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Ernest Michael and theory of continuous selections

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To follow the thoughts of a great man is the most interesting science.
A.S. Pushkin

1. Introduction

For a large number of those working in topology, functional analysis, multivalued analysis, approximation theory, convex geometry, mathematical economics, control theory, and several other areas, the year 1956 has always been strongly connected with the publication by Ernest Michael of two fundamental papers on continuous selections which appeared in the Annals of Mathematics [4,5].

With sufficient precision that year marked the beginning of the theory of continuous selections of multivalued mappings. In the last fifty years the approach to multivalued mappings and their selections, set forth by Michael [4,5], has well established itself in contemporary mathematics. Moreover, it has become an indispensable tool for many mathematicians working in vastly different areas.

Clearly, the principal reason for this is the naturality of the concept of selection. In fact, many mathematical assertions can be reduced to using the linguistic reversal "$\forall x \in X\ \exists y \in Y \ldots$". However, as soon as we speak of the validity of assertions of the type

$$\forall x \in X\ \exists y \in Y \quad P(x, y)$$

it is natural to associate to every x a nonempty set of all those y for which $P(x, y)$ is true. In this way we obtain a multivalued map which can be interpreted as a mapping, which associates to every initial data $x \in X$ of some problem P a nonempty set of solutions of this problem

$$F : x \mapsto \{y \in Y :\ P(x, y)\}, \qquad F : X \to Y.$$

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The question of the existence of selections in such a setting turns out to be the question about the unique choice of the solution of the problem under given initial conditions. Different types of selections are considered in different mathematical categories.

One could say that the key importance of Michael's theory is not so much in providing a comprehensive solution of diverse selection problems in the category of topological spaces and continuous maps, but rather the immediate inclusion of the obtained results into the general context of development of topology. In a remarkable number of cases, results of Michael on solvability of the selection problems turned out to be the final answers, i.e. they provided conditions which turned out to be necessary and sufficient.

Initially we were planning to write a survey paper, which would present the development of the theory in the last half of the century and its many applications. However, already our first attempts at such a project showed that the volume of such a survey would invariably fill an entire book, hence it would be inappropriate for this special issue.

After some deliberations we decided to limit ourselves to a survey of only the papers of Michael on the theory of selections and their mutual relations. For analogous reasons we do not give any precise references to many developments in the theory of selections—the number of papers in this area is by now around one thousand. A considerable number of facts on selections and theorems, which go beyond the present paper, can be found in books and surveys [R1–R15] listed at the end of the paper.

2. Bibliography

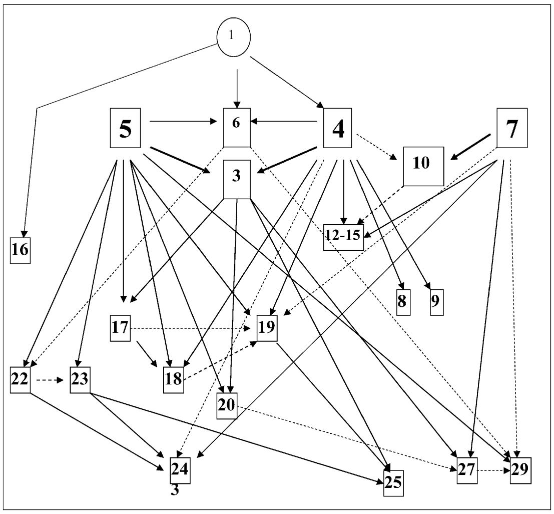
Papers in scientific journals usually end with the list of references. In our opinion, is it most reasonable to begin a survey dedicated to the work of a single person, on one special topic, spanning over 50 years, with a complete list of his papers on the subject.

List of all papers by E. Michael on selections

1. Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951) 152–182.
2. Selection theorems for continuous functions, Proc. Int. Congr. Math. 2 (1954) 241–242.
3. Selected selection theorems, Amer. Math. Monthly 63 (1956) 233–238.
4. Continuous selections I, Ann. of Math. (2) 63 (1956) 361–382.
5. Continuous selections II, Ann. of Math. (2) 64 (1956) 562–580.
6. Continuous selections III, Ann. of Math. (2) 65 (1957) 375–390.
7. A theorem on semi-continuous set valued functions, Duke Math. J. 26 (1959) 647–652.
8. Dense families of continuous selections, Fund. Math. 47 (1959) 174–178.
9. Paraconvex sets, Math. Scand. 7 (1959) 372–376.
10. Convex structures and continuous selections, Canadian J. Math. 11 (1959) 556–575.
11. Continuous selections in Banach spaces, Studia Math. Ser. Spec. (1963) 75–76.
12. A linear mapping between function spaces, Proc. Amer. Math. Soc. 15 (1964) 407–409.
13. Three mapping theorems, Proc. Amer. Math. Soc. 15 (1964) 410–415.
14. A short proof of the Arens–Eells embedding theorem, Proc. Amer. Math. Soc. 14 (1964) 415–416.
15. A selection theorem, Proc. Amer. Math. Soc. 17 (1966) 1404–1406.
16. Topological well-ordering, Invent. Math. 6 (1968) 150–158 (with R. Engelking and R. Heath).
17. A unified theorem on continuous selections, Pacific J. Math. 87 (1980) 187–188 (with C. Pixley).
18. Continuous selections and finite-dimensional sets, Pacific J. Math. 87 (1980) 189–197.
19. Continuous selections and countable sets, Fund. Math. 111 (1981) 1–10.
20. A parametrization theorem, Topology Appl. 21 (1985) 87–94 (with G. Mägerl and R.D. Mauldin).
21. A note on a selection theorem, Proc. Amer. Math. Soc. 99 (1987) 575–576.
22. Continuous selections avoiding a set, Topology Appl. 28 (1988) 195–213.
23. A generalization of a theorem on continuous selections, Proc. Amer. Math. Soc. 105 (1989) 236–243.
24. Some problems, in: J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 271–277.
25. Some refinements of a selection theorem with 0-dimensional domain, Fund. Math. 140 (1992) 279–287.
26. Selection theorems with and without dimensional restriction, in: Recent Developments of General Topology and its Applications, International Conference in Memory of Felix Hausdorff (1868–1942), Math. Res. 67, Berlin, 1992.
27. Representing spaces as images of 0-dimensional spaces, Topology Appl. 49 (1993) 217–220 (with M.M. Choban).
28. A note on global and local selections, Topology Proc. 18 (1993) 189–194.

29. A theorem of Nepomnyashchii on continuous subset-selections, Topology Appl. 142 (2004) 235–244.
30. Continuous selections, Encyclopedia of General Topology, c-8 (2004) 107–109.

We have selected the papers on selections [4,5,7] to serve as the basis of the classification of the entire list. Here is a reasonably precise diagram of relationship among the papers from the list:



Here the usual arrow means direct correlation and the dotted arrow means an implicit one.

Papers [2], [11], [26], [28], [30] are not included in this diagram, since they are either short announcements (or abstracts) on conferences or they are devoted to popularization of the subject.

3. Papers from 1956

A considerable number of fundamental mathematical papers can be divided into two types. In such papers, as a rule, a significant new theory is constructed or an important problem is solved. This division is of course, conditional—on the one hand, in constructions of new theories one often encounters difficult problems, on the other hand a solution of a difficult problem often gives rise to a development of a significant new theory.

The papers [4] and [5] are a clear cut example of such a division. In [4] an essentially new mathematical theory is constructed, in the form of a branched tree, which unifies a large number of sufficiently different theorems. To the contrary, in [5] the principal result consists of the proof of a single highly nontrivial theorem and all assertions and constructions in this paper are devoted to the solution of this problem.

Another, linguistic difference between [4] and [5] is connected with the notion of convexity: the formulations of practically all theorems of [4] use the term *convex*, where to the contrary, this word is practically absent from [5]. Finally, in [4] Lebesgue dimension is never used, while in [5], there are dimension restrictions on the domains of the multivalued maps everywhere.

One can say, with sufficient accuracy, that in [5] the finite-dimensional, purely topological analogue of such a nontopological notions as convexity and local convexity are presented and studied. Without any doubt, the best known assertion of [4] is Theorem 3.2″.

Theorem 1. *The following properties of a T_1-space X are equivalent*:

(a) *X is paracompact*; *and*
(b) *If Y is a Banach space, then every lower semicontinuous* (*LSC*) *carrier $\phi : X \to \mathcal{F}_c(Y)$ admits a singlevalued continuous selection.*

Here $\mathcal{F}_c(Y)$ denotes the family of all nonempty closed convex subsets of Y. Observe that in [4] Michael originally used the term "carrier" instead of "multivalued mapping". In mathematical practice the implication (a) $\Rightarrow$ (b) has the widest application and is in folklore known as the "Convex-Valued Selection Theorem". The implication (b) $\Rightarrow$ (a) gives a selection characterization of paracompactness.

The unusual numeration $3.2''$ for the theorem has a very simple explanation. In Chapter 3 of [4] Michael started by a citation of Theorem 3.1 (Urysohn, Dugundji, Hanner) and Theorem 3.2 (Dowker) on the extensions of *singlevalued* mappings and then presented the sequences:

Theorem 3.1, Theorem $3.1'$, Theorem $3.1''$, Theorem $3.1'''$ and

Theorem 3.2, Theorem $3.2'$, Theorem $3.2''$

of their analogs for *multivalued* mappings. To be more clear, let us unify Theorems 3.1 (a, b, c below) and $3.1'$ (a, d, e below) as follows:

Theorem 2. *The following properties of a T_1-space X are equivalent*:

(a) *X is normal*;
(b) *The real line $\mathbb{R}$ is an extensor for X*;
(c) *Every separable Banach space is extensor for X*;
(d) *Every LSC carrier $\phi : X \to \mathcal{C}(\mathbb{R})$ admits a singlevalued continuous selection*; *and*
(e) *If Y is a separable Banach space, then every LSC carrier $\phi : X \to \mathcal{C}(Y)$ admits a singlevalued continuous selection.*

Thus, playing with words, such a series shows that the selection theory in fact, extends the theory of extensors. Here $\mathcal{C}(Y) = \{Z \in \mathcal{F}_c(Y)\colon\ Z \text{ is compact or } Z = Y\}$.

As asserted by Michael, Theorem $3.2''$ was his very first selection theorem, the initial goal of which were generalizations of a theorem due to R. Bartle and L. Graves on sections of linear continuous surjections between Banach spaces. In particular, Proposition 7.2 of [4] states that such a section can be chosen in an "almost" linear fashion (scalar homogenous) and with the pointwise norm arbitrarily close to the "minimal" of all possible.

Thus the remaining Theorems $3.1''$–$3.2'$ are selection characterizations of other properties of the domain of a convex-valued mapping: normality, collectionwise normality, normality and countable paracompactness, and perfect normality. Many constructions and ideas from [4] later became the basis for subsequent research. For example, Lemma 5.2 in [4] was the first result in finding pointwise dense families of selections.

In comparison with [4], the paper [5] originally dealt only with the unique Theorem 1.2, the so-called "Finite-dimensional selection theorem":

Theorem 3. *Let X be a paracompact space, $A \subset X$ a closed subset with $\dim_X(X \setminus A) \leqslant n+1$, Y a complete metric space, $\mathcal{F}$ an equi-LC^n family of nonempty closed subsets of Y and $\phi : X \to \mathcal{F}$ an LSC map. Then every singlevalued continuous selection of $\phi|_A$ can be extended to a singlevalued continuous selection of $\phi|_U$, for some open subset $U \supset A$. If additionally every member of $\mathcal{F}$ is n-connected* (*briefly, C^n*) *then one can take $U = X$.*

Without any doubt, this is one of the most complicated topological theorems, the six-step proof in [5] is clearly a mathematical masterpiece. Various efforts were made by several people in the last 50 years to simplify this proof (or "improve" it), including ourselves. However, none of these versions turned out to be shorter or simpler. In our opinion, none of them reached the clarity of exposition in [5].

Two years later, in 1958, Dyer and Hamström applied this theorem to get the sufficient conditions for a regular map f to be a trivial fibration. Such a condition turned out to be local n-connectedness (LC^n) of the homeomorphisms group $H(M)$ of the fiber M of f. The problem when $H(M)$ is LC^n was one of the central in topology over a period of almost 20 years and served as one of the key sources for the development of infinite-dimensional topology, as a separate part of topology.

For the first encounter with the theory of selections, the papers [4] and [5] are too difficult and too voluminous. On the other hand, the short note [3] quickly tells the reader of the most popular method of selection theory—the method of outside approximation. The note consists of the proof of the Convex-valued and the 0-dimensional selection theorems. The last theorem is a particular case of the Finite-dimensional theorem, for $n = 0$.

Theorem 4. *If X is a zero-dimensional* ($\dim X = 0$) *paracompact space and Y is a complete metric space, then every LSC mapping $\phi : X \to \mathcal{F}(Y)$ admits a singlevalued continuous selection.*

In spite of its relative simplicity and clarity of its proof, the Zero-dimensional selection theorem has surprisingly many applications in selection theory and other areas of mathematics.

The last paper of the series [4–6] dealt mainly with restrictions on the displacement of a closed subset A in X. For example, as in the Borsuk pairs, when $X = Z \times [0; 1]$ and $A = (Z \times 0) \cup (B \times [0; 1])$ for an appropriate $B \subset X$. Also the lower semicontinuity assumption in [6] was strengthened by continuity in the corresponding Hausdorff metric h_ρ in $\exp X$. Here we reproduce a typical statement (Theorem 6.1):

Theorem 5. *Let X be a paracompact space with* $\dim X \leqslant n + 1$ *and $A \subset X$ a weak deformation retract of X. Let (Y, ρ) be a complete metric space, $\mathcal{F}$ a uniformly-LC^n family of nonempty closed subsets of (Y, ρ) and $\phi : X \to \mathcal{F}$ a continuous map with respect to h_ρ. Then every selection of $\phi|_A$ can be extended to a singlevalued continuous selection of ϕ.*

Surprisingly, deep constructions and results of [6] have until now had no essential applications.

4. Papers from 1959

We begin by the first paper of the series [7–10]. If one combines arbitrary paracompact domains, as in the Convex-valued selection theorem, and arbitrary complete metric ranges for closed-valued mappings, as in the Zero-dimensional selection theorem, then of course, there is no hope of obtaining a *singlevalued* continuous selection. It turned out that under those assumptions a sufficiently fine *multivalued* selections exist. It was rather an unexpected and "...curious result about semi-continuous..., [7]" selections. Below, 2^Y denotes the family of all nonempty subsets of a set Y:

Theorem 6. *(See [7, Theorem 1.1].) Let X be a paracompact space, Y a metric space, and $\phi : X \to 2^Y$ an LSC map with each $\phi(x)$ complete. Then there exist $\psi : X \to 2^Y$ and $\theta : X \to 2^Y$ such that*:

(a) $\psi(x) \subset \theta(x) \subset \phi(x)$ *for all $x \in X$*;
(b) $\psi(x)$ *and* $\theta(x)$ *are compact, for all $x \in X$*;
(c) ψ *is LSC*; *and*
(d) θ *is USC* (*upper semicontinuous*).

It appears that this was in principle, the very first theorem on multivalued selections. The proof of this Compact-valued selection theorem is based on the so-called method of inner approximations. Roughly speaking, one can inscribe into each value $\phi(x)$ a tree with a countable set of levels, with finite sets of vertices on each level, so that each maximal linearly ordered sequence of vertices will be fundamental.

Thus the sets $\psi(x)$ and $\theta(x)$ are constructed as the sets of limits of different kinds of such maximal paths in the tree. Shortly, $\psi(x)$ and $\theta(x)$ are limits of certain inverse (countable) spectra in the complete metric space $\phi(x)$. Beginning by [7], multivalued selections became by then a fully respected part of general selection theory.

The comprehensive fundamental paper [10] also had an important impact on the development of selection theory. In that paper the axiomatic theory of convexity in metric spaces was presented. As far as we know, this was also one of the first papers on axiomatic convexities. It served as the starting point for many investigations in this direction.

Also, the method of inner approximations from [7] was changed and applied in [10] to convex-valued maps. Roughly speaking, at each level of a tree above one can consider the barycenter of all vertices at that level, with respect to a suitable continuous partition of the unity of the domain. In this way it is possible to obtain a pointwise convergent sequence of singlevalued (discontinuous!) selections with degree of discontinuity uniformly tending to zero. Therefore the limit gives the desired continuous singlevalued selection.

In our experience, we have encountered several times the situations when the simpler and more direct smoothing method of outside approximations did not work, whereas the method of inner approximations successfully solved the problem at hand. Looking at the data on submission of the papers, one may perhaps infer that [10] was originally the source for [7].

Whereas Lemmas 5.1 and 5.2 and Theorem $3.1'''$ were proved in [4] for perfectly normal domains and separable Banach range spaces, a version was obtained in [8] for metric domains and any Banach range spaces. The proof was based on the replacement of the G_δ-property for closed subsets of a perfectly normal domain by the Stone theorem on the existence of σ-discrete closed basis in any metric space. Note also that Theorem 5.1 [8] on the one hand, used the ideas from the proofs in [6], and on the other hand was the basis for the later appearance of such notions as SEP and SNEP (*selection extension* and *selection neighborhood extension properties*) in [18].

While [10] estimates the relations and links between convex and metric structures on the set, the paper [9] deals with the degree of nonconvexity of a closed subset P of a Banach space, endowed with standard convex and metric structures. Simply put, imagine that we move the endpoints of a segment of length $2r$ over a set P. In this situation it is very natural to look for the distance between the points of segment and the set P.

So if all such distances are less than or equal to $\alpha \cdot r$ for some constant $\alpha \in [0; 1)$, then the set P is *paraconvex* in dimension 1. By passing to triangles, tetrahedra, and n-simplices, one obtains the notion of a *paraconvex* set. So, as it was proved in [9], the statement of the Convex-valued selection theorem [3,4] holds whenever one replaces the convexity assumption for the values $\phi(x)$ by their α-paraconvexity, for some common $\alpha \in [0; 1)$, for all $x \in X$.

Moreover, the proof looks as a double sequential "improvement" process of exactness of approximation, on the account of applying the Convex-valued selection theorem.

5. Papers from 1964–1979

One of the main purposes of the series [12–15] was to examine the metrizability assumption for the range space in the Convex-valued selection theorem. In the papers [12–14] improvements of the Arens–Eells embedding theorem were proved and a selection theorem for mappings from metric domains into completely metrizable subsets of locally convex topological vector (LCTV) spaces was established. It was shown in [15] that the statement holds for paracompact domains as well. Observe that for LCTV spaces *completness* is a delicate and in general, "multivalued" notion. Below, a LCTV space is said to be *complete* if the closed convex hull of any compact subset is also a compact subset.

Theorem 7. *(See [15, Theorem 1.2].) Let X be a paracompact space and (M, ρ) a metric subset of a complete LCTV space E. Let $\phi : X \to 2^M$ be an LSC map such that every $\phi(x)$ is ρ-complete. Then there exists a continuous singlevalued $f : X \to E$ such that for every $x \in X$, the value $f(x)$ belongs to the closed convex hull of the set $\phi(x)$.*

Note that one of the key ingredients of the proof is the Compact-valued selection theorem. Next, if ϕ is convex-valued and closed-valued, then completness of the entire E can be replaced by completness of the closed spans of $\phi(x)$, $x \in X$. Such a replacement can also be derived from the Zero-dimensional selection theorem and by the technique of pointwise integration (see [R13]).

In the joint paper with Engelking and Heath [16], Michael in some sense returned to his first selection publication [1]. Namely, by using embeddings into closed topologically well-ordered subspaces of the Baire space $B(\mathfrak{m})$, they proved [16, Corollary 2] that for any complete metric, zero-dimensional (with respect to dim or Ind) space (X, ρ) there exists a singlevalued continuous *selector* f on the family $\mathcal{F}(X)$ of all nonempty closed subsets of X.

Here $\mathcal{F}(X)$ is endowed with the Hausdorff topology, say τ_ρ, and $f : \mathcal{F}(X) \to X$ is a mapping with $f(A) \in A$ for every $A \in \mathcal{F}(X)$. The zero-dimensionality is the necessary restriction, because for example, there are no selectors for $\mathcal{F}(\mathbb{R})$ (see [16, Proposition 5.1]).

Note that formally, a selector is simply a selection of the multivalued evaluation mapping, which associates to each $A \in \mathcal{F}(X)$ the same A, but as a subset of X. However, historically the situation was reverse. In [1] Michael proposed a division of the problem about the existence of a selection $g : Y \to X$ for $G : Y \to 2^X$ into two separate problems: first, to check that G is continuous and second, to prove that there exists a selector on 2^X. Hence, the selection problem was originally reduced to a certain selector problem.

6. Papers from 1980–1990

Pick points $x_1, x_2, \dots, x_n$ in the domain X of a multivalued mapping ϕ and arbitrary select points $f(x_i) \in \phi(x_i)$, using the Axiom of choice. Thus we find a partial selection of ϕ over the closed subset $C = \{x_1, x_2, \dots, x_n\} \subset X$. By replacing the values $\phi(x_i)$ with the singletons $\{f(x_i)\}$ we once again obtain an LSC mapping, say ϕ_C. If all assumptions of a selection theorem hold for the new LSC mapping ϕ_C, then such a mapping admits a selection, and hence ϕ also admits a selection.

This simple observation shows that any restrictions for the value of ϕ over a finite subset $C \subset X$, like closedness, connectivity, convexity, etc. are inessential for the existence of a continuous selection of ϕ. But what can one say about such an omission for an *infinite* $C \subset X$? Clearly, C should be a sufficiently "small", "dispersed", etc. subset of X. At the International congress of mathematicians in Vancouver in 1974, Michael announced results for countable C. Based on this, the following result was published in 1981 (see [19, Theorem 1.4]):

Theorem 8. *Let X be a paracompact space, Y a Banach space, $C \subset X$ a countable subset and $\phi : X \to 2^Y$ an LSC map with closed and convex values $\phi(x)$ for all $x \notin C$. Then for every closed subset $A \subset X$, each selection of $\phi|_A$ admits an extension which is a selection of ϕ (shortly, ϕ has SEP).*

Briefly, over a countable subset of a domain we can simply omit any restriction for the values of LSC map $\phi : X \to 2^Y$. A year before, in a joint paper with Pixley [17], Michael proved that the convexity assumption can be omitted over any subset $Z \subset X$ with $\dim_X Z = 0$.

Roughly speaking, results of [17–19,23,25] are principally related to several possibilities for relaxing convexity in selection theorems and in particular, the closedness assumptions for values of multivalued mappings. For example, let us mention the following two results:

Theorem 9. *(See [19, Theorem 7.1].) Let X be a paracompact space, Y a Banach space, $C \subset X$ a countable subset, $Z \subset X$ a subset with* $\dim_X Z \leqslant 0$ *and $\phi : X \to 2^Y$ an LSC map such that $\phi(x)$ is closed for all $x \notin C$ and Clos$(\phi(x))$ is convex, for all $x \notin Z$. Then ϕ has SEP.*

Theorem 10. *(See [18, Theorem 1.2].) Let X be a paracompact space, Y a Banach space, $Z \subset X$ a subset with* $\dim_X Z \leqslant n+1$ *and $\phi : X \to \mathcal{F}(Y)$ an LSC map such that and $\phi(x)$ is convex, for all $x \notin Z$ and the family $\{\phi(x)\colon x \in Z\}$ is uniformly equi-LC^n. Then ϕ has SNEP. If moreover, $\phi(x)$ is n-connected for every $x \in Z$, then ϕ has SEP.*

Note that in [18] the technique of the proof in [5] was rearranged in a more structured form, with exact extracting of the useful properties like SEP, SNEP and SAP (*selection approximation property*).

The joint paper with Mägerl and Mauldin [20] formally contains no "selections" in the title or in the statements of the main theorems (1.1 and 1.2). Nevertheless, the essence of these theorems is contained in the selection result.

It is a classical fact that each metric compact X can be represented as the image of the Cantor set K under some continuous surjection $h : K \to X$. Theorem 5.1 of [20] states that if $\{X_\alpha\}$ is a family of subcompacta of a metric space X which is continuously parameterized by $\alpha \in A$ with $\dim A = 0$ then one can choose a family of surjections $h_\alpha : K \to X_\alpha$ continuously depending on the same parameter $\alpha \in A$. Such parameterized version of the Alexandrov theorem is in fact, derived from the Zero-dimensional selection theorem.

In general, the decade 1980–1990 was marked by Michael's very diverse set of papers on selections, practically every one of which contained new ideas of high quality. For one more example, the Finite-dimensional selection theorem from [5] was strengthened in [23] simultaneously in two directions. First, the assumption that $\{\phi(x)\}_{x\in X}$ is an equi-LC^n family in Y was replaced by the property that fibers $\{\{x\} \times \{\phi(x)\}_{x\in X}\}$ constitutes an equi-LC^n family in $X \times Y$. This answered the problem of Eilenberg stated in 1956 (see the comments in [5]). Next, the closedness assumption for $\phi(x) \subset Y$ can be weakened to the closedness of graph-fibers $\{x\} \times \phi(x)$ in some G_δ-subset of $X \times Y$.

The key ingredient of the proof was a "factorization" construction. Briefly, it turned out that the LSC mapping $\phi : X \to Y$ with weakened assumptions can be represented as a composition $\phi = h \circ \psi$ with singlevalued $h : Z \to Y$ and with $\psi : X \to Z$, where ψ satisfies the classical assumptions of the Finite-dimensional selection theorem [5]. Hence the composition of a selection of ψ with h gives the desired selection of ϕ.

We guess that the idea of the appearance of the G_δ-conditions was a corollary of constructions of selections, avoiding a countable set of obstructions, from the paper [22] which appeared one year earlier:

Theorem 11. *(See [22, Theorem 3.3].) Let X be a paracompact space, Y a Banach space and $\phi : X \to \mathcal{F}(Y)$ a LSC map with convex values. Let $\psi_i : X \to \mathcal{F}(Y)$, $i \in \mathbb{N}$, be continuous, $Z_i = \{x \in X\colon \phi(x) \cap \psi_i(x) \neq \emptyset\}$ and suppose that*

$$\dim X < \dim \phi(x) - \dim\bigl(\operatorname{conv}\bigl(\phi(x) \cap \psi_i(x)\bigr)\bigr),$$

for all $x \in Z_i$ and $i \in \mathbb{N}$. Then ϕ admits a selection f which avoids every ψ_i: $f(x) \notin \psi_i(x)$.

Briefly, in the values $\phi(x)$ there is sufficient "room" to avoid all sets $\psi_i(x)$.

Based on [22,23], Michael stated in "Open problems in topology, I" the "G_δ"-problem [24, Problem 396]: Does the Convex-valued selection theorem remain true if ϕ maps X into some G_δ-subset Y of a Banach space B with convex values which are closed in Y? In spite of numerous cases with affirmative answer this problem has in general a negative (as it was convected in [24]) solution, for details see the paper of Namioka and Michael in this issue.

7. Papers from 1992

In general, all papers [21,25,27] are related to "dispersed", mainly to zero-dimensional, (in dim-sense) domains of multivalued mappings.

Briefly, in [25] results of [17, Theorem 1.1] and [19, Theorem 1.3] are unified and generalized in the spirit of [23] to subsets $C \subset X$, which are unions of countable family of G_δ-subsets C_n of X and to a mappings ϕ, having SNEP at each C_n. In the paper written with Choban [27], the Compact-valued selection Theorem 6 was derived from the Zero-dimensional one (Theorem 4). In fact, a paracompact domain X was represented as the image $h(Z)$ of some zero-dimensional paracompact space Z with respect to some appropriate continuous (perfect or inductively open) mapping $h : Z \to X$. Theorem 4 applied to the composition $\phi \circ h$ gives a selection, say $s : Z \to Y$. So, the composition $s \circ h^{-1}$ will be a desired multivalued selection of $\phi : X \to \mathcal{F}(Y)$.

The pair of papers [26,28] is related to "the differences between selection theorems which assume that the domain is finite-dimensional and those which do not". More generally, based on the Pixley counterexample in [26] it was shown that a genuine dimension-free analogue of the Finite dimensional selection theorem does not exist or briefly, that there are no purely topological analogs of convexity. In comparison, in [28] a convexity, or connectivity type restrictions in the spirit of [10] for a *mapping* are presented and under such restrictions the equivalence is proved between the existence of global selections and the existence of selections locally.

The paper [29] on continuum-valued selections is an elegant simultaneous application of the "universality" idea from [27] and the one-dimensional selection theorem (special case $n = 0$ of Theorem 5). The key step can be described as follows. Due to a recent theorem of Pasynkov, each paracompact domain X can be represented in the form $h(Z)$, for some perfect, open surjection $h : Z \to X$ with pathwise connected fibers and for some paracompact space Z with $\dim Z \leqslant 1$. So, if the composition $\phi \circ h$ admits a selection, say $s : Z \to Y$ then the composition $s \circ h^{-1}$ will be a continuum-valued selection of $\phi : X \to \mathcal{F}(Y)$.

We should mention the thoughtfulness, exactness and perfectness of all Michael's papers. His laconic style of exposition is perfectly matched with the deepness of his results. In our opinion, *A man of few words but with great ideas* could well serve as a good description of his character. As a rule, all his papers are equipped with a considerable number of additional references, which were added at proofs, and which very precisely give correct accents to the paper needed for proper understanding. In conclusion of this survey of Michael's results on selections we wish our jubilant successful realization of many more projects.

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