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On Continuity Properties of the Modulus of Local Contractibility

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Let M_X be the set of all metrics compatible with a given topology on a locally contractible space X and let for each triple $z = (\rho, x, \varepsilon) \in M_X \times X \times (0, \infty)$, $\Delta(z)$ be the set of all positive δ such that the open δ-neighborhood of x is contractible in the open ε-neighborhood of x in metric ρ. We prove several continuity properties of the map $\Delta : M_X \times X \times (0, \infty) \to (0, \infty)$ and then, using a selection theorem for non-lower semicontinuous mappings, show that Δ admits a continuous singlevalued selection. Similar, but somewhat different properties are also demonstrated for the modulus Δ_n of local n-connectedness.

1. INTRODUCTION

A topological space X is said to be *locally contractible* if for every point $x \in X$ and for each of its neighborhoods $U \subset X$ there exists a neighborhood $V \subset X$ of x such that the inclusion $V \subset U$ is homotopically trivial. For a metric space X the notion of local contractibility admits a definition via the real-valued parameters, namely the radii of neighborhoods U and

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V. More precisely, let M_X be the set of all metrics on X compatible with the given topology on X. The space M_X is equipped with topology induced by the following metric of uniform convergence:

$$\operatorname{dist}(\rho, d) = \sup\{\min\{|\rho(x, y) - d(x, y)|, 1\} | x, y \in X\}.$$

For every triple $(\rho, x, \varepsilon) \in M_X \times X \times (0, \infty)$ we define $\Delta(\rho, x, \varepsilon)$ as the set of all positive numbers δ such that the δ-neighborhood $B(\rho, x, \delta)$ of the point x in metric ρ is contractible in the ε-neighborhood $B(\rho, x, \varepsilon)$ of the point x in the same metric ρ. In this way, we define a multivalued mapping $\Delta : M_X \times X \times (0, \infty) \to (0, \infty)$ with nonempty convex values which we call the *modulus of local contractibility* of the space X.

The definition of modulus Δ_n of local n-connectedness is very similar to the definition of modulus of local contractibility—one only needs to replace the contractibility of the ball $B(\rho, x, \delta)$ inside the ball $B(\rho, x, \varepsilon)$ with the contractibility inside the ball $B(\rho, x, \varepsilon)$ of every continuous image of the n-sphere S^n lying in $B(\rho, x, \delta)$.

The goal of the present paper is to investigate some continuity type properties of Δ. As a corollary we prove the existence of a continuous singlevalued choice (selection) of elements of the sets $\Delta(\rho, x, \varepsilon)$ (see Theorem 1.5).

EXAMPLE (1.1). Let X be the half-open interval $[0, 1)$ and let ρ be the standard metric on $\mathbb{R}$. Then $\Delta(\rho, 0, 1) = (0, \infty)$ and $\Delta(\rho, 0, \varepsilon) = (0, \varepsilon]$, for all $\varepsilon < 1$.

This example shows that the map Δ does not need to be lower semicontinuous and hence in general, the standard Michael selection theory techniques do not apply. However, for a locally compact space X it is always possible to find a lower semicontinuous selection of the map Δ:

THEOREM (1.2). *Let X be a locally contractible and locally compact metrizable space. Define the map $\nabla : M_X \times X \times (0, \infty) \to (0, \infty)$ as follows: let for each triple $(\rho, x, \varepsilon) \in M_X \times X \times (0, \infty)$*

$$\nabla(\rho, x, \varepsilon) = \{\delta \in \Delta(\rho, x, \varepsilon) \mid \textit{the closure of } B(\rho, x, \delta) \textit{ is compact}\}.$$

Then the map ∇ is lower semicontinuous.

In general we can only prove the quasi lower semicontinuity of the closure of the modulus of local contractibility Δ. We denote by $\delta_0(\rho, x, \varepsilon) = \sup \Delta(\rho, x, \varepsilon)$ and $\overline{\Delta}(\rho, x, \varepsilon) = (0, \delta_0(\rho, x, \varepsilon)]$. Clearly, $\overline{\Delta}(\rho, x, \varepsilon)$ is the closure of the set $\Delta(\rho, x, \varepsilon)$ in the complete metric space $\mathbb{R}^* = ((0, \infty) \cup \{\infty\}, c)$, where

$$c(t, s) = |t^{-1} - s^{-1}| \qquad \text{and} \qquad c(t, \infty) = t^{-1}.$$

THEOREM (1.3). *Let X be a locally contractible (locally n-connected) metrizable space. Then the map $\overline{\Delta}: M_X \times X \times (0,\infty) \to \mathbb{R}^*$ (respectively, the map $\overline{\Delta}_n$) is a quasi lower semicontinuous mapping into the complete metric space with closed, convex values.*

THEOREM (1.4) (Gutev [5]). *Any quasi lower semicontinuous closed valued mapping from a topological space into a complete metric space admits a lower semicontinuous closed-value selection.*

THEOREM (1.5). *Let X be a locally contractible (locally n-connected) metrizable space. Then there exists a singlevalued continuous function $\hat{\delta}: M_X \times X \times (0,\infty) \to (0,\infty)$ such that for every point $(\rho, x, \varepsilon) \in M_X \times X \times (0,\infty)$, the neighborhood $B(\rho, x, \hat{\delta}(\rho, x, \varepsilon))$ is contractible in the neighborhood $B(\rho, x, \varepsilon)$ (respectively, every continuous mapping of the n-sphere S^n into $B(\rho, x, \hat{\delta}(\rho, x, \varepsilon))$ is null homotopic in $B(\rho, x, \varepsilon)$).*

Having the values of the multivalued mapping Δ as convex subsets of $\mathbb{R}$ one can try to use the well-known Deutsch–Kenderov approach [3]. More precisely, it is natural to start by proving almost lower semicontinuity or 2-lower semicontinuity of Δ or $\overline{\Delta}$ (see [3] for definitions). Unfortunately, their selection theorems work only for compact convex-valued mappings. A theorem from [6] generalizes the Deutsch–Kenderov theorem, but only for closed convex-valued mappings into $\mathbb{R}$. So, we chose instead the Gutev selection theorem [5, Theorem 1.4].

2. PRELIMINARIES

Recall that a multivalued map $G: X \to Y$ is said to be a *selection* of a multivalued map $F: X \to Y$ if $G(x) \subset F(x)$, for every $x \in X$. A multivalued map $F: X \to Y$ is said to be *lower semicontinuous* if for every open subset $V \subset Y$, the set $F^{-1}(V) = \{x \in X \mid F(x) \cap V \neq \emptyset\}$ is open in the space X. For a metric space (Y, ρ), the lower semicontinuity of F at a point x can be reformulated as follows: for each positive γ the following implication holds:

$$y \in F(x) \Rightarrow x \in \operatorname{Int}\big(F^{-1}(B(\rho, y, \gamma))\big).$$

DEFINITION (2.1) [4, 5, 10]. A multivalued mapping $F: X \to Y$ of a topological space X into a metric space (Y, ρ) is said to be *quasi lower semicontinuous* at a point $x \in X$ if for each positive γ and for each neighborhood $V(x)$ there exists a point $q(x) \in V(x)$ such that the following implication holds:

$$y \in F(q(x)) \Rightarrow x \in \operatorname{Int}\big(F^{-1}(B(\rho, y, \gamma))\big).$$

Quasi lower semicontinuity of $F : X \to Y$ means that F is quasi lower semicontinuous at every point $x \in X$.

Clearly, quasi lower semicontinuity follows from the lower semicontinuity: it suffices to put $q(x) = x$. However, the converse does not hold. Let us consider an example:

EXAMPLE (2.2). Let a function $f : \mathbb{R} \to \mathbb{R}^*$ be monotone and increasing. Then the map $F : t \mapsto (0, f(t)]$ is quasi lower semicontinuous.

In this example it is easy to construct a lower semicontinuous selection of F in a straightforward manner: it suffices to put $H(t) = (0, f(t-0)] \subset F(t)$. We can apply the standard Michael selection theorem [8, Theorem 3.1‴] to the map H and find a singlevalued continuous selection for H and hence for F. In particular, such a selection exists for the modulus of the uniform local contractibility of a metric space (Y, ρ).

3. PROOFS OF THEOREMS

Proof of Theorem (1.2). Let us check the lower semicontinuity of the map ∇ at a point $(\rho, x, \varepsilon) \in M_X \times X \times (0, \infty)$. To this end, we fix $\delta \in \nabla(\rho, x, \varepsilon)$ and some of its σ-neighborhoods $(\delta - \sigma, \delta + \sigma)$. We need to find a neighborhood V of the point (ρ, x, ε), such that for all $(\rho', x', \varepsilon') \in V$, the sets $\nabla(\rho', x', \varepsilon')$ intersect with the interval $(\delta - \sigma, \delta + \sigma)$.

If $\delta - \sigma \leq 0$, then the sets $\nabla(\rho', x', \varepsilon')$ intersect with the interval $(\delta - \sigma, \delta + \sigma)$ for all triples $(\rho', x', \varepsilon')$. In the case when $\delta - \sigma > 0$ we choose numbers α and β so that

$$\delta - \sigma < \beta < \alpha < \delta \tag{1}$$

and we fix a homotopy

$$H : B(\rho, x, \delta) \times [0, 1] \to B(\rho, x, \varepsilon)$$

which contracts the neighborhood $B(\rho, x, \delta)$ into a point.

Let us consider the image $H(\overline{B(\rho, x, \alpha)} \times [0, 1])$, where $\overline{B}$ denotes the closure of B. Because of the compactness of the set $\overline{B(\rho, x, \delta)}$ this image lies in the open ball $B(\rho, x, \varepsilon)$ with some "freedom." More precisely, there exists a number $0 < \lambda < \varepsilon$ such that

$$H\big(\overline{B(\rho, x, \alpha)} \times [0, 1]\big) \subset B(\rho, x, \lambda). \tag{2}$$

Fix a number μ such that $\lambda < \mu < \varepsilon$ and put

$$r = \min\{(\alpha - \beta)/2, (\mu - \lambda)/2, 1\}. \tag{3}$$

Let V_1 be the r-neighborhood of the metric ρ in (M_X, dist), and let V_2 be the r-neighborhood of the point x in metric ρ and let $V_3 = (\mu, \infty)$ be the neighborhood of the number ε. Then by the triangle inequality

$$B(\rho', x', \beta) \subset B(\rho, x, \alpha) \tag{4}$$

and

$$B(\rho, x, \lambda) \subset B(\rho', x', \mu) \tag{5}$$

for all $(\rho', x') \in V_1 \times V_2$. Hence for each triple $(\rho', x', \varepsilon') \in V = V_1 \times V_2 \times V_3$, we have that

$$\begin{aligned} H(B(\rho', x', \beta) \times [0,1]) &\overset{(4)}{\subset} H(B(\rho, x, \alpha) \times [0,1]) \\ &\overset{(2)}{\subset} B(\rho, x, \lambda) \overset{(5)}{\subset} B(\rho', x', \mu) \subset B(\rho', x', \varepsilon'). \end{aligned}$$

Hence the homotopy H contracts the ball $B(\rho', x', \beta)$ into a point in the ball $B(\rho', x', \varepsilon')$. Compactness of $\overline{B(\rho', x', \beta)}$ follows from (1), (4), and from the compactness of $\overline{B(\rho, x, \delta)}$. Hence the number β lies in the intersection $\nabla(\rho', x', \varepsilon') \cap (\delta - \sigma, \delta + \sigma)$. Theorem (1.2) is thus proved. ∎

LEMMA (3.1). *Let Z be a Hausdorff space and suppose that the function $f : Z \to \mathbb{R}^*$ is locally positive. Then $F : z \mapsto (0, f(z)]$ is a quasi lower semicontinuous and closed-valued mapping into the complete metric space $\mathbb{R}^*$.*

Proof. Local positivity of f means that for each $z \in Z$, there exists a neighborhood $U(z)$ such that $\inf\{f(z') \mid z' \in U(z)\} > 0$. To check the quasi lower semicontinuity of F at a point $z \in Z$ it suffices to choose points $q(z)$ in which the values of the function f approximate the above infimum. More precisely, for a fixed $z \in Z$, fixed $\gamma > 0$, and a fixed neighborhood $V(z)$ we put

$$m = \inf\{f(z') \mid z' \in U(z) \cap V(z)\} > 0.$$

If $m = \infty$ then $F \equiv \mathbb{R}^*$ over the whole neighborhood $W = U(z) \cap V(z)$ and in this case we can set $q(z) = z$. If $m < \infty$ then we choose a positive λ such that $c(m, m + \lambda) < \gamma$ and then we choose a point $q(z) \in W$ such that $m < f(q(z)) < m + \lambda$.

Let us check that the following implication holds:

$$t \in F(q(z)) \Rightarrow W \subset F^{-1}(B(c,t,\gamma)).$$

For each point $z' \in W$, we have that $F(z') = (0, m] \cup (m, f(z')]$ and $F(q(z)) = (0, m] \cup (m, f(q(z))]$. Hence only three cases can occur: (a) $t \leq m$; (b) $f(z') > f(q(z))$; and (c) the numbers t and $f(z')$ lie in the segment $[m, f(q(z))]$. In the cases (a) and (b),

$$t \in F(z') \cap B(c,t,\gamma) \neq \varnothing,$$

whereas in the case (c), $c(t, f(z')) < c(m, m + \lambda) < \gamma$ and hence

$$f(z') \in F(z') \cap B(c,t,\gamma) \neq \varnothing. \qquad \blacksquare$$

Proof of Theorem (1.3). By Lemma (3.1) it suffices to check the local positivity of the function $\delta_0\colon (\rho, x, \varepsilon) \mapsto \sup \Delta(\rho, x, \varepsilon)$, i.e., the property that for every $z = (\rho, x, \varepsilon) \in Z = M_X \times X \times (0,\infty)$, there exists a neighborhood $U(z)$ such that

$$\inf\{\delta_0(z') \mid z' \in U(z)\} > 0.$$

For a fixed point $z \in Z$ choose the number 2δ from the set $\Delta(\rho, x, \varepsilon/2)$ and put $\alpha = \min\{\varepsilon/6, \delta/3, 1\}$. Let $U(z)$ be the Cartesian product of the α-neighborhoods of points $\rho \in (M_X, \mathrm{dist})$, $x \in (X, \rho)$, $\varepsilon(0,\infty)$. We claim that $\delta \in \Delta(z')$ for all $z' \in U(z)$, i.e., that

$$\inf\{\delta_0(z') \mid z' \in U(z)\} \geq \delta > 0.$$

In fact, let $z' = (\rho', x', \varepsilon')$ and $\mathrm{dist}(\rho, \rho') < \alpha \leq 1$, $\rho(x, x') < \alpha$, $|\varepsilon - \varepsilon'| < \alpha$. Then we can conclude from $\rho(x, y) < \varepsilon/2$ that

$$\begin{aligned} \rho'(x', y) &\leq \rho(x', y) + \mathrm{dist}(\rho, \rho') < \rho(x', x) + \rho(x, y) + \alpha \\ &< \rho(x, y) - 2\alpha < \varepsilon/2 + 2\alpha \leq \varepsilon - \alpha < \varepsilon' \end{aligned}$$

hence

$$B(\rho, x, \varepsilon/2) \subset B(\rho', x', \varepsilon'). \tag{6}$$

In an analogous manner we obtain from $\rho'(x', y) < \delta$ that

$$\begin{aligned} \rho(x, y) &\leq \rho'(x, y) + \mathrm{dist}(\rho, \rho') < \rho'(x, x') + \rho'(x', y) + \alpha \\ &< \rho'(x', y) + \alpha + \rho(x, x') + \mathrm{dist}(\rho, \rho') \\ &< \rho'(x', y) + 3\alpha < \delta + 3\alpha \leq 2\delta \end{aligned}$$

hence

$$B(\rho', x', \delta) \subset B(\rho, x, 2\delta). \tag{7}$$

By our choice of the number δ we have the ball $B(\rho, x, 2\delta)$ is contradictible to a point in the ball $B(\rho, x, \varepsilon/2)$ and in the ball $B(\rho', x', \varepsilon')$, because of (6). By (7), the ball $B(\rho', x', \delta)$ contracts into the same point (under the same homotopy) in the ball $B(\rho', x', \varepsilon')$. Hence $\delta \in \Delta(z')$, for all $z' \in U(z)$. Theorem (1.3) is thus proved. ∎

Proof of Theorem (1.5). Quasi lower semicontinuity of the map $\overline{\Delta}$ implies quasi lower semicontinuity of the map $\overline{\Delta}/2$. Note that $\overline{\Delta}/2$ is a selection of Δ. By Theorem (1.4), there exists a lower semicontinuous closed-value selection of the map $\overline{\Delta}/2$, say F. By Michael's selection theorem [8, Theorem 3.1‴], the map $\overline{\operatorname{conv} F}$ admits a singlevalued continuous selection. The last selection of the map $\overline{\operatorname{conv} F}$ will be a selection of the map $\overline{\Delta}/2$ and will be a selection of the modulus Δ of the local contractibility of the space X. Theorem (1.5) is thus also proved. ∎

4. EPILOGUE

We conclude with some remarks and observations:

(a) Let (X, d) and (Y, ρ) be metric spaces and let $C(X, Y)$ be the space of all continuous mappings from X into Y, endowed with the uniform topology. Then for each triple $(f, x, \varepsilon) \in C(X, Y) \times X \times (0, \infty)$ one can consider the set of all positive δ such that

$$d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon,$$

i.e., one can define a multivalued mapping

$$C(X, Y) \times X \times (0, \infty) \to (0, \infty).$$

In [11], it was proved that for *locally compact* X such a mapping is always lower semicontinuous and hence admits a selection. However, in general, "modulus of continuity" is non-lower semicontinuous, i.e., the situation is the same as in the present paper. (See also [7], where the inequality $\rho(f(x), f(x')) < \varepsilon$ is replaced by some suitable predicate in variables f, x, x', ε and ρ.)

(b) In the proof of Lemma (3.1) above, the neighborhood W does not depend on the choice of $t \in F(q(z))$. Hence we in fact, prove that in this lemma (and hence also in Theorem (1.3)) the multivalued mapping is weakly Hausdorff lower semicontinuous (for definition see [1]). But de Blasi and Myjak's theorem [1] is not directly applicable due to the incompleteness of the values $F(z)$ in $\mathbb{R}$.

(c) Formally, Theorem (1.5) can be proved using our proof of Theorem (1.3), avoiding Theorem (1.4) and the notion of quasi lower semicontinuity. In fact, local positivity of the function δ_0 implies the existence of a local continuous (constant) selection for a map Δ. Because of paracompactness of the space $Z = M_X \times X \times (0, \infty)$ we can obtain a global single-valued selection for Δ: it suffices to use a suitable continuous partition of unity.

(d) We conclude by stating interesting open problems. It is known that every quasi lower semicontinuous mapping F admits some "maximal" lower semicontinuous selection F_0. Moreover, such a selection coincides with the derived (in the sense of Brown [2]) mapping F' of the mapping F.

Problem (4.1). Under the assumptions of Theorem (1.2), is it true that $\nabla = \Delta'$?

Problem (4.2). Is it true that ∇_n is lower semicontinuous whenever X is locally compact?

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