed $R > 0$, the function $r = r(\rho)$ is bounded, then the left side in (iv) is bounded, whereas the right side is unbounded. Contradiction. Hence, $r \to +\infty$, when $\rho \to \infty$. Moreover, we claim that $r = o(\sqrt{\rho})$, $\rho \to +\infty$.

Suppose, to the contrary, that for some $\frac{1}{4} > c > 0$, the inequality $r^2 > c \cdot \rho$ holds, for sufficiently large $\rho$. Then for the left side in (iv), we have a *lower estimate*

$$\frac{\sqrt{c}}{3R}\sqrt{\rho} - \frac{4}{3}R^2.$$

But for the right side in (iv) we have the following *upper estimate*

$$\sqrt[3]{\rho}\left(\frac{1 - 4c}{3}\right)^{1/3}.$$

So, we see that for $\rho \to +\infty$, the lower estimate is greater than the upper estimate. Contradiction.

After having proved the relation $r = o(\sqrt{\rho})$, we calculate $\rho + y_\rho$ via $r$ and $\rho$. So, by Lemma 6.4, we know that $a = 1$, $D = 3\rho$, and that

$$z_\rho = \frac{2}{27}\sqrt{3\rho - 3r^2}(3\rho - 12r^2) = \frac{2\rho^{3/2}}{3^{3/2}}\sqrt{1 - \frac{r^2}{\rho}}\left(1 - \frac{4r^2}{\rho}\right). \tag{v}$$

Therefore

$$\rho + y_\rho = \rho - \frac{3\cdot 2^{1/3}}{2} z_\rho^{2/3} = \rho\left[1 - \left(1 - \frac{r^2}{\rho}\right)^{1/3}\left(1 - \frac{4r^2}{\rho}\right)^{2/3}\right].$$

Since $r^2/\rho \to 0$ when $\rho \to +\infty$, it follows that

$$\begin{aligned}\rho + y_\rho \underset{p\to\infty}{\sim} \rho\left[1 - \left(1 - \frac{r^2}{3\rho}\right)\left(1 - \frac{8r^2}{3\rho}\right)\right] &= \rho\left[\frac{3r^2}{\rho} - \frac{8r^4}{9\rho^2}\right] \\ &= 3r^2 - \frac{8}{9}\frac{r^2}{\rho}.\end{aligned}$$

But we know that $r \to +\infty$ and $r^2/\rho \to 0$, when $\rho \to +\infty$. Thus, Lemma 6.6 is proved. □

If we return to (ii) (see Figure 8), we see that $\min\{\tan\alpha, \tan\beta\} > (\rho+y_\rho)/2R \to +\infty$ and, hence, $\min\{\sin\alpha, \sin\beta\} \to 1$, when $\rho \to +\infty$. Due to the concavity of $y = y(x)$ over $[A, B]$, we see that $\delta(\{A, B\}, H \cap \Gamma(z)) \to 1$, when $\rho \to +\infty$. This completes steps (I)–(IV) of our plan. To realize step (V), we first prove that the middle point $M$ of $[A, B]$ practically coincides 'at infinity' with the point of local maximum of function $y(x)$ from (i).

LEMMA 6.7.

$$\lim_{\rho\to\infty} \frac{x_A + R}{x_\rho} = 1.$$

*Proof.* Due to (v), and to the fact that $r^2 = o(\rho)$ we have for $x_\rho$

$$\begin{aligned}x_\rho = (t_\rho)^{1/3} &= (-z_\rho/2)^{1/9} \\ &= \left[-(\rho/3)^{3/2}\sqrt{1 - \frac{r^2}{\rho}}\left(1 - \frac{4r^2}{\rho}\right)\right]^{1/9} \underset{\rho\to\infty}{\sim} -(\rho/3)^{1/6}.\end{aligned}$$

On the other hand,

$$\begin{aligned}x_A + R &= (t_{A'})^{1/3} + R = R - \left(r + \frac{\sqrt{\rho - r^2}}{\sqrt{3}}\right)^{1/3} \\ &= R - \left(\sqrt{\rho}\left[\frac{r}{\sqrt{\rho}} + \frac{\sqrt{1 - \frac{r^2}{\rho}}}{\sqrt{3}}\right]\right)^{1/3} \underset{\rho\to\infty}{\sim} -(\rho/3)^{1/6}.\end{aligned}$$

Lemma 6.7 is thus proved. □

To finish step (V), we fix $0 \leqslant \gamma < 1$, $\gamma < \hat{\gamma} < 1$ and draw two horizontal planes: $H_\rho$ and $H'_\rho$ which is an upper parallel shift of $H_\rho$ onto $R$ along the $z$-axis (see Figure 9).

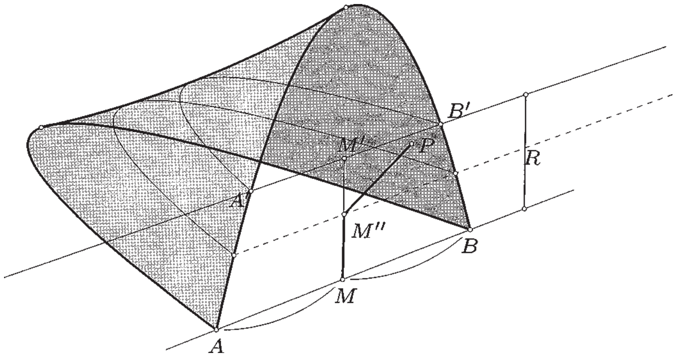



*Figure 9.*

We choose $\rho$ large enough so that $\mathrm{dist}(A', B') > \hat{\gamma}R$ and, hence, for every horizontal plane $H''_\rho$ between $H_\rho$ and $H'_\rho$, we have $\mathrm{dist}(A', B') > \hat{\gamma}R$. Then, using our estimates along horizontal planes, we choose $\rho$ so large, that $\mathrm{dist}(M'', P) > (\gamma/\hat{\gamma})\mathrm{dist}(M'', B'') > \gamma R$ for all points $P \in H''_\rho \cap \Gamma(z)$. Moreover, we can choose $\rho$ so large, that the triangle $[M, M'', P]$ will be practically rectangular – see Lemma 6.7. But then $\mathrm{dist}(M, P) \geqslant \mathrm{dist}(M'', P) > \gamma R$. We omit the technical details. Theorem 2.7 is thus proved. □

The key step of the proof of Theorem 2.7 is the equality (iv) and the dependence $r = \mathrm{o}(\sqrt{\rho})$. Such an approach is unsuccessful for third-degree polynomials. More precisely, instead of the statement of Lemma 6.6 that $\rho + y_\rho \to +\infty$, we have for the case of the polynomial $P_\mu(x, y) = x^3 + \mu xy$ (for a fixed $R > 0$) that $\rho + y_\rho \to 3R^2/\mu$, when $\rho \to +\infty$. So, by passing to $\mu \to 0$, we obtain the same result as in Lemma 6.6, however, for *a family* of polynomials $\{P_\mu\}$, rather than for *a single* polynomial as in Lemma 6.6. The remaining steps in the proof of Theorem 2.8 differ in corresponding places in the proof of Theorem 2.7 only by routine technical changes. We omit the details. □

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