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On Functions of Nonconvexity for Graphs of Continuous Functions

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For any subset P of a normed space we introduce the concept of a function h_P of non-convexity of the set P. We investigate the case when P lies in the Euclidean plane and P is the graph of some continuous function of one variable. One of the applications for example is that in the well-known E. Michael Selection Theorem the condition of convexity in this case can be replaced by the condition that the values of the many-valued map are graphs of polynomials $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, $|a_i| \leq C$. Here, the coordinate system is not fixed: it may be different for different values of the many-valued map.

0. INTRODUCTION

Let P be a subset of a normed space $(E, \|\cdot\|)$. We define a function h_P: $(0, \infty) \to [0, 2]$ which is (approximately) the "characteristic function of nonconvexity" of this set P. For a closed P the equality $h_P \equiv 0$ is equivalent to the convexity of P. The definition of the function h_P is reminiscent of the definition of the moduli convexity of the unit sphere of the Banach space [1].

So, let $B(r)$ be the set of all open balls with radius r in the normed space E. For every $D \in B(r)$, let

$$\delta(D, P) = \sup\{\rho(y, P) \,|\, y \in \overline{\text{conv}}(D \cap P)\}/r$$

where, as usual,

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$$\rho(y, P) = \inf\{\|y - x\| \mid x \in P\}$$

and $\overline{\text{conv}}$ denotes the closure of the convex hull. It is easy to check that for inner product space E the following inequality holds: $0 \le \delta(D, P) \le 1$. Indeed, if $y_1, y_2, \ldots, y_n \in D \cap P$ and if $y \in \text{conv}\{y_1, \ldots, y_n\}$ then the distance from y to one of the vertexes $y_1, y_2, \ldots, y_n$ is less than the radius of the ball D. For any normed space E, the example of the space ℓ_∞ shows that (in general) $0 \le \delta(D, P) \le 2$.

DEFINITION 0.1. Let P be a subset of the normed space E. Then for any $r > 0$ put

$$h_P(r) = \sup\{\delta(D, P) \mid D \in B(r)\}.$$

The function h_p: $(0, \infty) \to [0, 2]$ is called the function of non-convexity of the set P.

For a closed set P the equality $h_P \equiv 0$ is equivalent to the convexity of the set P. Examples (in the Euclidean plane except e) and f)) are:

(a) For the set $P = \{a, b\}$, $\|a - b\| = 2R$, we have that

$$h_P(r) = \begin{cases} 0, & r \le R \\ R/r, & r > R. \end{cases}$$

(b) For the circle P of radius R,

$$h_P(r) = \begin{cases} (R - \sqrt{R^2 - r^2})/r, & r < R \\ R/r, & r \ge R. \end{cases}$$

We get the same answer for the half-circle.

(c) If P is an arc of the circle with the central angle $0 < 2\varphi < \pi$ then

$$h_P(r) = \begin{cases} (R - \sqrt{R^2 - r^2})/r, & r < R \sin \varphi \\ R(1 - \cos \varphi)/r, & r \ge R \sin \varphi. \end{cases}$$

i.e., h_P is strongly increasing on $(0, R \sin \varphi)$ from 0 to $\alpha = \tan(\varphi/2) < 1$ and h_P is strongly decreasing on $[R \sin \varphi, \infty)$ from α to 0.

(d) For the boundary P of the square with the inscribed circle of radius R we have that

$$h_P(r) = \begin{cases} 1/2, & r \le R \\ R/r, & r > R. \end{cases}$$

(e) For a closed set P the concept of α-paraconvexity introduced in [3] by Michael is equivalent to the function of non-convexity h_P being bounded from above by the constant α on the whole interval $(0, \infty)$, $0 \le \alpha < 1$. The result of Klee [2] shows that if for an arbitrary subset $P \subset E$ we have

$$h_P \le 1,$$

then E is two-dimensional or an inner product space.

(f) Clearly, for every point $y \in \overline{\text{conv}}(D \cap P)$, where $D \in B(r)$, we have the inequality

$$\rho(y, P) \le h_P(r) \cdot r$$

1. Preliminaries

In this section we shall explain why the graphs of continuous functions on the Euclidean plane are preferable to arbitrary sets on the plane. We shall also give sufficient conditions on the set of functions to have a common non-decreasing majorant h: $(0, \infty) \to [0, 1]$ for the set of functions of nonconvexity of graphs of these functions.

In Section 2 we demonstrate that from the existence of such a common non-decreasing majorant one can obtain a selection theorem for lower semicontinuous, many-valued maps from a paracompact to the plane, with values which are graphs of such functions. We also give a concrete example for graphs of polynomials.

Recall that a single-valued map φ: $X \to Y$ is called a *selection* of a many-valued map Φ: $X \to Y$ if $\varphi(x) \in \Phi(x)$ for every $x \in X$. A many-valued map Φ: $X \to Y$ is said to be *lower semicontinuous* if for every open set $G \subset Y$, the set $\{x \in X \mid \Phi(x) \cap G \neq \varnothing\}$ is open in X (see [3]).

We begin with the following lemma (see [5, Theorem $(n = 1)$]):

Lemma 1.1. *Let A, B, C be points on a graph Γ_f of a continuous function f defined on an interval. Let D be a point in the triangle ΔABC. Then the point D lies in some segment $[E, F]$, where $E \in \Gamma_f$, $F \in \Gamma_f$, and the distance EF is less than or equal to* $\max\{AB, BC, AC\}$.

To find the function of non-convexity for a subset of the Euclidean plane it suffices to consider only the triangles with vertices from this set and to control the distance between the set and the points of these triangles. In fact, any polygon can be divided into a union of triangles (Carathéodory theorem). Lemma 1.1 shows that for graphs of continuous functions with convex domains of definition it suffices to consider only the segments and to control the distance between graph and points of these segments.

LEMMA 1.2. *Let A and B be points from the subset P of a linear metric space F, $\operatorname{dist}(A, B) = 2r$, and let $\operatorname{dist}(O, P)$ between the middle point O of the segment $[A, B]$ and the set P be less than $\alpha \cdot r$, for some $0 \leq \alpha < 1$. Then for any point $X \in [A, B]$ we have $\operatorname{dist}(X, P) < \tilde{\alpha} r$, where $\tilde{\alpha} = (1 + \alpha)/2 \in [0, 1)$.*

Proof. If $\operatorname{dist}(X, O) > ((1 - \alpha)/2)r$ then $\operatorname{dist}(X, A) < \tilde{\alpha} r$ or $\operatorname{dist}(X, B) < \tilde{\alpha} r$. On the other hand, if $\operatorname{dist}(X, O) \leq ((1 - \alpha)/2)r$ and $\operatorname{dist}(O, Y) < \alpha r$, for some $Y \in P$ then by the triangle inequality we have that $\operatorname{dist}(X, P) \leq \operatorname{dist}(X, Y) < \tilde{\alpha} r$. ∎

Lemma 1.2 shows that if we want to find only a majorant for the functions of nonconvexity of graphs then we can consider only the middle points of the segments. As an application we have the following lemma:

LEMMA 1.3. (a) *Let P be a graph of a Lipschitz function with a constant $k \geq 0$ and with a convex domain of definition. Then P is an α-paraconvex subset of the plane, where $\alpha = (1 + \sin(\arctan k))/2$.*

(b) *Let P be a graph of a monotone continuous function with convex domain of definition. Then P is a β-paraconvex subset of the plane, where $\beta = (1 + (\sqrt{2}/2))/2$.*

Proof. (a) See Fig. 1 below. Here $0 \leq \varphi \leq \gamma = \arctan k$, $OC = OD = r \sin \varphi$ and the set P intersects either the segment $[O, C]$ or the segment $[O, D]$.

(b) It suffices to remark that if we rotate the coordinate system by 45° counterclockwise (clockwise) then the graph of the monotone non-decreasing (non-increasing) continuous function passes to the graph of a Lipschitz function with the constant 1. ∎

DEFINITION 1.2. Let $0 \leq \alpha < 1$ and let β be any function from $(0, \infty)$ into $[0, 1)$. We define the set $\mathcal{F}(\alpha, \beta)$ of functions from $\mathbb{R}$ to $\mathbb{R}$ with closed convex domains of definitions such that there exist a $< b$ such that for any $f \in \mathcal{F}(\alpha, \beta)$ the following conditions hold:

(1) The graphs of restrictions $\Gamma(f|_{(-\infty,a]})$ and $\Gamma(f|_{[b,\infty)})$ are either empty or α-paraconvex.

(2) For any $R > 0$, the graphs of restrictions $\Gamma(f|_{[a-2R,b+2R]})$ are either empty or are $\beta(R)$-paraconvex.

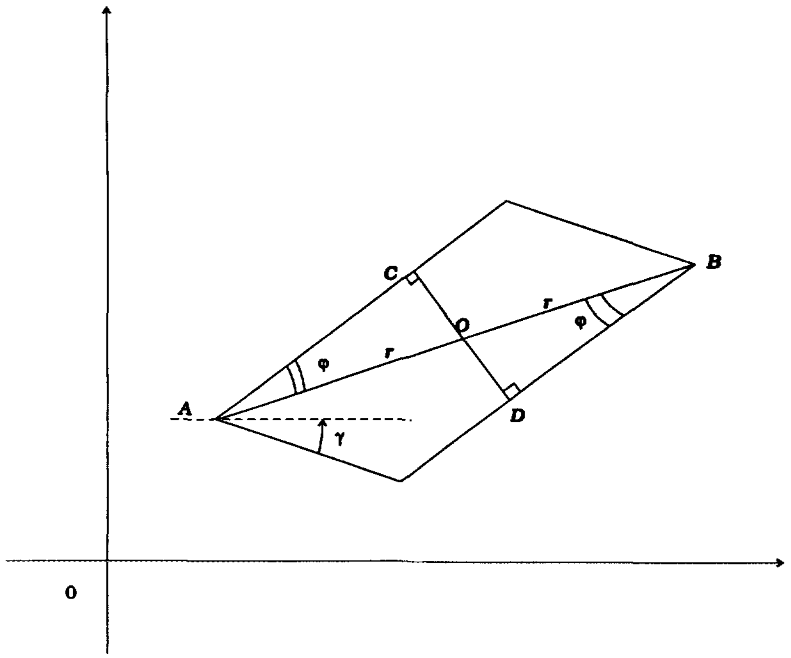

FIGURE 1

THEOREM 1.1. *Let $0 \le \alpha < 1$ and let β be any function from $(0, \infty)$ into $[0, 1)$. Then there exists a non-decreasing function h: $(0, \infty) \to [0, 1)$ such that for any f from the set $\mathcal{F}(\alpha, \beta)$, the function $h_{\Gamma(f)}$ of nonconvexity of the graph $\Gamma(f)$ is less than h on the whole ray $(0, \infty)$.*

Proof. We fix $f \in \mathcal{F}(\alpha, \beta)$ and $R > 0$. It suffices to show that

$$\sup\{h_{\Gamma(f)}(r) \mid 0 < r \le R\} \le \gamma(R) < 1 \qquad (*)$$

where $\gamma(R)$ is some constant which does not depend on f. After such an estimate we may put

$$h(R) = (1 + \sup\{\sup\{h_{\Gamma(f)}(r) \mid 0 < r \le R\} \mid f \in \mathcal{F}(\alpha, \beta)\})/2.$$

Indeed, $h(R) \le (1 + \gamma(R))/2 < 1$, $h(R)$ is a nondecreasing function, and $h_{\Gamma(f)}(R) < h(R)$ for any $f \in \mathcal{F}(\alpha, \beta)$ and for any $R > 0$.

To check the condition $(*)$ we consider any segment $[A, B]$ with $A, B \in \Gamma(f)$ and $AB = 2r$. Let $a < b$ be the numbers from the definition of the set of functions $\mathcal{F}(\alpha, \beta)$. There are only three possibilities: (a) the left

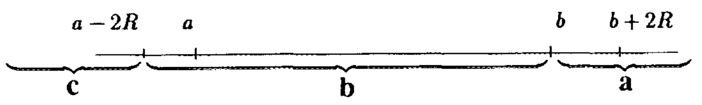

FIGURE 2

endpoint A of the segment $[A, B]$ lies on the right-hand side of the number b; (b) A lies on the left-hand side of the number b and lies on the right-hand side of the number $a - 2R$; (c) A lies on the left-hand side of the number $a - 2R$. (See Fig. 2.)

In the case (a) points A and B lie on the graph $\Gamma(f|_{[b,\infty)})$ and from condition (1) of the definition of the set $\mathcal{F}(\alpha, \beta)$ we have that for any point $X \in [A, B]$, $\text{dist}(X, \Gamma(f)) \le \alpha r$. From the inequality $r \le R$ we have that in the case (b) points A and B lie on the graph $\Gamma(f|_{[a-2R,b+2R]})$ and from condition (2) of the definition of the set $\mathcal{F}(\alpha, \beta)$ we have that for any point $X \in [A, B]$, $\text{dist}(X, \Gamma(f)) \le \beta(R) \cdot r$. Finally, case (c) is analogous to case (a).

Hence we may, in fact, put $\gamma(R) = \max(\alpha, \beta(R))$ and hence Theorem 1.1 is proved. ∎

2. THE MAIN THEOREMS

DEFINITION 2.3 Let α and β be as in Definition 1.2. We denote by $\Gamma(\alpha, \beta)$ the set of all subsets P of the Euclidean plane such that for any $P \in \Gamma(\alpha, \beta)$, there exists an orthonormal coordinate system and there exists a function $f \in \mathcal{F}(\alpha, \beta)$ such that P is graph of f in this coordinate system.

THEOREM 2.2. *Let α and β be as in Definition 1.2. Then every lower semicontinuous map from a paracompact space X into the Euclidean plane with values from $\Gamma(\alpha, \beta)$ has a continuous single-valued selection.*

Proof. We modify the proof of the theorem about the existence of selections of maps with paraconvex values from [4].

Let $h: (0, \infty) \to [0, 1)$ be a non-decreasing strong majorant for the set of all functions of nonconvexity of elements from $\Gamma(\alpha, \beta)$ (for existence of h, see Theorem 1.1).

LEMMA 2.4. *For any lower semicontinuous map F from a paracompact space X into the Euclidean plane with $F(x) \in \Gamma(\alpha, \beta)$, for any $r > 0$, and for any continuous single-valued map $g: X \to \mathbb{R}^2$ with $\text{dist}(g(x), F(x)) < r$, for every $x \in X$, there exists a continuous single-valued selection $f: X \to \mathbb{R}^2$ of the many-valued map F such that $\text{dist}(g(x), f(x)) < H(r)r$, where $H(r) = 1 + \sum_{n=0}^{\infty} h_n(r)$, and $h_0 \equiv 1$, $h_1 \equiv h$, and for every $n \ge 1$, $h_{n+1}(r) = h(h_n(r) \cdot r) \cdot h_n(r)$.*

Proof. We construct the required map f as the uniform limit of a sequence of continuous maps. First, set $f_0 \equiv g$.

Step 1. Let $F_1(x) = \overline{\text{conv}}\{D(f_0(x), r) \cap F(x)\}$, where $D(y, R)$ denotes the open ball with center y and radius R. By the hypothesis, $F_1(x) \neq \varnothing$; by the construction, $F_1(x)$ is closed and convex; and by standard methods (see [3]), F_1 is lower semicontinuous. Hence, by Michael's classical selection theorem [3], F_1 admits a continuous single-valued selection $f_1 : X \to \mathbb{R}^2$, $f_1(x) \in F_1(x)$. But the set $F(x)$ lies in the family $\Gamma(\alpha, \beta)$ and hence $h_{F(x)}(r) < h(r)$. Therefore we have that (see the property (f) from the Introduction):

(i$_1$) $\operatorname{dist}(f_1(x), F(x)) \leq h_{F(x)}(r) \cdot r < h(r) \cdot r = h_1(r) \cdot r.$

By construction, we also have that

(ii$_1$) $\operatorname{dist}(f_1(x), f_0(x)) \leq r = h_0(r) \cdot r.$

Step 2. Let $F_2(x) = \overline{\text{conv}}\{D(f_1(x), h_1(r) \cdot r) \cap F(x)\}$. By Step 1, we have that $F_2(x) \neq \varnothing$, $x \in X$. Hence, as in Step 1, we can find a continuous selection f_2: $X \to \mathbb{R}^2$, $f_2(x) \in F_2(x)$ such that

$$\begin{aligned}\operatorname{dist}(f_2(x), F(x)) &\leq h_{F(x)}(h_1(r) \cdot r) \cdot h_1(r) \cdot r \\ &< h(h_1(r)r)h_1(r)r = h_2(r)r\end{aligned} \qquad (\text{i}_2)$$

and

$$\operatorname{dist}(f_2(x), f_1(x)) \leq h_1(r)r. \qquad (\text{ii}_2)$$

Step n. At the nth step we find a continuous single-valued map f_n: $X \to \mathbb{R}^2$ such that

(i$_n$) $\operatorname{dist}(f_n(x), F(x)) < h_n(r) \cdot r$ and

(ii$_n$) $\operatorname{dist}(f_n(x), f_{n-1}(x)) \leq h_{n-1}(r) \cdot r.$

So, in order to prove Lemma 2.4 it suffices to check that the series $\sum_{n=0}^{\infty} h_n(r)$ is convergent for every $r > 0$.

For a fixed $r > 0$ and for $\alpha = (1 + h(r))/2 \in [0, 1)$ we have that

(iii$_1$) $h_1(r) = h(r) < \alpha$ and $h(h_1(r)r) < h(\alpha r) < h(r) < \alpha$;

(iii$_2$) $h_2(r) = h(h_1(r)r)h_1(r) < \alpha^2$ and $h(h_2(r)r) < h(\alpha^2 r) < h(r) < \alpha$; and

(iii$_3$) $h_3(r) = h(h_2(r)r)h_2(r) < \alpha^3$ and $h(h_3(r)r) < h(\alpha^3 r) < h(r) < \alpha$. Hence $\sum_{n=0}^{\infty} h_n(r) < \sum_{n=0}^{\infty} \alpha^n < \infty$, for every $r > 0$.

Lemma 2.4 is proved. ▌

The rest of the proof of Theorem 2.2 now practically coincides with the corresponding part from [4] with only one modification. We must replace the constant $\beta > 1$ and the sequence $\beta^n \to \infty$ when $n \to \infty$ (see [4]) by the functional sequence $H_0 \equiv 1$, $H_1 \equiv H$ (see Lemma 2.4), $H_{n+1}(r) = H(H_n(r)r)H_n(r)$, $n \geq 1$, and use the fact that $H_n(r) \to \infty$ when $n \to \infty$ for every $r > 0$. ▌

As an application, we consider the case of the graphs of a polynomial. For $n \in \mathbb{N}$ and for $C \geq 0$ we denote by

$$\mathrm{Pol}(n, C) = \{g: \mathbb{R} \to \mathbb{R} \mid g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, |a_i| \leq C\}$$

and Γ Pol(n, C) = $\{P \subset \mathbb{R}^2 \mid P$ is the graph of some element of Pol(n, C) in some orthogonal coordinate system$\}$.

THEOREM 2.3. *For every $n \in \mathbb{N}$ and for every $C \geq 0$ there exist $\alpha \in [0, 1)$ and β: $(0, \infty) \to [0, 1)$ such that* Pol(n, C) $\subset \mathcal{F}(\alpha, \beta)$.

As a corollary, every lower semicontinuous map from a paracompact space X into the Euclidean plane with values from the family Γ Pol(n, C) admits a continuous single-valued selection.

Proof. From the equality

$$g'(x) = nx^{n-1}\left(1 + \sum_{i=0}^{n-1} (i/n)a_ix^{i-n}\right)$$

we have that $g'(x) \neq 0$ for $|x| \geq \max\{1, n \cdot \max_i\{|a_i|\}\}$. Hence, if we put $b = \max\{1, nC\}$ and $a = -b$ then, for every $g \in$ Pol(n, C), g is monotone and continuous on both rays $[b, \infty)$ and $(-\infty, a]$. So, by Lemma 1.3, graphs of restrictions of g onto these rays are, in fact, α-paraconvex sets, where $\alpha = (1 + (\sqrt{2}/2))/2 \in [0, 1)$. Hence, the condition (1) from Definition 1.2 holds.

From the inequality

$$|g'(x)| = \left|\sum_{i=0}^{n-1} (i + 1)a_{i+1}x^i\right| \leq C\sum_{i=0}^{n-1} (i + 1)|x|^i = C\varphi(|x|),$$

where φ is a continuous monotone, increasing function on the ray $(0, \infty)$, we have that on the segment $[a - 2R, b + 2R]$, $R > 0$, any function $g \in$ Pol(n, C) is, in fact, a Lipschitz function with constant $C\varphi(b + 2R)$.

So, by Lemma 1.3, the graph of the restriction of any element $g \in$ Pol(n, C) onto the segment $[a - 2R, b + 2R]$ is a $\beta(R$-paraconvex set where $\beta(R) = (1 + \arctan(C\varphi(\max\{1, nC\} + 2R))/2$.

Hence, the condition (2) from Definition 1.2 holds, too. Theorem 2.3 is proved. ▮

Using a similar argument, an analoguous result can be proved for graphs of polynomials $g(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, such that $|a_i| \leq C$ and $|a_i/a_n| \leq C$, for every $i \in \{1, 2, \ldots, n\}$.

Problem 2.1. Find the analogue of families $\mathcal{F}(\alpha, \beta)$ for a function of n variables and prove the analogue of Theorem 2.2 for such a function.

Problem 2.2. Is it possible to omit the condition $|a_i| \leq C$ from Theorem 2.3? What about the case of polynomials of n variables?

Problem 2.3. Find the criteria for the function h_P of non-convexity of the set P which gives a way to prove a generalization of Theorem 2.2 about selections of paraconvex-valued maps.

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