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# On Paraconvexity of Graphs of Continuous Functions

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**Abstract.** The concept of paraconvexity of a subset $P \subset E$ of a normed space $E$ was first introduced by E. Michael. Roughly speaking, it consists of a controlled weakening of the convexity assumption for $P$, where the control is guaranteed via some parameter $\alpha \in [0, 1)$. In this paper, we consider the case when $P$ is a subset of some $(n+1)$-dimensional Euclidean space $E$ and $P$ is the graph of some continuous function $f : V \to \mathbb{R}$, where $V \subset E$ is some convex $n$-dimensional subset of $E$. Our key result is that paraconvexity of such a set $P$ follows from the paraconvexity of sections of $P$ by two-dimensional planes, orthogonal to $V$. As an application, we prove a selection theorem for graph-valued mappings whose values have Lipschitzian (with a fixed constant) or monotone two-dimensional sections.

**Key words:** Selection, lower semicontinuous map, $\alpha$-paraconvexity, graph-valued map.

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## 1. Introduction

We shall denote by $E_k$ the Euclidean $k$-dimensional space without any prescribed coordinate system, whereas $\mathbb{R}^k$ will denote the product space

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ copies}}$$

with the standard coordinate system. We shall denote by $\mathrm{Conv}(\mathbb{R}^n)$ the class of all convex subsets of the space $\mathbb{R}^n$ and we shall fix some class $F$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ with convex domains of definition, i.e.

$$F \subset \{f : \mathbb{R}^n \to \mathbb{R} \mid \mathrm{Dom}(f) \in \mathrm{Conv}(\mathbb{R}^n)\}\,.$$

Let $\Gamma(F)$ be the class of graphs of all elements $f \in F$: each element of $\Gamma(F)$ is a subset of $\mathbb{R}^{n+1}$. We shall denote by $\Gamma(F, E_{n+1})$ the following class of subsets of the space $E_{n+1}$: subset $P \subset E_{n+1}$ belongs to $\Gamma(F, E_{n+1})$ if and only if there

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exist $f \in F$ and an isometry $T$ of the Euclidean space $\mathbb{R}^{n+1}$ onto the Euclidean space $E_{n+1}$ such that $P = T(\Gamma_f)$, where $\Gamma_f$ is the graph of the function $f \in F$. In other words, $\Gamma(F, E_{n+1})$ is the class of graphs of all continuous functions in $n$ variables. Note that $\Gamma(F)$ is a subset of $\Gamma(F, \mathbb{R}^{n+1})$ but $\Gamma(F) \neq \Gamma(F, \mathbb{R}^{n+1})$ since $\Gamma(F, \mathbb{R}^{n+1})$ consists of graphs of functions over all orthonormal coordinate systems in the space $\mathbb{R}^{n+1}$.

In this paper, we shall consider the multivalued maps ($m$-maps) whose values are elements of the class $\Gamma(F, E_{n+1})$. More precisely, we are interested in finding conditions for the class $F$ which guarantee the existence of singlevalued continuous selections for such $m$-maps.

Recall that a singlevalued map $g : X \to Y$ is said to be a *selection* of an $m$-map $G : X \to Y$ if and only if $g(x) \in G(x)$, for every point $x \in X$. An $m$-map $\Phi : X \to Y$ between topological spaces $X$ and $Y$ is said to be *lower semicontinuous* if and only if for every open subset $V \subset Y$, the set $\Phi^{-1}(V) = \{x \in X : \Phi(x) \cap V \neq \emptyset\}$ is open in $X$.

We shall use the following E. Michael's selection theorem for nonconvex-valued $m$-maps (see [1]):

THEOREM 1.1 (E. Michael). *Let $\alpha \in [0, 1)$. Then every lower semicontinuous $m$-map $\Phi : X \to B$ from any paracompact space $X$ into any Banach space $B$ whose values are $\alpha$-paraconvex subsets of $B$ admits a singlevalued continuous selection.* □

Recall that a nonempty closed subset $P \subset B$ of a Banach space $B$ is said to be *$\alpha$-paraconvex* if for every open ball $D$ with radius $r$ and for every point $q$ from the closed convex hull $\overline{\text{conv}}(D \cap P)$ of the intersection $D \cap P$, the following inequality holds:

$$\text{dist}(q, P) \leq \alpha \cdot r \ .$$

For graphs of continuous functions, i.e. for elements of the class $\Gamma(F)$, we need some special versions of the notion of paraconvexity. We define for $k \in \{1, 2, \ldots, n\}$ the following subclasses $F_k$ of the class $F$:

$$F_k = \{f|_\Delta | f \in F, \Delta \subset \text{Dom}(f), \Delta \in \text{Conv}(\mathbb{R}^n), \dim \Delta = k\} \ .$$

In summary, the main point of this paper is that for paraconvexity of elements of the class $\Gamma(F)$ it suffices to verify only the paraconvexity of elements of the class $\Gamma(F_1)$, i.e. it suffices to consider only the one-dimensional case.

We shall denote:

(a) by $[x_1, x_2, \ldots, x_m]$ the closed convex hull $\overline{\text{conv}}\{x_1, x_2, \ldots, x_m\}$ of the points $x_1, x_2, \ldots, x_m$;

(b) by $R[x_1, x_2, \ldots, x_m]$ the minimum of the radii of all closed balls which contain all points $x_1, x_2, \ldots, x_m$; and

(c) by $c[x_1, x_2, \ldots, x_m]$ the center of the single closed ball with the radius $R[x_1, x_2, \ldots, x_m]$ which contains all points $x_1, x_2, \ldots, x_m$.

DEFINITION 1.2. Let $0 \le \alpha < 1$ and $0 \le \beta < 1$. The graph $\Gamma_f$ of the element $f \in F_k$ is said to be $(\alpha, \beta)$-paraconvex if for every $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in \mathrm{Dom}(f)$, there exists a point $q \in \Gamma_f$ such that

$$\|c[p_1, p_2, \ldots, p_{k+1}] - q\| \le \alpha \cdot R[p_1, p_2, \ldots, p_{k+1}]$$

and

$$\|c[p_1, p_2, \ldots, p_{k+1}] - q'\| \le \beta \cdot R[p_1, p_2, \ldots, p_{k+1}]\,,$$

where $p_i = (x_i, f(x_i)) \in \Gamma_f$, $i = 1, 2, \ldots, k+1$, and $q'$ is the point of the intersection of the line orthogonal to $\mathrm{Dom}(f)$ and passing through the point $q$ and the plane $\Pi[p_1, p_2, \ldots, p_{k+1}] = p_1 + \mathrm{span}\{p_2 - p_1, \ldots, p_{k+1} - p_1\}$.

DEFINITION 1.3. Let $0 \le \alpha < 1$. The graph $\Gamma_f$ of the element $f \in F_k$ is said to be $(\alpha, +)$-paraconvex if for every $k+1$ points $x_1, x_2, \ldots, x_{k+1} \in \mathrm{Dom}(f)$, there exists a point $q \in \Gamma_f$ such that

$$\|c[p_1, p_2, \ldots, p_{k+1}] - q\| \le \alpha \cdot R[p_1, p_2, \ldots, p_{k+1}]$$

and the points $q$ and $p$ are on the same side with respect to the plane $\Pi[p_1, p_2, \ldots, p_{k+1}]$, where $p$ is the point of the graph $\Gamma_i$ which lies on the vertical line passing through the point $c[p_1, p_2, \ldots, p_{k+1}]$.

Definitions 1 and 2 are, on the one hand, stronger than the original notion of paraconvexity: there are two inequalities or two conditions which make the controlled failure of convexity. But on the other hand, these definitions are weaker restrictions than the original notion of paraconvexity. First, we estimate the distance $\mathrm{dist}(q, \Gamma_f)$ only for the center $q = c[p_1, p_2, \ldots, p_{k+1}]$ of the simplex $[p_1, p_2, \ldots, p_{k+1}]$. Second, and this is in fact, more essential, we consider the simplices $[p_1, p_2, \ldots, p_{k+1}]$ with a fixed dimension $k$ which equals the dimension of the domain of the definition of the function $f \in F_k$.

THEOREM 1.4. *Let $F \subset \{f : \mathbb{R}^n \to \mathbb{R} \mid \mathrm{Dom}(f) \in \mathrm{Conv}(\mathbb{R}^n)\}$ and suppose that for some $\alpha_1, \beta_1 \in [0,1)$, all elements from the class $\Gamma(F_1)$ are $(\alpha_1, \beta_1)$-paraconvex subsets of the space $\mathbb{R}^{n+1}$. Then for every $k \in \{1, 2, \ldots, n\}$, there exist $\alpha_k$ and $\beta_k$ from $[0,1)$ such that all elements from the class $\Gamma(F_k)$ are $(\alpha_k, \beta_k)$-paraconvex subsets of the space $\mathbb{R}^{n+1}$.*

THEOREM 1.5. *Let $F \subset \{f : \mathbb{R}^n \to \mathbb{R} \mid \mathrm{Dom}(f) \in \mathrm{Conv}(\mathbb{R}^n)\}$ and suppose that for some $\alpha_1 \in [0,1)$, all elements from the class $\Gamma(F_1)$ are $(\alpha_1, +)$-paraconvex subsets of the space $\mathbb{R}^{n+1}$. Then for every $k \in \{1, 2, \ldots, n\}$, there exists $\alpha_k \in [0,1)$*

*such that all elements from the class* $\Gamma(F_k)$ *are* $(\alpha_k, +)$*-paraconvex subsets of the space* $\mathbb{R}^{n+1}$.

COROLLARY 1.6. *Let* $F$ *be as in Theorem 1.4 or in Theorem 1.5. Then there exists* $0 \le a < 1$ *such that all closed elements of the class* $\Gamma(F)$ *are* $\alpha$*-paraconvex subsets of the space* $\mathbb{R}^{n+1}$.

COROLLARY 1.7. *Let* $F$ *be as in Theorem 1.4 or in Theorem 1.5 and let* $\Phi : X \to E_{n+1}$ *be a lower semicontinuous map from a paracompact space* $X$ *to the Euclidean space* $E_{n+1}$ *with closed values such that* $\Phi(x) \in \Gamma(F, E_{n+1})$, *for all* $x \in X$. *Then* $\Phi$ *admits a singlevalued continuous selection.*

COROLLARY 1.8. *Let* $\mathrm{Lip}(C) = \{f : \mathbb{R}^n \to \mathbb{R} \mid \mathrm{Dom}(f) \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $|f(x) - f(y)| \le C\|x - y\|$ *for all* $x, y \in \mathrm{Dom}(f)\}$ *and let* $\Phi : X \to E_{n+1}$ *be a lower semicontinuous map from a paracompact space* $X$ *to the Euclidean space* $E_{n+1}$ *with closed values such that* $\Phi(x) \in \Gamma(\mathrm{Lip}(C))$ *for all* $x \in X$. *Then* $\Phi$ *admits a singlevalued continuous selection.*

COROLLARY 1.9. *Let* $\mathrm{Mon} = \{f : \mathbb{R}^n \to \mathbb{R} \mid \mathrm{Dom}(f) \in \mathrm{Conv}(\mathbb{R}^n)$ *and restriction* $f|_l$ *is a monotone function for every line* $l\}$ *and let* $\Phi : X \to E_{n+1}$ *be a lower semicontinuous map from a paracompact space* $X$ *to the Euclidean space* $E_{n+1}$ *with closed values such that* $\Phi(x) \in \Gamma(\mathrm{Mon})$, *for all* $x \in X$. *Then* $\Phi$ *admits a singlevalued continuous selection.*

## 2. **Proof of the Theorem 1.4**

We proceed by induction on $k$. The base of induction coincides with the condition of the theorem. Suppose now that the theorem holds for $1 \le m < k$. We shall verify the theorem for $m = k$.

We fix $f \in F_k$ and points $p = (x_i, f(x_i)) \in \Gamma_f$, $i \in \{1, 2, \ldots, k+1\}$ and we denote by $\Delta$ the simplex $[p_1, \ldots, p_{k+1}]$. We may assume that $\dim(\Delta) = k$ since in the opposite case it suffices to use the inductive hypothesis.

*Case A. The center* $c(\Delta)$ *of the simplex* $\Delta$ *belongs to one of the boundary simplices* $\nabla$ *of this simplex.*

In this case $c(\Delta) = c(\nabla)$, $R(\Delta) = R(\nabla)$ and we may use the inductive hypothesis to the restriction $f$ on $\nabla$.

*Case B. The center* $c(\Delta)$ *of the simplex* $\Delta$ *is its interior point.*

We choose $\varepsilon > 0$ such that $\alpha_{k-1} + \varepsilon < 1$ and $\beta_{k-1} + \varepsilon < 1$ and denote by $d_i$ the distance between the center $c(\Delta)$ and the boundary simplex $\nabla_i$ of the simplex

$\Delta$; $i \in \{1, 2, \ldots, k+1\}$.

*Case BA. There exists $i$ such that $d_i \leq \varepsilon \cdot R(\Delta)$.*

We note that $d_i$ equals to the distance between the center $c(\Delta)$ and the center $c(\nabla_i)$ of the boundary symplex $\nabla_i$. So one can put in this case $\alpha_k = \alpha_{k-1} + \varepsilon$ and $\beta_k = \beta_{k-1} + \varepsilon$ and use the triangle inequality.

*Case BB. For all $i = 1, 2, \ldots, k+1$, the inequality $d_i > \varepsilon \cdot R(\Delta)$ holds.*

First, we will prove that in this case there is an upper estimate that $d_i \leq \delta(\varepsilon) \cdot R(\Delta)$ for certain $\delta(\varepsilon) < 1$. Let $S(R, \varepsilon)$ be the set of all $(k+1)$-simplices $\sigma$ contained in a fixed closed ball with radius $R(\Delta) = R$, centered at the point $c(\Delta) = c$ and which contains the closed ball with radius $\varepsilon \cdot R$, centered at the same point $c$. It's easy to check that in the Hausdorff metric, the set $S(R, \varepsilon)$ is compact (see Blaschke's Choice Theorem). For every $\nabla \in S(R, \varepsilon)$, we define the number $\delta(\nabla)$ to be the product of the number $1/R$ and the distance between the point $c$ and the boundary of the simplex $\nabla$. Then $\delta : S(R, \varepsilon) \to [0, 1)$ is a continuous function on a compact set and therefore

$$\delta(\varepsilon) = \max\{\delta(\nabla) \mid \nabla \in S(R, \varepsilon)\} < 1 .$$

Next, we consider for every $i \in \{1, 2, \ldots, k+1\}$ the restriction $g_i = f|_{\nabla_i}$ of the function $f$ onto the simplex $\nabla_i$. Then $g_i \in F_{k-1}$ and by the inductive hypothesis, we can find points $q_i \in \Gamma_{g_i} \subset \Gamma_f$ such that

$$\|c(\nabla_i) - q_i\| \leq \alpha_{k-1} \cdot R(\nabla_i) \text{ and } \|c(\nabla_i) - q_i'\| \leq \beta_{k-1} \cdot R(\nabla_i)$$

where $q_i'$ is the result of the vertical projection of the point $q_i$ onto the simplex $\nabla_i$; $i \in \{1, 2, \ldots, k+1\}$. We remark that

$$\begin{aligned}
&\|c(\Delta) - q_i'\|^2 \\
&= d_i^2 + \|c(\nabla_i) - q_i'\|^2 \leq d_i^2 + \beta_{k-1}^2 (R^2(\Delta) - d_i^2) \\
&= (1 - \beta_{k-1}^2) \cdot d_i^2 - \beta_{k-1}^2 R^2(\Delta) \leq R^2(\Delta) \cdot ((1 - \beta_{k-1}^2) \cdot \delta^2(\varepsilon) + \beta_{k-1}^2) \\
&= R^2(\Delta) \cdot B_k^2 , \ 0 \leq B_k < 1 .
\end{aligned}$$

Now exactly two cases are possible.

*Case BBA. There exists $i \neq j$ such that the points $q_i$ and $q_j$ lie on different sides of the plane $\Pi[p_1, p_2, \ldots, p_{k+1}] = p_1 + \operatorname{span}\{p_2 - p_1, \ldots, p_{k+1} - p_1\}$;* or

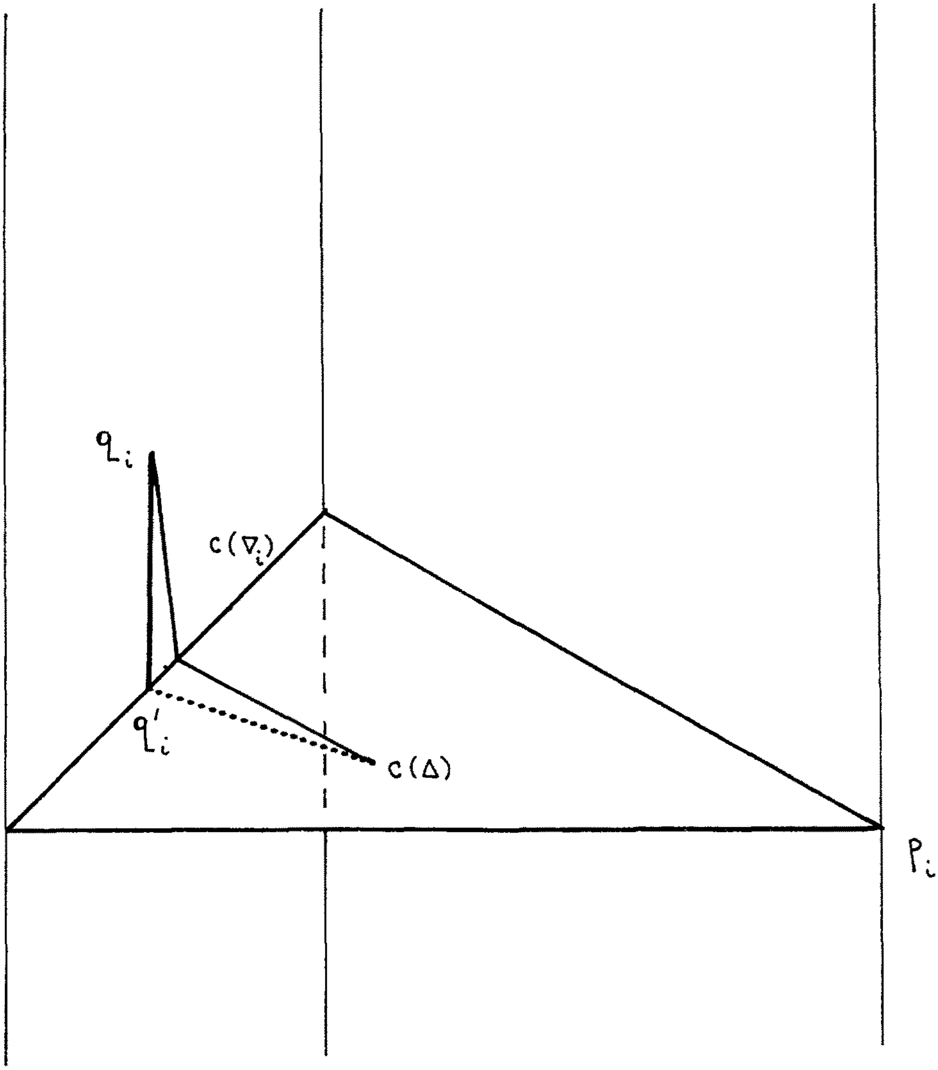


Fig. 1.

*Case BBB. All points $q_i$ lie on the same side of this plane.*

In the case BBA we can find, by the continuity of function $f$ over the segment $[q_i', q_j']$, a point $q \in (\Gamma_f \cap \Delta)$ such that

$$\|c(\Delta) - q\| \leq \max\{\|c(\Delta) - q_i'\|, \|c(\Delta) - q_j'\|\} \leq R(\Delta) \cdot B_k .$$

Moreover, in this case the point $q'$ coincides with the point $q$ and hence the distance $\|c(D) - q'\|$ allows the same upper estimate.

In the case BBB we define vectors $v_i = c(\Delta) - c(\nabla_i)$ and $w_i = q_i - c(\nabla_i)$; $i \in \{1, 2, \dots, k+1\}$. Let all points $q_i$ lie above the plane $\Pi[p_1, p_2, \dots, p_{k+1}]$. From the obvious fact that $c(\Delta) \in \operatorname{conv}\{c(\nabla_1), \dots, c(\nabla_{k+1})\}$ we obtain that one of the angles between vectors $v_i$ and $w_i$ is less than or equal to $\pi/2$ and for such $i$ we have that

$$\|c(\Delta) - q_i\|^2 \leq d_i^2 + \|c(\nabla_i) - q_i\|^2 \leq d_i^2 + \alpha_{k-1}^2 (R^2(\Delta) - d_i^2)$$

$$= (1 - \alpha_{k-1}^2) \cdot d_i^2 + \alpha_{k-1}^2 R^2(\Delta) \leq R^2(\Delta) \cdot ((1 - \alpha_{k-1}^2) \cdot \delta^2(\varepsilon) + \alpha_{k-1}^2)$$

$$= R^2(\Delta) \cdot A_k^2 ; \quad 0 \leq A_k < 1 .$$

We have found the upper estimate for distance $\|c(\Delta) - q_i'\|$ in the part BB above.

In summary, we have the following:

| | Case A | Case B | | |
|---|---|---|---|---|
| | | Case BA | Case BB | |
| | | | Case BBA | Case BBB |
| $\alpha_k$ | $\alpha_{k-1}$ | $\alpha_{k-1} + \varepsilon$ | $B_k$ | $A_k$ |
| $\beta_k$ | $\beta_{k-1}$ | $\beta_{k-1} + \varepsilon$ | $B_k$ | $B_k$ |

where $B_k = ((1 - \beta_{k-1}^2) \cdot \delta^2(\varepsilon) + \beta_{k-1}^2)^{1/2} < 1$, and $\max\{\alpha_{k-1} + \varepsilon,\ \beta_{k-1} + \varepsilon\} < 1$.

$$A_k = ((1 - \alpha_{k-1}^2) \cdot \delta^2(\varepsilon) + \alpha_{k-1}^2)^{1/2} < 1 ,$$

Theorem 1.4 is thus proved. $\square$

## 3. Proof of the Theorem 1.5

The idea of this proof is similar to the proof of the previous theorem. We shall make only the following changes (we may assume that the point $p$ lies above the plane

$$\Pi[p_1, p_2, \dots, p_{k+1}] = \Pi[\Delta] = p_1 + \operatorname{span}\{p_2 - p_1, \dots, p_{k+1} - p_i\}) :$$

*Case BA.* By the triangle inequality, we can find points $q_i \in \Gamma_{g_i}$, where $g_i = f|_{\nabla_i} \in F_i$ such that

$$\|c(\Delta) - q_i\| \leq (\alpha_{k-1} + \varepsilon) \cdot R(\Delta) ; \quad i \in \{1, 2, \dots, k+1\} .$$

If one of the points $q_i$ lies above the plane $\Pi[\Delta]$ then we may set $\alpha_k = \alpha_{k-1} + \varepsilon$ and obtain the $(\alpha_k, +)$-paraconvexity of the graph $\Gamma_f$. Otherwise we conclude, by the definition of $(\alpha_{k-1}, +)$-paraconvexity of the graphs $\Gamma_{k-1}$ that the points $p_i \in \Gamma_f$ belonging to the vertical lines passing through the points $c(\nabla_i)$ are below the plane $\Pi[\Delta]$. So by the continuity of the function $f$ we can find a point $q_i^* \in [c(\Delta), c(\nabla_i)]$ such that $q_i^* \in \Gamma_f$. Moreover, we have the following estimate:

$$\|c(\Delta) - q_i^*\| \leq \|c(\Delta)) - c(\nabla_i)\| \leq \varepsilon \cdot R(\Delta) \leq (\alpha_{k-1} + \varepsilon) \cdot R(\Delta) .$$

*Case BBA*. There exists an integer $i \in \{1, 2, \ldots, k+1\}$ such that the point $q_i$ lies below the plane $\Pi[\Delta]$ and by an argument similar to the argument in the above paragraph, there exists a point $q_i^* \in [c(\Delta), c(\nabla_i)]$ such that $q_i^* \in \Gamma_f$. Therefore,

$$\|c(\Delta) - q_i^*\| \leq \|c(\Delta) - c(\nabla_i)\| \leq \delta(\varepsilon) \cdot R(\Delta)) .$$

*Case BBB*. If all points $q_i$ lie above the plane $\Pi[\Delta]$ the proof coincides with the analogous proof of Theorem 1.4. If all points $q_i$ lie below the plane $\Pi[\Delta]$ then the proof coincides with the proof of the case BBA above.

| | Case A | Case B | | |
|---|---|---|---|---|
| | | Case BA | Case BB | |
| | | | Case BBA | Case BBB |
| $\alpha_k$ | $\alpha_{k-1}$ | $\alpha_{k-1} + \varepsilon$ | $\delta(\varepsilon)$ | $A_k$ |

where $A_k = ((1 - \alpha_{k-1}^2) \cdot \delta^2(\varepsilon) + \alpha_{k-1}^2)^{1/2} < 1$ and $\alpha_{k-1} + \varepsilon < 1$. Theorem 1.5 is thus also proved. □

## 4. **Proofs of Corollaries**

*Proof of Corollary 1.6.* By the definition, it is clear that

$$F \subset F_1 \cup F_2 \cup \ldots \cup F_n .$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the constants of paraconvexity of the elements of the classes $F_1, F_2, \ldots, F_n$ respectively (see Theorem 1.4). Let $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\} < 1$. We fix a function $f \in F$ and let $z$ be an arbitrary point from the convex hull $\mathrm{conv}(D \cap \Gamma_f)$ of the intersection of some open ball $D$ with radius $r$ and the graph $\Gamma_f$ of the function $f$. By Carathéodory's theorem we can find points $y_1, y_2, \ldots, y_{n+2} \in D \cap \Gamma_f$ such that

$$z \in \mathrm{conv}\{y_1, y_2, \ldots, y_{n+2}\} .$$

The key ingredient in the sequel of the proof is the following result from [3], reformulated in accordance with our notations above:

THEOREM 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with a convex domain and let $y_1, y_2, \ldots, y_{n+2} \in \Gamma_f$ and $z \in [y_1, y_2, \ldots, y_{n+2}]$ be arbitrary points. Then there exist points $p_1, p_2, \ldots, p_{n+1} \in \Gamma_f$ such that $z \in [p_1, p_2, \ldots, p_{n+1}]$ and the simplex $[p_1, p_2, \ldots, p_{n+1}]$ can be moved into one of the faces of the simplex $[y_1, y_2, \ldots, y_{n+2}]$.* □

So, if $p_1, p_2, \ldots, p_{n+1}$ are the points provided by Theorem 4.1, then

$$R = R[p_1, p_2, \ldots, p_{n+1}] \leq R[y_1, y_2, \ldots, y_{n+2}] \leq r .$$

By Theorem 1.4, we can find a point $q \in \Gamma_f$ such that

$$\|c[p_1, p_2, \ldots, p_{n+1}] - q\| \leq \alpha R \leq \alpha \cdot r .$$

If $\|c[p_1, p_2, \ldots, p_{n+1}] - z\| \leq \varepsilon \cdot R$ then $\|q - z\| \leq (\alpha + \varepsilon) \cdot R \leq (\alpha + \varepsilon) \cdot r$. If for the center $c = c[p_1, p_2, \ldots, p_{n+1}]$ of the simplex $[p_1, p_2, \ldots, p_{n+1}]$ we have that $\|c - z\| > \varepsilon \cdot R$ then by convexity of the simplex $[p_1, p_2, \ldots, p_{n+1}]$ we can find a point $p_i$ such that the triangle $\Delta c z p_i$ has an obtuse angle at the vertex $z$. Hence

$$\|z - p_i\|^2 \leq \|c - p_i\|^2 - \|c - z\|^2 \leq R^2 - \varepsilon^2 R^2 \leq (1 - \varepsilon^2) \cdot r .$$

Therefore the distance between the point $z$ and the graph $\Gamma_f$ is less than or equal to the product $r \cdot \max\{\alpha + \varepsilon, (1 - \varepsilon^2)^{1/2}\}$. So we can define $\varepsilon > 0$ as a root of the equation

$$(\alpha + x)^2 = 1 - x^2$$

and hence the graph $\Gamma_f$ of the function $f \in F$ is an $\alpha_0$-paraconvex subset of the Euclidean space $\mathbb{R}^{n+1}$, where $\alpha_0 = \alpha + \varepsilon$ and $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\} < 1$; here $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the constants of paraconvexity of the elements of the classes $F_1, F_2, \ldots, F_{n+1}$, respectively (see Theorem 1.4). Corollary 1.6 is thus proved. □

*Proof of Corollary 1.7.* This corollary is a direct consequence of Corollary 1.6 and E. Michael's theorem on selections of paraconvex-valued maps mentioned above. □

*Proof of Corollary 1.8.* It is easy to check that the graph $\Gamma_f$ of an arbitrary function $f(x)$ with a convex, closed domain of definition such that $f \in \mathrm{Lip}(C)$, is an $(\alpha_1, \beta_1)$-paraconvex subset of the Euclidean plane, where $\alpha_1 = \sin(\operatorname{arc\,tg} C)$ and $\beta_1 = \alpha_1^2$. Consequently, Corollaries 1.6 and 1.7 yield a proof of Corollary 1.8. □

*Proof of Corollary 1.9.* We remark that if $f : \mathbb{R} \to \mathbb{R}$ is a monotone continuous function with a convex domain of definition then $f$ is a Lipschitz map, with the Lipschitz constant 1, in some other coordinate system. To see this it suffices to rotate the standard coordinate system by an angle of $\pi/4$ or $-\pi/4$. Then we can use the previous corollary. □

PROBLEM 4.2. Do these results hold for graphs of continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ if $m > 1$?

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