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FUNCTIONAL ANALYSIS

On Spaces of Continuous Functions on Nonmetrizable Compacta

by

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Summary. We consider the following three classical examples of nonmetrizable compacta: the 2-arrows space T, the lexicographical square L, and the double K_2 of K, where K is an uncountable metric compactum. We study the Banach spaces of continuous functions on these compacta and we express them *via* more standard Banach spaces. We prove the existence of a sufficiently large family of projectors whose images are isomorphic to the whole space of functions. Such an existence yields the contractibility of the linear groups of these Banach spaces of functions. For the space T the proof is based on the well-known conditions of infinite divisibility and smallness of operator blocks. For the spaces L and the double K_2 we use different arguments to prove the contractibility of the linear groups.

1. Introduction. By theorem of A. A. Milyutin [8], for each uncountable metric compactum K the Banach space $C(K)$ of all continuous functions $f : K \to \mathbb{R}$ with the usual sup-norm metric is isomorphic to the Banach space $C(D^\omega)$ of all continuous functions $f : D^\omega \to \mathbb{R}$ defined on the Cantor set D^ω, where $D = \{0, 1\}$. The existence of such an isomorphism guarantees in the space $C(K)$ the existence of a sufficiently large family of projectors whose images are isomorphic to the whole space $C(K)$. For example, one can consider in the space $C(D^\omega)$ the projectors which are induced by the restriction onto the subsets of the Cantor set, homeomorphic to the whole Cantor set. Precisely, such a family of projectors was used in [3, 9] for a proof

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of the contractibility of the linear group $GL(C(K))$ of all isomorphisms of the Banach space $C(K)$, in the uniform topology.

For nonmetrizable compacta K, the problem of the isomorphic classification of spaces $C(K)$ is more complicated than in the metric case (see [12]). The purpose of our paper is to investigate from this point of view the simplest examples of nonmetrizable compacta: the so-called "two arrows" space (T), the "lexicographic sphere" (L) and the "double of K" (K_2) (see [1]). There exists a principal distinction between these nonmetrizable compacta. Any open subset of T has a subset which is homeomorphic to the whole compactum T. However, such a herediterity does not hold in L or K_2. Thus, for the proof of contractibility of $GL(C(T))$ we shall use the approach of ([9], Theorem 1) where the topological property of the contractibility of the linear group of the Banach space was reduced to the geometrical properties of that space, namely to the properties of the *infinite divisibility* (ID) and the *smallness of operator blocks* (SB). For $GL(C(L))$ and for $GL(C(K_2))$ we shall use some other construction in order to avoid the problems with direct verification of the property SB.

The space $C(T)$ has been considered in different papers on weak topology in Banach spaces. For example, it is known that $C(T)$ is not weakly compactly generated (WCG) but nevertheless it admits an equivalent locally uniform convex norm. Moreover, $C(T)$ is not separable, but its conjugate space has a countable total subset, etc. (see [2, 6, 13]). In order to check the properties ID and SB for $C(T)$ and to construct a suitable family of projectors we exploit the existence of an isomorphism between $C(T)$ and some Banach space of functions on the unit interval $I = [0, 1]$. We use the convex structure of I instead of using the zero-dimensionality of the Cantor set D^{ω}.

An analysis of the proof below shows that such an approach gives the contractibility of $GL(C(I))$ without the Milyutin theorem and gives the contractibility of $GL(B)$ for some other Banach spaces of functions on I.

THEOREM 1.1. *The linear group $GL(B)$ of all isomorphisms of a Banach space B is contractible in the uniform topology in the following cases*:
(a) $B = C(T)$,
(b) $B = C(L)$,
(c) $B = C(K_2)$ *where K is any uncountable metric compactum.*

Clearly, for B from Theorem 1.1 we obtain that the space $\Phi(B)$ of all Fredholm operators is the classifying space for the K-theory of compact metric spaces [4, 5].

2. Isomorphic types of $C(T)$, $C(L)$, $C(K_2)$. We shall use the notation $c_0(\Gamma; B)$ for the Banach space of all continuous mappings $f : \Gamma \to B$ from the discrete space Γ which vanish "at infinity", i.e. $c_0(\Gamma; B) = \{f : \Gamma \to B \mid$ for any $\varepsilon > 0$ the set $\{\gamma \in \Gamma \mid \|f(\gamma)\| \geqslant \varepsilon\}$ is finite$\}$. Obviously, $c_0(\Gamma; B)$ is isomorphic to $c_0(|\Gamma|; B)$, where $|\Gamma|$ is the cardinality of Γ. For a countable Γ we put $c_0(\mathbb{N}; B) = c_0(B)$ and for $B = \mathbb{R}$ we put $c_0(\Gamma; \mathbb{R}) = c_0[\Gamma]$. In the space $c_0[\Gamma]$ we fix a notation e_γ for the mapping $\Gamma \to B$ such that $e_\gamma(\alpha) = 0$ for $\alpha \neq \gamma$ and $e_\gamma(\gamma) = 1$. Finally, we define $\mathfrak{c} = |I|$ and $C = C(I)$.

Let $<$ be the usual linear ordering on the unit interval I and let $\prec$ be the lexicographical ordering on the Cartesian square I^2, i.e.

$$(x, a) \prec (y, b) \Leftrightarrow (x < y) \vee (x = y, a < b).$$

It is well known [1] that I^2, with the topology generated by such an ordering $\prec$, is a nonmetrizable compactum, called the "lexicographical square" L. The space called "two arrows" T is the following closed subset of L:

$$T = [0, 1] \times \{0, 1\}.$$

Finally, let K be a compactum and suppose that in the Cartesian product $K \times \{0, 1\}$ the following local basis of topology is introduced:

(i) any point $(x, 1)$ is an open set, for every $x \in K$,

(ii) if $V(x)$ is an open neighbourhood of $x \in K$ then

$$V(x, 0) = (V(x) \times \{0\}) \cup ((V(x) \times \{1\}) \setminus (x, 1))$$

is an open neighbourhood of the point $(x, 0)$ in $K \times \{0, 1\}$.

Then the "double of K", denoted as K_2, is the product $K \times \{0, 1\}$, endowed with the topology described above.

LEMMA 2.1. (a) *The Banach space $C(T)$ is isomorphic to the Banach space $C_r = C_r(I)$ of all functions $f : I \to \mathbb{R}$ which are left continuous and which have the right limit at every point $x \in I$ (We use the notation of H. Corson* [2]),

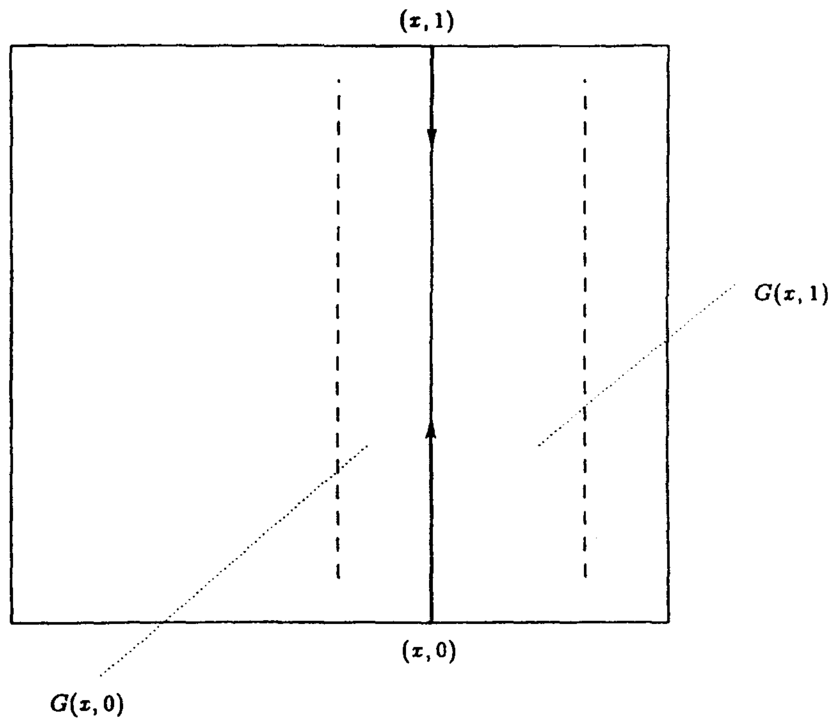
(b) *the Banach space $C(L)$ is isoporphic to the direct sum $C_r \oplus c_0(\mathfrak{c}; C)$,*

(c) *The Banach space $C(K_2)$ is isoporphic to the direct sum $C \oplus c_0[\mathfrak{c}]$ if K is an uncountable metrizable compactum.*

Proof. (a) Let for each $f \in C(T)$ and each $x \in I$, $f_0(x) = f(x, 0)$ and $f_1(x) = f(x, 1)$. Clearly, $f_0 \in C_r$ and f_1 may be obtained from f_0 by setting $f_1(x) = \lim f_0(t)$, as $t \to x + 0$, i.e. f_1 is continuous from the right and has the left limit at every point $x \in I$. It is easy to check that the correspondence $f \mapsto f_0$ gives the desired isomorphism between $C(T)$ and C_r.

(b) Let $R : C(L) \to C(T)$ be the operator of restriction onto the subset $T \subset L$ and let $E : C(T) \to C(L)$ be the operator of the extension, i.e. for each $f \in C(T)$ the function Ef coincides with f onto T and the function Ef is linear over each vertical interval $I_x = \{(x, t) \mid 0 \leqslant t \leqslant 1\}$. Thus, for each $f \in C(L)$, we define $Af = (Rf, f - E(Rf))$. Then A is the desired isomorphism between $C(L)$ and $C_r \oplus c_0([0,1]; C)$.

Linearity, injectivity and continuity of A are evident. Let us check that $g = f - E(Rf)$ is in fact in $c_0([0,1]; C)$. Here, $[0,1]$ denotes the horizontal interval $\{(x,0) \mid 0 \leqslant x \leqslant 1\}$. By the construction, g is identically zero over T and restrictions $g|_{I_x}$ are elements of $C(I_x) = C$. Suppose that, on the contrary, for some $\varepsilon > 0$, there exists an infinite sequence $\{x_n\}$, $x_n \in [0,1]$ such that $\|g_n\| \geqslant \varepsilon$, where g_n is the restriction of g onto the vertical interval I_{x_n}. Let x be the limit point of $\{x_n\}$ and $G = G(x,0) \cup G(x,1)$, where $G(x,0)$ and $G(x,1)$ are basic neighbourhoods of the points $(x,0)$ and $(x,1)$ in which the modulus of g is less than ε. Then G contains one of the vertical intervals I_{x_n}, and thus contradicts the inequality $\|g_n\| \geqslant \varepsilon$ (see Figure).



Figure

(c) We divide the proof into two steps. First, we prove that $C(K_2)$ is isometric to the space $C_\delta(K)$ of all functions on K with removable points of discontinuity. Let $f \in C(K_2)$ and $f_0 = f|_{K\times\{0\}}$, $f_1|_{K\times\{1\}}$. The topology on $K \times \{0\}$ induced from K_2 coincides with the original metric topology on K. Hence $f_0 \in C(K)$. Moreover, from the definition of topology on

K_2 we obtain (see (ii) above) that $\lim_{y\to x} f(y,1) = f(x,0)$, for every pair of points $x, y \in K$, $x \neq y$. Hence f_1 is a function with removable points of discontinuity and f_0 is the "same" function with points of discontinuity removed. Thus, the correspondence $f \mapsto f_1$ gives the desired isometry. Next, we prove that $C_\delta(K)$ is isomorphic to $C(K) \oplus c_0[|K|]$. To see this it is sufficient to define the projector $P : C_\delta(K) \to C_\delta(K)$ by the formula

$$(Pf)(x) = \lim_{y\to x} f(y)$$

and note that the kernel $\operatorname{Ker} P$ is isometric to $c_0[|K|]$ and the image $\operatorname{Im} P$ of this projector is isometric to $C(K)$. Finally, we have that $|K| = c$ and by Milyutin's Theorem, $C(K)$ is isomorphic to C. □

3. Proof of Theorem 1.1(a). Recall, that a Banach space X is said to have the property of infinite divisibility (ID), ([9], Definition 1) if there exists a series $\sum P_n$ of pairwise disjoint projectors $P_n : X \to X$ such that:

(i) this series is unconditionally and pointwisely convergent to the identity operator id_X,

(ii) there exist isomorphisms $\tau_n : \operatorname{Im} P_n \to X$ for every $n \in \mathbb{N}$,

(iii) the operators of the "left" shift $L = \sum i_{n-1}\tau_{n-1}^{-1}\tau_n P_n$ and the "right" shift $R = \sum i_{n+1}\tau_{n+1}^{-1}\tau_n P_n$ are continuous on X (here $i_k : \operatorname{Im} P_k \hookrightarrow X$ are the identity inclusions),

(iv) for every continuous linear operator $A : X \to X$ the "diagonal" operator $\widetilde{A} = \sum i_n\tau_n^{-1}A\tau_n P_n$ is also continuous on X.

Recall also, that a Banach space X is said to have the property of smallness of operator blocks (SB) ([9], Definition 2), if for every $\varepsilon > 0$ and for every compactum S of linear continuous operators in X, $S \subset L(X)$, there exist projectors $P : X \to X$ and $Q : X \to X$ such that:

(i) P and Q are disjoint, i.e. $PQ = QP = 0$,

(ii) the images $\operatorname{Im} P$ and $\operatorname{Im} Q$ are isomorphic to the whole space X,

(iii) $\|PAQ\| < \varepsilon$, for all $A \in S$.

It is easy to see that if one chooses projectors from some bounded (by norm) set of operators then the property SB for finite sets $\{A_1, A_2, \ldots, A_m\} \subset L(X)$ implies the property SB for any compactum $S \subset L(X)$. In our construction the norm of every projector will be at most 2.

We shall use the following family of projectors in the space $C_r = C(T)$. For each segment $\Delta \subset I$ and for each function f on I we set the function $Q_\Delta f$ equal to f outside the interior of the segment Δ and $Q_\Delta f$ equal to a linear function over Δ. Note that $Q_\Delta f$ coincides with f at the ends of Δ and hence $Q_\Delta f$ is correctly defined as a linear function over Δ. Then the function $P_\Delta f$ will be defined by $P_\Delta f = f - Q_\Delta f$. It is clear that the support of the function $P_\Delta f$ lies in Δ, that Q_Δ and P_Δ are continuous linear projectors in C_r, $\|Q_\Delta\| = 1$, $\|P_\Delta\| \leqslant 2$, and that Q_Δ and P_Δ are continuous linear projectors in the closed subspace:

$$C_{r,0} = \{f \in C_r \mid f(0) = f(1) = 0\}$$

of the space C_r. Moreover, the image $\operatorname{Im} P_\Delta$ of the projector P_Δ is isomorphic to $C_{r,0}$.

LEMMA 3.1. (a) *The Banach space $C_{r,0}$ is isomorphic to the Banach space $c_0(C_{r,0})$,*

(b) *the Banach spaces C_r and $C_{r,0}$ are isomorphic.*

Proof. (a) Let $X = C_{r,0}$ and $Y = c_0(X)$. Then Y is isomorphic to $c_0(Y)$ and hence Y is infinitely divisible. Moreover, X is isomorphic to the complementary subspace of Y (the inclusion as the first coordinate). By the decomposition principle ([9], Lemma 7, [11], Proposition 4) it suffices to check the complement of Y in X in order to establish the existence of an isomorphism between X and Y.

Let $\Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$ be a sequence of mutually disjoint segments in I converging to the right end of I. Then for each function $f \in C_{r,0}$, the series $\sum P_{\Delta_n} f$ converges to some function $g = Pf \in C_{r,0}$. Clearly, the image $\operatorname{Im} P$ of the projector P is isomorphic to $Y = c_0(C_{r,0})$.

(b) Invoking (a) and once more the decomposition principle, it suffices to check that C_r and $C_{r,0}$ are isomorphic to the complementary subspaces of each other.

For each $f \in C_r$, we set Qf equal to a linear function on I which coincides with f at the ends of I and put $Pf = f - Qf$. Then $P : C_r \to C_r$ is a continuous projector and $\operatorname{Im} P = C_{r,0}$.

Let $\Delta = [\frac{1}{3}, \frac{2}{3}]$. For each $g \in C_{r,0}$ we put Rg equal to g over Δ and Rg linear on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]$. Then $R : C_{r,0} \to C_{r,0}$ is a continuous projector and the image $\operatorname{Im} R$ is isomorphic to $C_r(\Delta)$ and hence to C_r. □

The Mityagin's theorem ([9], Theorem 1) states that the properties ID and SB are sufficient for contractibility of linear groups GL. Here we really need only one partial case of infinite divisibility. More precisely, we use Lemma 8 from [9] which asserts that if X is isomorphic to $\ell_p(X)$, $1 \leqslant p \leqslant \infty$, or X is isomorphic to $c_0(X)$, then X has the property ID. Thus, from Lemmas 2.1 and 3.1 we get the following corollary.

COROLLARY 3.2. *The spaces $C(T)$, $C(L)$, $C(K_2)$ are infinitely divisible.*

LEMMA 3.3. *Let $\varepsilon > 0$ and let Δ and ∇ be subsegments of $(0,1]$ such that $\Delta \subset \operatorname{Int} \nabla$. If for function $f \in C_r$ the inequality $\|P_\Delta f\| > \varepsilon$ holds, then there exists a subsegment $\square \subset I$ such that:*

(i) $\square \subset \operatorname{Int} \nabla$,

(ii) $f|_{\square}$ *has a constant sign,*
(iii) $|f(x)| > \varepsilon/2$ *for all* $x \in \square$.

P r o o f. There exists $x \in \operatorname{Int} \nabla$ such that

$$|f(x) - Q_{\Delta} f(x)| = |P_{\Delta} f(x)| > \varepsilon,$$

because of inclusion $\operatorname{supp}(P_{\Delta} f) \subset \Delta$ and the inequality $\|P_{\Delta} f\| > \varepsilon$.

If $|f(x)| > \varepsilon/2$ then we can find the desired subsegment $\square$ using left-continuity of f at the point x. If $|f(x)| \leqslant \varepsilon/2$ then $|Q_{\Delta} f(x)| > \varepsilon/2$ and from the linearity of $Q_{\Delta} f$ over Δ we obtain that $|Q_{\Delta} f|$ is more than $\varepsilon/2$ at one of the ends of the segment Δ. However, at the ends of Δ function $Q_{\Delta} f$ coincides with f. Finally, we once more use the continuity from the left of f. □

LEMMA 3.4. *The Banach space C_r has the property SB.*

P r o o f. Let $\varepsilon > 0$ and let $\{A_1, A_2, \ldots, A_m\} \subset L(C_r)$. Pick a sequence $\nabla'_1, \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$ of mutually disjoint subsegments on interval $(0, 1]$ and choose any subsegment $\nabla_1 \subset \operatorname{Int} \nabla'_1$. If for all $1 \leqslant i \leqslant m$ we have that

$$\|P_{\nabla_1} A_i P_{\Delta_1}\| < \varepsilon$$

then the projectors $P = P_{\nabla_1}$ and $Q = P_{\Delta_1}$ give the property SB for the given finite set of operators $\{A_1, A_2, \ldots, A_m\} \subset L(C_r)$. In the opposite case, there exists an index $1 \leqslant i(1) \leqslant m$ such that

$$\|P_{\nabla_1} A_{i(1)} P_{\Delta_1}\| \geqslant \varepsilon > \varepsilon/2.$$

This means that for some $h \in C_r$ with $\|h\| = 1$ we have

$$\|P_{\nabla_1} A_{i(1)} P_{\Delta_1}(h)\| > \varepsilon/2.$$

Apply Lemma 3.3 to the segments ∇_1 and ∇'_1 and to the function $A_{i(1)}(g_1)$, where $g_1 = P_{\Delta_1}(h)$ and hence $\|g_1\| \leqslant 2$, $\operatorname{supp}(g_1) \subset \Delta_1$. In this way we find a subsegment $\square_1$ such that

(i_1) $\square_1 \subset \operatorname{Int} \nabla'_1$,
(ii_1) $A_{i(1)}(g_1)|_{\square_1}$ has a constant sign,
(iii_1) $|A_{i(1)}(g_1)(x)| > \varepsilon/4$, for all $x \in \square_1$.

We can always replace h with $-h$ and hence we can assume that, in addition

(iv_1) $A_{i(1)}(g_1)(x) > \varepsilon/4$ for all $x \in \square_1$.

Next, we pick subsegments ∇_2 and ∇'_2 such that

$$\nabla_2 \subset \operatorname{Int} \nabla'_2 \subset \nabla'_2 \subset \operatorname{Int}(\square_1) \subset \square_1$$

and we repeat the above procedure for the pair of projectors P_{∇_2} and P_{Δ_2}. In this way we find a subsegment $\square_2$ such that:

(i_2) $\square_2 \subset \operatorname{Int} \nabla'_2 \subset \square_1$, (iv$_2$) for some index $1 \leqslant i(2) \leqslant m$ and for some function $g_2 \in C_r$ with $\|g_2\| \leqslant 2$ and $\operatorname{supp}(g_2) \subset \Delta_2$, the following inequality holds

$$A_{i(2)}(g_2)(x) > \varepsilon/4 \quad \text{for all } x \in \square_2.$$

Iterating such a procedure mN times we find some index, say $i(0) \in \{1, 2, \ldots, m\}$ which is repeated at least N times. Let

$$g = \sum_{\substack{k \in \{1,2,\ldots,N_m\} \\ i(k)=i(0)}} g_k.$$

Then $\|g\| \leqslant 2$ because we have, for every $k \geqslant 1$, the inclusion $\operatorname{supp}(g_k) \subset \Delta_k$. By construction, we obtain

$$(A_{i(0)}(g))(x) = \sum (A_{i(0)}(g_k))(x) > N \cdot (\varepsilon/4)$$

for all $x \in \square_{mN} \subset \cdots \subset \square_2 \subset \square_1$. To get a contradiction it is sufficient to choose N so that $N > (8/\varepsilon) \cdot \max\{\|A_1\| : 1 \leqslant i \leqslant m\}$. □

Theorem 1.1(a) is a direct corollary of Lemma 3.4, Corollary 3.2 and Mityagin's Theorem. Let us formulate an abstract version of Theorem 1.1(a):

THEOREM 3.5. *Let B be a closed subspace of Banach space of all bounded functions on the unit interval I with sup-norm. Suppose that*

(i) *for each $f \in B$ and for each $x \in I$, the function f is either left- or right-continuous at the point x,*

(ii) *for each $f \in B$ and each subsegment $\Delta \subset I$, the function $P_\Delta f \in B$, too.*

Then $GL(B)$ is contractible in the uniform topology.

4. Proof of Theorem 1.1(b) and (c). Let a Banach space B be isomorphic to a direct sum of Banach spaces X_1 and X_2, $B = X_1 \oplus X_2$. Then any operator $A \in L(B)$ has the standard (2×2)-matrix representation

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{ij} : X_j \to X_i$ are linear continuous operators. We shall prove that it is possible to find X_1 and X_2 such that $A_{12} = 0$ for a given invertible operator A and, moreover, such that $A_{11} \in GL(X_1)$, $A_{22} \in GL(X_2)$. Note that the invertibility is essential here, which is in contrast with the properties ID and SB which, in turn, are related to any operators from $L(B)$. We start from the case $B = C \oplus c_0[\Gamma] = C(K_2)$, i.e. from Theorem 1.1(c). As in Section 2, we denote e_γ, $\gamma \in \Gamma$, the standard γ-th ort in the space $c_0[\Gamma]$, i.e. $e_\gamma(\alpha) = 0$ for $\alpha \neq \gamma$ and $e_\gamma(\gamma) = 1$.

LEMMA 4.1. *Suppose that* $A \in L(C \oplus c_0[\Gamma])$, *that* Γ *is uncountable, and that* $y_\gamma = A_{12}e_\gamma$. *Then* $|\{\gamma \in \Gamma \mid y_\gamma \neq 0\}| \leqslant \aleph_0$.

Proof. Suppose, on the contrary, that $\Gamma' = \{\gamma \in \Gamma \mid y_\gamma \neq 0\}$ is uncountable. We can assume that $C \subset l_\infty$ and hence in the set $\mathbb{N}$ of all coordinate indices of elements of l_∞ we have an uncountable family of nonempty subsets; namely, the supports of y_γ, $\gamma \in \Gamma'$. Therefore, for some $n \in \mathbb{N}$ and some $\varepsilon > 0$, there exists an uncountable $\Gamma'' \subset \Gamma'$ such that $y_\gamma(n) \geqslant \varepsilon$, for all $\gamma \in \Gamma''$ or $y_\gamma(n) \leqslant -\varepsilon$, for all $\gamma \in \Gamma''$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_k, \ldots\} \subset \Gamma''$. Then

$$x = \sum (1/k) \cdot e_{\gamma k} \in c_0[\Gamma]$$

but $A_{12}x$ has $+\infty$ or $-\infty$ as the n-th coordinate. This is a contradiction. □

Note that Lemma 4.1 holds for any closed subspace of l_∞ e.g. for the space C_r since any function $f \in C_r$ is uniquely determined by its values at the rational points.

LEMMA 4.2. *Suppose that* $A \in L(C \oplus c_0[\Gamma])$, *that* Γ *is uncountable and that* A_{22} *is the corresponding matrix element of* A. *Then* A_{22} *is a* $(\Gamma \times \Gamma)$-*matrix such that in each one of its columns and in each one of its rows there are at most countably many nonzero numbers.*

Proof. Let $a_{\mu\gamma}$ be the μ-th coordinate of $A_{22}e_\gamma$. Then $(a_{\mu\gamma})$ is the desired $(\Gamma \times \Gamma)$-matrix; $\mu, \gamma \in \Gamma$. In fact, $A_{22}e_\gamma \in c_0[\Gamma]$ and hence in each column there is at most a countable set of nonzero numbers. If for some $\mu \in \Gamma$ in the μ-row of the matrix $(a_{\mu\gamma})$ there is an uncountable set of nonzero numbers, then we can repeat the proof from Lemma 4.1 to obtain a contradiction.

LEMMA 4.3. *Under the hypothesis of Lemma 4.2 in the uncountable set* Γ *there exists an equivalence relation* $\sim$ *such that for any equivalence class* $\widetilde{\Gamma}$:

(i) $|\widetilde{\Gamma}| \leqslant \aleph_0$,

(ii) $A_{22}(c_0[\widetilde{\Gamma}]) \subset c_0[\widetilde{\Gamma}]$.

Proof. This is one of the variants of G. Neubauer's construction [10]. Briefly, the $(\Gamma \times \Gamma)$-matrix A_{22} may be divided into some diagonal operator $(\aleph_0 \times \aleph_0)$-blocks so that all elements outside these blocks are zero.

LEMMA 4.4. *Let* $B = C \oplus c_0[\Gamma]$ *and* $A \in GL(B)$. *Then there exists an at most countable subset* $\Gamma_0 \subset \Gamma$ *such that under the representation*

$$B = (C \oplus c_0[\Gamma_0]) \oplus c_0[\Gamma \setminus \Gamma_0]$$

the operator A has the (2×2)-matrix (A_{ij}) for which $A_{12} = 0$, $A_{11} \in GL(C \oplus c_0[\Gamma_0])$, $A_{22} \in GL(c_0[\Gamma \setminus \Gamma_0])$.

Proof. Let Γ' be a subset of Γ from Lemma 4.1 and Γ_{01} be the union of all equivalence classes of all elements from Γ' under the equivalence relation from Lemma 4.3. Then $|\Gamma_{01}| \leqslant \aleph_0$ and $c_0[\Gamma \setminus \Gamma_{01}]$ is an A-invariant subspace of the space B. Let A_0 be the restriction of A onto $c_0[\Gamma \setminus \Gamma_{01}]$. Invoking Lemma 4.3, A_0 is the union of some diagonal A-invariant $(\aleph_0 \times \aleph_0)$-blocks. We claim that at most countable set of these blocks are noninvertible operators.

Suppose that, on the contrary, there are nonsurjective operators in some uncountable set of these blocks. Then in the space $c_0[\Gamma \setminus \Gamma_{01}]$ there exists an uncountable set of norm one elements with mutually disjoint supports, which are images (under operator A) of some elements from the space $C = C \oplus \{0\} \subset B$. However, then the inverse operator A^{-1} maps the uncountable set of norm one elements with mutually disjoint supports into $C \setminus \{0\}$ which contradicts Lemma 4.1.

It is now sufficient to define Γ_{02} as the union of all equivalence classes of all elements from above "non-invertible" blocks and set $\Gamma_0 = \Gamma_{01} \cup \Gamma_{02}$. □

Let us finish the proof of Theorem 1.1(c). First, we note that Lemma 4.4 holds for an arbitrary finite set of invertible operators $A_1, A_2, \ldots, A_m$ in the space $B = C \oplus c_0[\Gamma]$. Hence, Lemma 4.4 holds for each finite simplicial complex $S \subset GL(B)$. Now, the map

$$H(A, t) = \begin{bmatrix} A_{11} & 0 \\ tA_{21} & A_{22} \end{bmatrix}, \quad 0 \leqslant t \leqslant 1.$$

gives a homotopy $H : S \times [0, 1] \to GL(B)$ such that $H|_{S \times \{1\}} = \mathrm{id}|_S$ and all operators from $H(S \times \{0\})$ have a diagonal form.

The contractibility of the linear groups of the spaces $C \oplus c_0[\Gamma_0]$ (which is isomorphic to C) and $c_0[\Gamma \setminus \Gamma_0]$ is a well-known fact [3, 10]. In this way, we have proved that any finite simplicial complex in $GL(B)$ can be shrunk to the point $\{\mathrm{id}|_B\}$. Finally, we use two standard facts. The first one is the following Milnor's lemma:

LEMMA 4.5 ([7]). *Let A be a Banach algebra with a unit and G the group of all its invertible elements. Then the following conditions are equivalent:*

(a) *All homotopy groups $\pi_n(G)$ are trivial, $n \in \{0, 1, 2, \ldots\}$,*

(b) *G is contractible.*

The second one asserts that each subcompactum of $GL(B)$ can be homotopically deformed (in $GL(B)$) into some finite simplicial complex see ([5], § 2, Lemma 1). Theorem 1.1(c) is thus proved, because one can put $A = GL(B)$. □

The proof of Theorem 1.1(b) differs from the one above only in technical details. In fact it suffices to check that:

(1) $c_0(\mathfrak{c}, C)$ has a contractible linear group,

(2) $C_r \oplus c_0(C) = C_r \oplus C$ has a contractible linear group.

The first statement follows from the contractibility of the linear group of the space $c_0(C) = C$ because of separability of C which gives the possibility to use Neubauer's construction. To prove the second one we need to check the property SB for the space $B = C_r \oplus C$. Let

$$C_r = \operatorname{Im} P_1 \oplus \operatorname{Im} Q_1 \oplus \operatorname{Im}(I - P_1 - Q_1)$$

and

$$C = \operatorname{Im} P_2 \oplus \operatorname{Im} Q_2 \oplus \operatorname{Im}(I - P_2 - Q_2)$$

where (P_1, Q_1) and (P_2, Q_2) are pairs of disjoint projectors. Then $B = C_r \oplus C = \operatorname{Im} P \oplus \operatorname{Im} Q \oplus \operatorname{Im}(I - P - Q)$, where $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ and each operator $A \in L(B)$ has the (3×3)-matrix representation under the triple of projectors $(P, Q, I - P - Q)$. Moreover, each block in such a (3×3)-matrix representation has its own (2×2)-matrix representation. For example, the block PAQ has the representation

$$\begin{bmatrix} P_1AQ_1 & P_1AQ_2 \\ P_2AQ_1 & P_2AQ_2 \end{bmatrix}.$$

Next, one can modify the proof of Lemmas 3.3 and 3.4, and find disjoint subsegments Δ_1, ∇_1, Δ_2 and ∇_2 such that for projectors $P_1 = P_{\Delta_1}$, $Q_1 = P_{\nabla_1}$, $P_2 = P_{\Delta_2}$, $Q_2 = P_{\nabla_2}$ the norms of the four operators P_1AQ_1, P_1AQ_2, P_2AQ_1, P_2AQ_2 are less than a given positive ε. □

The referee suggested the following problem: for which classes of $C(K)$-spaces would an analogous proof work? It seems to us that the class of linear ordered compacta K with "sufficiently large" families of subcompacta homeomorphic to K is suitable. However, at the moment we have no positive answers. Hence, as a replacement we formulate the abstract version of Theorem 1.1(b), (c).

THEOREM 4.6. *Let $B_1, B_2, \ldots, B_n$ be Banach spaces for which assertions of Theorem* 3.5 *hold. Then $GL(B_1 \oplus B_2 \oplus \ldots \oplus B_n)$ is contractible in the uniform topology.*

THEOREM 4.7. *Let B_1 and B_2 be Banach spaces for which assertions from Theorem* 3.5 *hold and let Γ be an uncountable set. Then $GL(B_1 \oplus c_0(|\Gamma|, B_2))$ is contractible in the uniform topology.*

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REFERENCES

[1] P. S. Aleksandrov, P. S. Uryson, *A Memoir On Compact Topological Spaces,* (in Russian) Nauka, Moscow 1971.

[2] H. Corson, *The weak topology of Banach space,* Trans. Amer. Math. Soc., **101** (1961) 1-15.

[3] I. Edelstein, B. Mityagin, E. Semenov, *The linear groups of C and L_1 are contractible,* Bull. Pol. Ac.: Math., **18** (1970) 27-33.

[4] K. D. Elworthy, *Fredholm maps and $GL_c(E)$-structures,* Bull. Amer. Math. Soc., **74** (1968) 341-373.

[5] N. Kuiper, *The homotopy type of unitary group of Hilbert space.* Topology, **3** (1965) 19-30.

[6] J. Lindenstrauss, *Weakly compact sets — their topological properties and Banach spaces they generate,* Annals Math. Stud., **69** (1972) 235-273.

[7] J. Milnor, *On spaces having the homotopy type of CW-complex,* Trans. Amer. Math. Soc., **90** (1959) 272-280.

[8] A. A. Milyutin, *Isomorphism of spaces of continuous functions over compact sets of the cardinality of continuum,* (in Russian) Teor. Funkc. Anal. Prilozh., **2** (1966) 150-156.

[9] B. S. Mityagin, *Homotopical structure of linear group of Banach spaces,* (in Russian) Uspekhi Mat. Nauk, **25** (1970) 63-106.

[10] G. Neubauer, *Der Homotopietyp der automorphismengruppe der Raumen c_0 und l_p,* Math. Ann., **174** (1970) 33-40.

[11] A. Pełczyński, *Projections in certain Banach spaces,* Studia Math., **19** (1960) 209-228.

[12] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions,* Dissert. Math., **58** (1968).

[13] R. Pol, *On question of H. Corson and some related problems,* Fund. Math., **109** (1980) 143-152.