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J. Math. Anal. Appl. 314 (2006) 537–545
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A minimax theorem for functions with possibly nonconnected intersections of sublevel sets

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Received 1 January 2005
Available online 10 May 2005
Submitted by Steven G. Krantz

Abstract

We apply the selection theorem for multivalued mappings with paraconvex values (rather than various versions of KKM-principle) to prove several minimax theorems. In contrast with well-known minimax theorems for coordinatewise semicontinuous functions, in our theorems finite intersections of sublevel or uplevel sets can be nonempty and nonconnected.

Keywords: Paraconvexity; Convex-valued mapping; Continuous selection; Banach space; Lower semicontinuous and upper semicontinuous map

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[1] The first author was supported by the MHEST research program No. P1-0292-0101-04.
[2] The second author was supported by the RFBR grant No. 05-01-00993.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.04.002

0. Introduction

Every function $f : X \times Y \to \mathbb{R}$ defined on the Cartesian product of two sets satisfies the inequality

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) \leqslant \inf_{y\in Y} \sup_{x\in X} f(x, y).$$

John von Neumann [10] proved the equality

$$\max_{x\in X}\Big\{\min_{y\in Y}\big\{f(x, y)\big\}\Big\} = \min_{y\in Y}\Big\{\max_{x\in X}\big\{f(x, y)\big\}\Big\}$$

for any finite-dimensional simplices X and Y and any bilinear function f. Ky Fan [1] and Sion [14] minimax theorems prove the above minimax equality for any pair of convex subcompacta X and Y of locally convex topological linear spaces and for any real-valued function $f : X \times Y \to \mathbb{R}$ which satisfies the following assumptions:

(1) for each $c \in \mathbb{R}$ and each $x_0 \in X$ the set $\{y \in Y\colon\ f(x_0, y) \leqslant c\}$ is convex and compact;
(2) for each $d \in \mathbb{R}$ and each $y_0 \in Y$ the set $\{x \in X\colon\ f(x, y_0) \geqslant d\}$ is convex and compact.

Note, that (1) implies openness of all sets of the form $\{y \in Y\colon\ f(x_0, y) > c\}$, i.e. lower semicontinuity of the function f with respect to the second coordinate. By compactness of Y, the function

$$x \mapsto \min\big\{f(x, y)\colon\ y \in Y\big\}$$

is well defined and upper semicontinuous on the compactum X. Thus the number $b = \max_{x\in X}\{\min_{y\in Y}\{f(x, y)\}\}$ indeed exists. In the same manner one can check the existence of the number $a = \min_{y\in Y}\{\max_{x\in X}\{f(x, y)\}\}$, $b \leqslant a$.

There exist many generalizations of this fundamental theorem of von Neumann–Ky Fan–Sion. Most of them deal with various kinds of generalized, topological, or axiomatically defined convexities in (1) and (2) (see [3,5,6]). After the fundamental result of Ky Fan [2], the key role in all approaches to minimax theorems was played by the so-called Knaster–Kuratowski–Mazurkiewicz (KKM) principle concerning finite intersection property of values of KKM-mappings [7]. Having in mind this principle, many authors did not exploit precisely the convexity assumption, but only the basic hereditary property that the intersection of convex sets is also convex. Moreover, due to results [15,17] it is clear that convexity of such intersections can as a rule be simply replaced by connectedness, the sets

$$\bigcap_{i=1}^{n}\Big\{\big\{x \in X\colon\ f(x, y_i) \geqslant d\big\}\colon\ y_i \in Y\Big\},$$

$$\bigcap_{j=1}^{k}\Big\{\big\{y \in Y\colon\ f(x_i, y) \leqslant c\big\}\colon\ x_j \in X\Big\}$$

are connected whenever they are nonempty [3,6,7].

The present paper deals with another principal property, somewhat symmetrical to *intersections*, namely that the *union* of directly ordered family of arbitrary convex sets is also convex. So as a base for obtaining minimax theorems we shall use the selection theory of multivalued mappings instead of versions of the KKM-principle. More precisely, we shall use the selection theorem for multivalued mappings with *α-paraconvex* values [12,13]. Therefore our minimax theorem includes cases when the latter finite intersection of sublevel and uplevel sets is nonempty but not connected.

1. Preliminaries

We shall denote the open ball in a Banach space, centered at the point y, of radius r by $D(y,r)$. Let P be a nonempty closed subset of a normed space B. The number

$$\delta(P, D) = \sup\bigl\{\mathrm{dist}(q, P)/r\colon\ q \in \mathrm{conv}(P \cap D)\bigr\}$$

is a natural upper estimate for the relative precision of nonconvexity of the intersection of the set P with the open ball D of radius r.

Definition 1.1. The function of nonconvexity $\alpha_P(\cdot)$ of the set P associates to each number $r > 0$ the supremum of the set $\{\delta(P, D)\}$ over all open balls of radius r.

Clearly, the identity $\alpha_P(\cdot) \equiv 0$ means that the closed set P is convex. The more function $\alpha_P(\cdot)$ differs from zero, the "less convex" is the set P.

Definition 1.2. A nonempty closed subset P of a Banach space is said to be α-paraconvex provided that function $\alpha(\cdot)$ pointwisely majorates the function of nonconvexity $\alpha_P(\cdot)$ of this set: $\alpha_P(r) < \alpha(r)$, $r > 0$.

Geometrically, α-paraconvexity of a subset $P \subset B$ means that for every open ball D of radius r which intersects P and for each point q of the closed convex hull $\mathrm{conv}(P \cap D)$, the distance $\mathrm{dist}(q, P)$ between q and P is less than $\alpha(r) \cdot r$.

The following selection theorem was proved in [13].

Theorem 1.3. *Suppose that the right upper limits of the function $\alpha : (0,\infty) \to (0,1)$ are less than* 1 *over the closed ray* $[0,\infty)$. *Let $\Phi : E \to B$ be a lower semicontinuous mapping from a paracompact space E into a Banach space B with all values α-paraconvex. Then Φ admits a continuous single-valued selection.*

For a constant function α this theorem was proved by Michael [9]—he introduced the notion of α-paraconvexity for a *constant* α. As a corollary, we prove that α-paraconvex sets have the same maximally fine topological properties as the usual convex closed sets: they are contractible and locally contractible and moreover, the collection $\Pi_\alpha(B)$ of all α-paraconvex subsets of a given Banach space B is equilocally connected family ($\Pi_\alpha(B) \in ELC$). Graphs of Lipschitz functions of several real variables are typical examples of paraconvex sets [11]. Clearly, intersection of two such sets can have several components of connectedness.

In what follows we shall assume that $\alpha : (0, \infty) \to (0, 1)$ is a fixed function with the right upper limits less than 1 over the closed ray $[0, \infty)$. We shall also use the notation $\alpha(\cdot) < \beta(\cdot)$ for pointwise inequality between real valued functions.

Theorem 1.4. *Let $f : X \times Y \to \mathbb{R}$ be a real-valued function on Cartesian product of two AR subcompacta X and Y of a Banach space and suppose that*:

(1) *for each $c \in \mathbb{R}$ and each $x_0 \in X$, the set $\{y \in Y\colon\ f(x_0, y) \leqslant c\}$ is α-paraconvex compact*; *and*
(2) *for each $d \in \mathbb{R}$ and each $y_0 \in Y$, the set $\{x \in X\colon\ f(x, y_0) \geqslant d\}$ is α-paraconvex compact for a fixed $\alpha : (0, \infty) \to [0, 1)$.*

Then $\max_X\{\min_Y\{f(x, y)\}\} = \min_Y\{\max_X\{f(x, y)\}\}$.

Recall that a multivalued mapping $F : X \to Y$ is said to be *lower semicontinuous* (LSC) if the set $F^{-1}(U) = \{x \in X\colon\ F(x) \cap U \neq \emptyset\}$ is open in X whenever U is open in Y. A single-valued mapping $f : X \to Y$ is a *selection* of multivalued mapping F whenever $f(x) \in F(x)$ for all $x \in X$. If a multivalued mapping F maps a set X into itself then $x_0 \in X$ is said to be a *fixed point* of F provided that $x_0 \in F(x_0)$.

Recall also that each *AR* compactum has the fixed point property for single-valued continuous mappings into itself. In fact, such compacta are closed subsets of suitable Tikhonov cubes and moreover are retracts of these cubes. Finally, a real-valued single-valued function $h : X \to \mathbb{R}$ is said to be *upper* (*lower*) *semicontinuous* if all preimages $h^{-1}(-\infty, c)$ (respectively, $h^{-1}(d, \infty)$) are open subsets of X, for any $c, d \in \mathbb{R}$. We preserve the term “function” for mappings to real line and use the abbreviation LSC only for nonsingle-valued mappings.

2. Two lemmas

Lemma 2.1. *For any functions $\alpha(\cdot) < \beta(\cdot) < 1$ and sequence $P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots$ of α-paraconvex subsets of a Banach space B, the closure of the union $P = \mathrm{Cl}(\bigcup_n P_n)$ is a β-paraconvex subset of B.*

Proof. It suffices to check that $\alpha_P(\cdot) \leqslant \alpha(\cdot)$. Suppose to the contrary, that for some $r > 0$ the inequality $\alpha_P(r) > \alpha(r)$ holds. Then there exist an open ball D of radius r and a point $q \in \mathrm{conv}(P \cap D)$ such that $\mathrm{dist}(q, P) > \alpha(r) \cdot r$. We can assume then that $\mathrm{dist}(q', P) > \alpha(r) \cdot r$ for all points q' from some sufficiently small neighborhood $D(q, \varepsilon) \subset D$ of the point q. If

$$q = \sum_{i=1}^{m} \lambda_i y_i, \quad y_i \in P \cap D,\ \lambda_i \geqslant 0,\ \sum \lambda_i = 1,$$

then every point $y_i \in P \cap D$ can be represented as the limit of a sequence $\{y_i^k\}_{k=1}^{\infty}$ of points from the intersection $D \cap (\bigcup_{n=1}^{\infty} P_n)$.

For every $i \in \{1, 2, \ldots, m\}$ one can find a point

$$y_i^{k_i} \in D \cap D(y_i, \varepsilon) \cap \left(\bigcup_{n=1}^{\infty} P_n \right).$$

Let $N = \max\{k_1, k_2, \ldots, k_m\}$ and $q' = \sum_{i=1}^{m} \lambda_i y_i^{k_i}$, where coefficients λ_i are taken from the above representation of the point q as the convex combination. Then

$$P_1 \subset P_2 \subset \cdots \subset P_N, \qquad q' \in D(q, \varepsilon) \cap \operatorname{conv}(P_N \cap D)$$

and therefore

$$\alpha_{P_N}(r) \cdot r \geqslant \operatorname{dist}(q', P_N) \geqslant \operatorname{dist}(q', P) > \alpha(r) \cdot r$$

which contradicts with the inequality $\alpha_{P_N}(r) < \alpha(r)$. $\square$

Lemma 2.2. *For any functions $\alpha(\cdot) < \beta(\cdot) < 1$, the Cartesian product of any two α-paraconvex subsets of Banach spaces B_1 and B_2 is a β-paraconvex subset of the Cartesian product $B_1 \times B_2$ with respect to the norm*

$$\big\| (x_1, x_2) \big\| = \max\big\{ \|x_1\|_1, \|x_2\|_2 \big\}.$$

Proof. Pick an open ball $D = D((x_0, y_0), r)$ in the Banach space $B = B_1 \times B_2$ which intersects the Cartesian product $P = P_1 \times P_2$ of two α-paraconvex sets. Choose any points $(x_i, y_i) \in P \cap D$, $i = 1, 2, \ldots, m$, and consider their convex combination

$$q = \sum_{i=1}^{m} \lambda_i (x_i, y_i), \quad \lambda_i \geqslant 0, \ \sum \lambda_i = 1.$$

Denote $q_1 = \sum_{i=1}^{m} \lambda_i x_i$ and $q_2 = \sum_{i=1}^{m} \lambda_i y_i$. Due to the definition of the norm in B, we have that $\max\{\|x_i - x_0\|, \|y_i - y_0\|\} < r$, for every $i = 1, 2, \ldots, m$. By definition of functions of nonconvexity, one can find for every positive ε points $p_1 \in P_1$ and $p_2 \in P_2$ such that

$$\|q_i - p_i\| < \operatorname{dist}(q_i, P_i) + \varepsilon \leqslant a_{P_i}(r) \cdot r + \varepsilon < \alpha(r) \cdot r + \varepsilon, \quad i \in \{1, 2\}.$$

Hence we can find for every $\varepsilon > 0$ a point $p = (p_1, p_2) \in P$ such that

$$\|q - p\| = \max\big\{ \|q_1 - p_1\|, \|q_2 - p_2\| \big\} < \alpha(r) \cdot r + \varepsilon.$$

If $\varepsilon \to 0$, then $\operatorname{dist}(q, P) \leqslant \alpha(r) \cdot r$. Passing to the supremum over all q and all open balls of fixed radius r, we conclude that $\alpha_P(r) \leqslant \alpha(r)$ for all positive r. $\square$

Note, that in Lemmas 2.1 and 2.2 we can assume that all right upper limits of function β are also less than 1. Hence Theorem 1.3 also applies to Cartesian products and to closures of increasing sequences of α-paraconvex sets.

3. Proof of Theorem 1.4

We shall follow the usual strategy of the proof in which the inequality

$$b = \max_{x\in X}\Bigl\{\min_{y\in Y}\bigl\{f(x,y)\bigr\}\Bigr\} < a = \min_{y\in Y}\Bigl\{\max_{x\in X}\bigl\{f(x,y)\bigr\}\Bigr\}$$

implies a contradiction. But instead of a single one we shall use *two* separation numbers between b and a. Recall that b and a exist and $b \leqslant a$, see the Introduction.

Suppose to the contrary that $b < d < c < a$. Define the multivalued mapping $F: X \times Y \to X \times Y$ by setting

$$F(x,y) = \bigl\{x' \in X\colon\ f(x',y) > c\bigr\} \times \bigl\{y' \in Y\colon\ f(x,y') < d\bigr\}$$

and let $\Phi: X \times Y \to X \times Y$ be its pointwise closure:

$$\Phi(x,y) = \mathrm{Cl}\bigl(H(x,y)\bigr), \quad (x;y) \in X \times Y.$$

We claim that then:

(i) Theorem 1.3 applies to the mapping Φ, i.e. Φ admits a single-valued continuous selection $\varphi: X \times Y \to X \times Y$;
(ii) φ has a fixed point $(x_0, y_0) \in X \times Y$; and
(iii) the inequality $c \leqslant \varphi(x_0, y_0) \leqslant d$ holds, which contradicts our assumption that $b < d < c < a$.

Let us verify (i)–(iii). First, note that (ii) holds because φ is a continuous mapping of the *AR* compactum $X \times Y$ into itself. Second, $(x_0, y_0) \in \mathrm{Cl}(F(x_0, y_0))$ implies that

$$(x_0, y_0) = \lim_{n\to\infty}(x_n, y_n), \qquad (x_n, y_n) \in F(x_0, y_0).$$

In other words, for each $n \in \mathbb{N}$ we have that $f(x_n, y_0) > c$ and $f(x_0, y_n) < d$. However, the function $f: X \times Y \to \mathbb{R}$ is upper semicontinuous with respect to the first coordinate and is lower semicontinuous with respect to the second coordinate. Therefore, by passing to the limit with $n \to \infty$, we obtain (iii): $c \leqslant f(x_0, y_0) \leqslant d$.

So, it only remains to verify (i). Suppose that $F(x,y) = \emptyset$, for some $(x,y) \in X \times Y$ and to be certain, let the second factor of $F(x)$, i.e. the set $\{y' \in Y\colon\ f(x,y') < d\}$ be empty. Then $f(x,y') \geqslant d$ for all $y' \in Y$ and $\min\{f(x,y')\colon\ y' \in Y\} \geqslant d$. Hence

$$b = \max_X\Bigl\{\min_Y\bigl\{f(x,y)\bigr\}\Bigr\} \geqslant d > b.$$

Contradiction. So all values of F are indeed nonempty.

The assumptions (1) and (2) together with the equalities

$$\begin{aligned} F(x,y) &= \bigl\{x' \in X\colon\ f(x',y) > c\bigr\} \times \bigl\{y' \in Y\colon\ f(x,y') < d\bigr\} \\ &= \Bigl(\bigcup_{n\in\mathbb{N}}\bigl\{x' \in X\colon\ f(x',y) \geqslant c + n^{-1}\bigr\}\Bigr) \\ &\quad \times \Bigl(\bigcup_{n\in\mathbb{N}}\bigl\{y' \in Y\colon\ f(x,y') \leqslant d - n^{-1}\bigr\}\Bigr) \end{aligned}$$

show that the values of F are Cartesian products of increasing sequences of α-paraconvex sets. By applying Lemmas 2.1 and 2.2 we conclude that all values of Φ are nonempty β-paraconvex subsets of $X \times Y$.

Therefore in order to complete the proof we must check that $\Phi = \mathrm{Cl}(F)$ is a LSC mapping. It certainly suffices to show that F is a LSC mapping. Checking this is a well-known verification that all point preimages of the mapping F are open subsets of $X \times Y$ and hence $F^{-1}(A)$ is open for every $A \subset X \times Y$:

$$\begin{aligned} F^{-1}(x, y) &= \left\{(x', y')\colon\ (x, y) \in F(x', y')\right\} \\ &= \left\{x' \in X\colon\ f(x', y) < d\right\} \times \left\{y' \in Y\colon\ f(x, y') > c\right\}. \end{aligned}$$

However, the latter two factors are open subsets of X and Y, respectively, by upper semicontinuity of $f(\cdot, y)$ and lower semicontinuity of $f(x, \cdot)$. This completes the proof. $\square$

Clearly, in the proof of Theorem 1.4 we never used any specific (geometric or topological) property of paraconvex sets. We simply reduced the proof to Lemmas 2.1, 2.2, selection Theorem 1.2 and to the fixed-point property of *AR* compacta. This is why Theorem 1.4 holds for arbitrary classes of subsets, provided that such classes satisfy hypothesis of these lemmas and theorem.

4. Some generalizations

Here is an abstract version of Theorem 1.4. For a family Ω of nonempty sets denote by $\Omega_{\sigma\uparrow}$ the family which consists of all unions of countable increasing (with respect to the inclusion) sequences of elements of Ω. For a family Ω of subsets of a topological space denote by $\mathrm{Cl}(\Omega)$ the family of all closures of the elements of Ω.

We also say that a family Ω of nonempty closed subsets of a topological space Y is *selectionable* in Y whenever for each paracompact space E, every lower semicontinuous mapping $\Phi : E \to Y$ with values from Ω, admits a single-valued continuous selection. The families of all nonempty convex closed subsets or all α-paraconvex subsets of a Banach spaces are typical examples of selectionable families.

Theorem 4.1. *Let Ω and Γ be families of nonempty closed subsets of Fréchet spaces B_1 and B_2, respectively, such that the family $\mathrm{Cl}(\Omega_{\sigma\uparrow} \times \Gamma_{\sigma\uparrow})$ is selectionable in $B_1 \times B_2$. Let $f : X \times Y \to \mathbb{R}$ be a real-valued function on the Cartesian product of two AR subcompacta of B_1 and B_2, respectively, and suppose that*:

(1) *for each $c \in \mathbb{R}$ and each $x_0 \in X$, the set $\{y \in Y\colon\ f(x_0, y) \leqslant c\}$ is an element of Γ; and*
(2) *for each $d \in \mathbb{R}$ and each $y_0 \in Y$, the set $\{x \in X\colon\ f(x, y_0) \geqslant d\}$ is an element of Ω.*

Then $\max_X\{\min_Y\{f(x, y)\}\} = \min_Y\{\max_X\{f(x, y)\}\}$.

The proof of Theorem 4.1 repeats the proof of Theorem 1.4 above. We believe that some other specific examples of families, satisfying the hypothesis of Theorem 4.1 can be constructed.

Finally, we present a generalization of another type of minimax theorems [8]: we simply replace the hypothesis "compact finite-dimensional ANR" on the second factor Y by the assumption that Y is compact and has the C-property. A space Y is said to have the *C-property* if for any sequence $\{\mu_n\colon n \in \mathbb{N}\}$ of open coverings of Y there exists a sequence $\{\lambda_n\colon n \in \mathbb{N}\}$ of disjoint families of open sets in Y such that each λ_n refines μ_n and the union $\bigcup_n \lambda_n$ is a covering of Y.

Every finite-dimensional paracompact and every countably-dimensional metric space has the C-property. It is still an open problem whether (for the metric case) the class of spaces with C-property coincides with the class of all weakly infinite-dimensional spaces.

Theorem 4.2. *Let X be an acyclic ANR space, Y a compact space with C-property, and f a real valued function on $X \times Y$. Suppose that*

$$b = \sup_{x \in X}\Big\{\min_{y \in Y}\{f(x, y)\}\Big\}, \qquad a = \min_{y \in Y}\Big\{\sup_{x \in X}\{f(x, y)\}\Big\}.$$

and that the following conditions are satisfied:

(1) *f is lower semicontinuous with respect to the second coordinate*;
(2) *the set $\{(x; y)\colon f(x; y) > b\}$ is open*;
(3) *for each $y \in Y$, the set $\{x \in X\colon f(x; y) > b\}$ is contractible or empty*; *and*
(4) *for each $x \in X$, the set $\{y \in Y\colon f(x; y) \leqslant b\}$ is acyclic.*

Then the equality $a = b$ holds.

The proof of Theorem 4.2 repeats the argument used in the proof of McClendon's theorem, as it was presented, for example, in [4, p. 332], and reduces the minimax theorem to a statement concerning intersections of two subsets in $X \times Y$.

However, in proving that statement we use a recent Uspenskii's selection theorem [16]—he has established the existence of continuous selections for arbitrary open-graph mapping with domain having the C-property and with all values infinitely connected. Unfortunately, it seems that this method does not work outside the class of C-domains because the selection theorem [16] actually gives a *characterization* of the C-property of domains.

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