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ON CONTINUOUS CHOICE IN THE DEFINITION OF CONTINUITY

D. Repovš[1] and P. V. Semenov[2], Ljubljana, Slovenia

Abstract. Using a classical theorem of C. H. Dowker on continuous separation of lower and upper semicontinuous functions, we prove the following result: *Let* (X, d) *and* (Y, ρ) *be metric spaces and suppose that* X *is locally compact. Then there exists a continuous function* $\widehat{\delta} : \mathscr{C}(X, Y) \times X \times (0, \infty) \to (0, \infty)$ *such that for every* $(f, x, \varepsilon) \in \mathscr{C}(X, Y) \times X \times (0, \infty)$ and for every $x' \in X$ the following implication holds: $d(x, x') < \hat{\delta}(f, x, \varepsilon) \implies \rho(f(x), f(x')) < \varepsilon$. As a corollary, we obtain that the Cantor theorem on uniform continuity follows from the Weierstrass theorem on boundedness of continuous functions on compacta.

1. Introduction

Recall the definition of continuity of a map f between metric spaces (X, d) and (Y, ρ):

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in X : \\ d(x, x') < \delta \implies \rho(f(x), f(x')) < \varepsilon).$$

The purpose of this note is to show that, for a locally compact space X (and for any Y), it is possible to choose $\delta > 0$, which continuously depends on the triple $(f, x, \varepsilon) \in \mathscr{C}(X, Y) \times X \times (0, \infty)$ and for which the above implication holds, where $\mathscr{C}(X, Y)$ is the set of all continuous maps from X into Y, endowed with the metric of *uniform convergence*:

$$\operatorname{dist}(f, g) = \sup\{\min\{1, \rho(f(x), g(x))\} \mid x \in X\}.$$

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[2] Permanent address: Moscow State Pedagogical University, 1 M. Pirogovskaya St., 119882 Moscow, Russia.

THEOREM 1. *Let (X, d) and (Y, ρ) be metric spaces and suppose that X is locally compact. Then there exists a continuous function $\widehat{\delta} : \mathscr{C}(X, Y) \times X \times (0, \infty) \to (0, \infty)$ such that for any $(f, x, \varepsilon) \in \mathscr{C}(X, Y) \times X \times (0, \infty)$ and for any $x' \in X$ the following implication holds*

$$d(x, x') < \widetilde{\delta}(f, x, \varepsilon) \implies \rho(f(x), f(x')) < \varepsilon.$$

Here is the outline of the proof:

(a) For any $z = (f, x, \varepsilon) \in Z = \mathscr{C}(X, Y) \times X \times (0, \infty)$ define $D = D(z)$ to be the set of all $\delta > 0$ such that
   (*) the closure of the δ-neighbourhood of the point x is compact and
   (**) for every $x' \in X$, $d(x, x') < \delta \implies \rho(f(x), f(x')) < \varepsilon$.

(b) Put $\Delta(z) =$ the supremum of the set $D(z)$.

(c) Prove that Δ is a lower semicontinuous function from Z to $(0, \infty)$.

(d) Apply C. H. Dowker's theorem about the existence of continuous separating functions on paracompacta.

Remarks. (1) Without condition $(*)$ from (a), the assertion in (c) is false. (2) If the metric d in X is unbounded then it is possible to have $\Delta(z) = \infty$ (cf. (b) above). In this case one can use a slight strenghtening of Dowker's theorem for functions which admit infinite values. Another possibility is to replace Dowker's theorem by one of E. Michael's selection theorems for maps with convex but non-closed values ([2], Theorem 3.1‴]) for multivalued maps $D : Z \to \mathbf{R}$.

2. Proof of Theorem 1

Neighbourhoods of points $x \in X$ of radius δ we shall denote by $V(x; \delta)$ and the closure of a subset $A \subset X$ we shall denote by $\overline{A}$. Define a function $\Delta : \mathscr{C}(X, Y) \times X \times (0, \infty) \to (0, \infty) \cup \{\infty\}$ as follows: for $z \in Z = \mathscr{C}(X, Y) \times X \times (0, \infty)$ let $\Delta(z) = \Delta(f, x, \varepsilon) = \sup\{\delta \in (0, \infty) \mid \overline{V(x; \delta)}$ is compact and for every $x' \in X$, $(d(x, x') < \delta \implies \rho(f(x), f(x')) < \varepsilon)\}$.

By hypotesis, we have that $\Delta(z) > 0$, for all $z \in Z$. So in order to prove Theorem 1 we only need to check that the function $\Delta : Z \to (0, \infty)$ is lower semicontinuous since then we shall apply the following result of Dowker (cf. [1, Ch. VIII, Theorem (4.3)]).

THEOREM 2. *(C. H. Dowker) Let $u : X \to \mathbf{R}$ and $v : X \to \mathbf{R}$ be real-valued functions, defined on a paracompact space X and suppose that for every $x \in X$, $u(x) > v(x)$, that u is lower semicontinuous* *and that* v *is upper semicontinuous. Then there exists a continuous function* $w : X \to \mathbf{R}$ *such that for all* $x \in X$, $v(x) < w(x) < u(x)$.

We shall apply this theorem in our case for $u \equiv \Delta$, $v \equiv 0$, $X = Z$ and hence we shall find a continuous function $\hat{\delta} : Z \to \mathbf{R}$ such that $0 < \hat{\delta}(z) < \Delta(z)$, for all $z \in Z$. In addition, we shall also use the fact that $\mathscr{C}(X, Y)$ is a metrizable space and we shall invoke the Stone theorem [1, Ch.IX, Theorem (5.3)] about paracompactness of metrizable spaces.

Suppose that, on the contrary, the function $\Delta : Z \to (0, \infty) \cup \{\infty\}$ isn't lower semicontinuous, i.e. that there exists

(i) a point $z_0 = (f_0, x_0, \varepsilon_0) \in Z$;
(ii) a number $0 < \alpha < \Delta(z_0)$; and
(iii) a sequence $z_n = (f_n, x_n, \varepsilon_n) \in Z$, $z_n \to z_0$ such that

$$\Delta(z_n) \leqslant \alpha. \tag{1}$$

We fix numbers β and γ such that

$$\Delta(z_n) \leqslant \alpha < \beta < \gamma < \Delta(z_0). \tag{2}$$

Since $x_n \to x_0$ we may assume that for any point x_n there exists a δ_n-neighbourhood $V(x_n, \delta_n)$ such that

$$\Delta(z_n) \leqslant \alpha \leqslant \delta_n \tag{3}$$

and

$$V(x_n, \delta_n) \subset V(x_0, \beta). \tag{4}$$

We remark that, by (4), $\overline{V(x_n, \delta_n)}$ are compact sets. Hence, by (3), it follows that there exists x'_n such that

$$x'_n \in V(x_n, \delta_n). \tag{5}$$

However,

$$\rho(f_n(x_n), f_n(x'_n)) \geqslant \varepsilon_0. \tag{6}$$

Due to the compactness of $\overline{V(x_0, \beta)}$, we may assume that

$$x'_n \to x' \in \overline{V(x_0, \beta)} \subset V(x_0, \gamma)$$

From the inequality $\gamma < \Delta(z_0)$, we have that

$$\rho(f_0(x_0), f_0(x')) < \varepsilon_0. \tag{7}$$

On the other hand, if we pass in (6) to the limit when $n \to \infty$, we have that

$$\rho(f_0(x_0), f_0(x')) \geqslant \varepsilon_0 \tag{8}$$

which contradicts (7).

To verify (8), it suffices to check that $f_n(x_n) \to f_0(x_0)$ and that $f_n(x'_n) \to f_0(x')$. But we have that

$$\rho(f_n(x_n), f_0(x_0)) \leqslant \rho(f_n(x_n), f_0(x_n)) + \rho(f_0(x_n), f_0(x_0)). \qquad (9)$$

The first term on the right hand side of (9) converges to zero because f_n is uniformly converging to f_0. The second term on the right hand side of (9) converges to zero because f_0 is continuous. The convergence $f_n(x'_n) \to f_0(x')$ may be checked in an analoguous manner.

3. Epilogue

COROLLARY 1. *The Cantor theorem on uniform continuity is a corollary of the Weierstrass theorem on boundedness of continuous functions on compacta.*

Proof. For a fixed $f \in \mathscr{C}(X, \mathbf{R})$ and for a fixed $\varepsilon > 0$ it suffices to choose $\delta > 0$ such that

$$\delta < \min\{\widehat{\delta}(f, x, \varepsilon) \mid x \in X\},$$

where X is a compact metric space and $\widehat{\delta}$ is a continuous function from Theorem 1.

COROLLARY 2. *Let (X, d) be a compact metric space, (Y, ρ) be a metric space and the set $\mathscr{C}(X, Y)$ be endowed with the metric*

$$\operatorname{dist}(f, g) = \max\{\rho(f(x), g(x)) \mid x \in X\}.$$

Then any precompact set $\mathscr{F} \subset \mathscr{C}(X, Y)$ is an equicontinuous set of maps from X into Y.

Proof. For a fixed $\varepsilon > 0$ it suffices to choose $\delta > 0$ such that

$$\delta < \min\{\widehat{\delta}(f, x, \varepsilon) \mid f \in \overline{\mathscr{F}},\ x \in X\}.$$

Question. Is Theorem 1 true for arbitrary (non-locally compact) metric spaces X?

We can only give an example which shows that without condition $(*)$, that $\overline{V(x, \delta)}$ is compact, from the definition of the function Δ, Dowker's theorem doesn't apply.

Example. Let $X = Y = (-1, 1)$, $f_0(x) = x$ and let the set $\Delta_1(f, x, \varepsilon)$ equal $\sup\{\delta \in (0, \infty) \mid \forall x' \in X, (d(x, x') < \delta \implies \rho(f(x), f(x')) < \varepsilon)\}$. Then $\Delta_1(f_0, 0, 1) = \infty$, but for any non-zero $x \in (-1, 1)$, we have that $\delta_1(f_0, x, 1) = 1$. Hence, the map Δ_1 is not lower semicontinuous. In the example, $\overline{V(0,1)} = V(0,1)$ is not compact. So condition $(*)$ from the definition of the map Δ is necessary for applicability of Dowker's theorem.

If in this example we do not omit condition (1), we get a lower semicontinuous function Δ such that $\Delta(z) \leqslant \Delta_1(z)$, for any $z \in Z$. Therefore, in this example Theorem 1 is of course true.

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REFERENCES

[1] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
[2] E. Michael, *Continuous selections I*, Ann. of Math. (2) 63 (1956), 361–382.

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Institute for Mathematics,
Physics and Mechanics
University of Ljubljana
P.O.Box 64, Ljubljana 61111
Slovenia

O NEPREKIDNOM IZBORU U DEFINICIJI NEPREKIDNOSTI

D. Repovš i *P. V. Semenov*, Ljubljana, Slovenija

Sadržaj

Koristeći klasični Dowkerov teorem o neprekidnom razdvajanju donje i gornje poluneprekidne funkcije dokazano je sljedeće: Neka su (X,d) i (Y,ρ) metrički prostori i neka je X lokalno kompaktan. Tada postoji neprekidna funkcija $\hat{\delta} : \mathscr{C}(X,Y)\times X\times(0,\infty) \to (0,\infty)$ takva da za proizvoljan $(f,x,\varepsilon) \in (\mathscr{C}(X,Y) \times X \times (0,\infty)$ i proizvoljan $x' \in X$ vrijedi implikacija $d(x,x') < \hat{\delta}(f,x,\varepsilon) \implies \rho(f(x),f(x')) < \varepsilon$. Kao posljedica, pokazano je da Cantorov teorem o uniformnoj neprekidnosti slijedi iz Weierstrassovog teorema o ograničenosti neprekidne funkcije na kompaktu.